Mathematics 623 – Complex Analysis
Fall 2014

Assignment 8
Due Wednesday, November 5

1. Let f be holomorphic in a domain containing $\{z : |z| \leq 1\}$ and suppose that $|f(z)| < |z|$ if $|z| = 1$. Show that f has a unique fixed point in $\{z : |z| < 1\}$, i.e. there is a unique z with $f(z) = z$ and $|z| < 1$.

2. (i) Prove that all zeros of $z^5 - 2z + 16$ are contained in the annulus $\{z : 1 < |z| < 2\}$.
(ii) How many of the zeros of $z^3 - 3z + 1$ are contained in $\{z : 1 < |z| < 2\}$?

3. Suppose that f_n are holomorphic functions in the domain[1] Ω and f_n converges to f uniformly on every compact subset of Ω. Suppose that $f_n(z) \neq 0$ for all $z \in \Omega$. Prove that either $f \equiv 0$ on Ω or f has no zeros on Ω.

4. Compute (with proof)

$$\int_0^\infty \frac{\sqrt{x}}{(x^2+4)^2} dx \, .$$

5. Problem No. 10, p. 104, in the textbook.

6. Problem No. 16, p. 106, in the textbook.

7. Problem No. 17, p. 106, in the textbook.

Additional (recommended) problem:
8.* One can prove the familiar formula

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

using an (admittedly complicated) complex analytic argument. Do this using the following outline:

Define $b = (1+i)\sqrt{\pi/2}$ and let

$$g(z) = \frac{e^{-z^2}}{1+e^{-2bz}} \, .$$

Show that e^{-2bz} is periodic with period b and that $g(z) - g(z+b) = e^{-z^2}$. For large r, s integrate $g(z)$ over the rectangle with lower corners $-r$, s, and height $\sqrt{\pi/2}$.

[1] Recall: This means Ω is open and connected.