Taylor's theorem

Theorem 1. *Let f be a function having $n+1$ continuous derivatives on an interval I. Let $a \in I$, $x \in I$. Then*

$$(*_n) \qquad f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x,a)$$

where

$$(**_n) \qquad R_n(x,a) = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt.$$

Proof. For $n = 0$ this just says that

$$f(x) = f(a) + \int_a^x f'(t)\,dt$$

which is the fundamental theorem of calculus.

For $n = 1$ we use the formula $(*_0)$ and integrate by parts. That is we apply the formula

$$\int_a^x u(t)v'(t)dt = u(x)v(x) - u(a)v(a) - \int_a^x u'(t)v(t)dt$$

with $u(t) = f'(t)$, $v(t) = t - x$.

We then get

$$\begin{aligned}\int_a^x f'(t)\,dt &= \Big[(t-x)f'(t)\Big]_a^x - \int_a^x (t-x)f''(t)dt \\ &= (x-a)f'(a) + \int_a^x (x-t)f''(t)dt.\end{aligned}$$

Therefore by $(*_0), (**_0)$

$$\begin{aligned}f(x) &= f(a) + \int_a^x f'(t)\,dt \\ &= f(a) + (x-a)f'(a) + \int_a^x (x-t)f''(t)dt.\end{aligned}$$

We prove the general case using induction. We show that the formula $(*_n)$ implies the formula $(*_{n+1})$. Suppose we have already proved the formula for a certain number $n \geq 0$. Then we integrate by parts in the remainder term $R_n(x,a)$ (cf. the above formula with $u(t) = f^{(n+1)}(t)$, $v(t) = (x-t)^{n+1}/(n+1)$). We obtain

$$\begin{aligned}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt &= \Big[\frac{-(x-t)^{n+1}}{n+1} f^{(n+1)}(t)\Big]_a^x - \int_a^x \frac{-(x-t)^{n+1}}{n+1} f^{(n+2)}(t)dt \\ &= \frac{(x-a)^{n+1}}{n+1} f^{(n+1)}(a) + \int_a^x \frac{(x-t)^{n+1}}{n+1} f^{(n+2)}(t)dt.\end{aligned}$$

Deviding by $n!$ yields

$$R_n(x,a) = \frac{(x-a)^{n+1}}{(n+1)!} + R_{n+1}(x,a).$$

Assuming the correctness of $(*_n)$, $(**)_n$ we may deduce $(*)_{n+1}$, $(**_{n+1})$:

$$\begin{aligned} f(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^n(a)}{n!}(x-a)^n + R_n(x,a) \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + R_{n+1}(x,a). \quad \square \end{aligned}$$

with $R_{n+1}(x,a) = \int_a^x \frac{(x-t)^{n+1}}{(n+1)!} f^{(n+2)}(t)dt$.

We now want to estimate the remainder term R_n.

Theorem 2. *Let f be as in Theorem 1 and R_n as in $(**_n)$. Let*

$$M = \max\{|f^{(n+1)}(t)| : t \text{ between } a \text{ and } x\}.$$

Then

$$|R_n(x,a)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}.$$

Proof. Let $I(a,x)$ be the interval with endpoints a and x

$$\begin{aligned} |R_n(x,a)| &\leq \int_{I(a,x)} \left|\frac{(x-t)^n}{n)!} f^{(n+1)}(t)\right| dt \\ &\leq \int_a^x \frac{(x-t)^n}{n!} M dt = \frac{M}{(n+1)!}(x-a)^{n+1}. \end{aligned}$$

The last theorem can be strengthened as follows.

Theorem 3. *Let f be as in Theorem 1. There is a number γ between a and x such that*

$$R_n(x,a) = \frac{f^{(n+1)}(\gamma)}{(n+1)!}(x-a)^{n+1}$$

Proof. Suppose first $a < x$.

Let k be the minimum of $f^{(n+1)}(t)$ in the interval $[a,x]$ (as above) and let K be the maximum of $f^{(n+1)}$ in this interval. Then

$$k \int_a^x \frac{(x-t)^n}{n!} dt \leq R_n(x,a) \leq K \int_a^x \frac{(x-t)^n}{n!} dt.$$

Evaluating the integral (as above) and deviding by the integral yields

$$\frac{k}{(n+1)!} \leq \frac{R_n(x,a)}{(x-a)^{n+1}} \leq \frac{K}{(n+1)!}.$$

An application of the intermediate value theorem to the function $\frac{f^{(n+1)}}{(n+1)!}$ shows that there exists a number γ between a and x such that

$$\frac{f^{(n+1)}(\gamma)}{(n+1)!} = \frac{R_n(x,a)}{(x-a)^{n+1}}.$$

Now modify this argument for the case $x \le a$! $\square$

Alternative expression of the remainder term: The remainder term can also be expressed by the following formula:

$$R_n(x,a) = \frac{(x-a)^{n+1}}{n!} \int_0^1 (1-s)^n f^{(n+1)}(a + s(x-a))\, ds.$$

It is obtained from $(**)_n$ by making the substitution $t = a + s(x-a)$ (so dt becomes $(x-a)ds$ and the integral from a to x is changed to an integral over the interval $[0,1]$.

In Math 521 I use this form of the remainder term (which eliminates the case distinction between $a \le x$ and $x \ge a$ in a proof above).

Remark: The conclusions in Theorem 2 and Theorem 3 are true under the assumption that the derivatives up to order $n+1$ exist (but $f^{(n+1)}$ is not necessarily continuous). For this version one cannot longer argue with the integral form of the remainder. See Rudin's book for the proof.