Mathematics 522
Review problems

1. (i) Let V, W be normed vector spaces. Let $T : V \to W$ be a linear bounded operator. Let $v \in V$ be a fixed vector. Determine the total derivative DT_v (the derivative of T at v).

(ii) Let Ω be the subset of $\mathbb{R}^2$ consisting of all vectors $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ for which $x_1 > 0$. Let $F : \Omega \to \mathbb{R}^3$ be given by $F(x) = \begin{pmatrix} x_1^2 \\ x_2 - \sqrt{x_1} \\ x_2^4 - (x_1 + x_2)^2 \end{pmatrix}$.

The total derivative DF_a at $a = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$. Compute it.

Also, for $u = (1, -3)$ compute $\lim_{t \to 0} \frac{F(a+tu)-F(a)}{t}$.

(iii) Let $\mathfrak{M}_{2,2}$ be the space of 2×2 matrices and let $\det : \mathfrak{M}_{2,2} \to \mathbb{R}$ be the map that assigns to A its determinant. That is,

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Determine $D\det_B$ for $B = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$, and also for general B. Also determine the total derivatives at B for the functions $F_1 : A \mapsto A^2$, $F_2 : A \mapsto AB$, $F_3 : A \mapsto \det(A^2)$ (for the last one compute it in two ways, using the chain rule in one form, and then by using $\det(A^2) = (\det(A))^2$ and using the chain rule in another way).

For any invertible C determine the derivative at C for $F_4 : A \mapsto (AB)^{-1}$, $F_5 : A \mapsto \det(AB)^{-1}$.

(iv) For $f \in C([0,2])$ define $S : C([0,2] \to \mathbb{R}$ and $G : C([0,2]) \to C([0,2])$ by

$$Sf = f(2)^3, \qquad G[f](x) = \int_0^x f(t)^2 dt$$

Determine the total derivative $D[S \circ G]_{\sin}$ (chain rule!).

2. Define $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{x_1x_2 + x_1x_3 + x_2x_3}{(x_1^2 + x_2^2 + x_3^2)^{1/4}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(i) Determine whether the partial derivatives of f exist at 0.
(ii) Determine whether f is differentiable at 0.
(iii) Determine whether f is a C^1 function on $\mathbb{R}^3$.

Repeat the same problem with

$$g(x) = \begin{cases} \frac{x_1x_2+x_1x_3+x_2x_3}{(x_1^2+x_2^2+x_3^2)^{1/2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

3. *Practice problem for the implicit function theorem:*
Let $F : \mathbb{R}^5 \to \mathbb{R}^3$ be a C^1-function such that $F(0) = (3,1,1)$ (here 0 denotes the origin in $\mathbb{R}^5$) and such that $\frac{\partial F_1}{\partial x_1}(0) = 1$, $\frac{\partial F_1}{\partial x_2}(0) = 0$, $\frac{\partial F_1}{\partial x_3}(0) = 0$, $\frac{\partial F_1}{\partial x_4}(0) = 2$, $\frac{\partial F_1}{\partial x_5}(0) = 1$, $\frac{\partial F_2}{\partial x_1}(0) = 1$, $\frac{\partial F_2}{\partial x_2}(0) = 2$, $\frac{\partial F_2}{\partial x_3}(0) = 1$, $\frac{\partial F_2}{\partial x_4}(0) = 0$, $\frac{\partial F_2}{\partial x_5}(0) = 1$, $\frac{\partial F_3}{\partial x_1}(0) = 1$, $\frac{\partial F_3}{\partial x_2}(0) = 0$, $\frac{\partial F_3}{\partial x_3}(0) = 1$, $\frac{\partial F_3}{\partial x_4}(0) = 3$ and $\frac{\partial F_3}{\partial x_5}(0) = 1$.
a) Show that near the origin the equation $F(x) = (3,1,1)$ can be solved in terms of x_2, x_3, x_5; that means there are functions $\tilde{x}_2$, $\tilde{x}_3$ and $\tilde{x}_5$ defined near $(0,0)$ such that

$$F(x_1, \tilde{x}_2(x_1,x_4), \tilde{x}_3(x_1,x_4), x_4, \tilde{x}_5(x_1,x_4)) = (3,1,1).$$

b) Compute

$$\frac{\partial \tilde{x}_3}{\partial x_4}(0,0).$$

4. (i) Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function and $g : \mathbb{R} \to \mathbb{R}$ a C^1 function. Derive a formula for the derivative of

$$G(x) = \int_0^{g(x)} F(x,t)dt\,.$$

(*Hint*: Take for granted the formula (98) on p. 236 in Rudin.)
(ii) Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function such that $F(0,0) = 3$ and define $G : \mathbb{R} \to \mathbb{R}$ by

$$G(x) = \int_0^{\sin 2x} F(x,t)dt.$$

Show that there are open intervals I, J containing 0 such that $G : I \to J$ is an invertible C^1 function. Compute $[G^{-1}]'(0)$.

5. (i) Show that the initial value problem

$$g'(x) = \frac{g(x)}{1 + x^2 + [g(x)]^2}$$
$$g(x_0) = y_0$$

has a unique solution $g \in C^2(\mathbb{R})$ defined for all $x \in \mathbb{R}$.
(ii) Can you compute $g''(x_0)$?
(iii) What is the solution for the initial datum $y_0 = 0$?

6. Let $g \in C([0,1])$ and let K be a continuous function on the unit square $[0,1]^2$, and let $|K(x,y)| \le A$ for $(x,y) \in [0,1]^2$ and $|g(x)| \le B$ for $0 \le x \le 1$. For $f \in C([0,1])$ define Tf by

$$Tf(x) = \int_0^x K(x,t)f(t)\,dt + g(x)$$

(i) Use the contraction principle on $C[0,1]$, with a suitably adjusted metric to show that the *Volterra integal equation*

$$f(x) = \int_0^x K(x,t)f(t)\,dt + g(x)$$

has a unique solution on $[0,1]$.

(ii) For the iteration defined by $f_0 = 0$, and $f_n = Tf_{n-1}$ for $n \ge 1$ prove that that

$$|f_{n+1}(x) - f_n(x)| \le B\frac{(Ax)^n}{n!}$$

and deduce without using (i) that f_n converges to a function f satisfying the above integral equation.

Show that for $0 \le x \le 1$,

$$|f(x) - f_n(x)| \le B\frac{(Ax)^n}{n!}e^{Ax}.$$

(iii) What can we say about the solution when $g(x) = 0$ on $[0,1]$?