Mathematics 522
Some practice problems

Below $C([a,b])$ denotes the space of continuous real-valued functions on the interval $[a,b]$, with norm

$$\|f\|_\infty := \max_{a\le x\le b} |f(x)|.$$

1. Let $K : [a,b] \times [a,b] \to \mathbb{R}$ be a continuous function. For $f \in C([a,b])$ define

$$Sf(x) = \int_a^b K(x,t)f(t)dt$$

and

$$Tf(x) = \int_a^x K(x,t)f(t)dt\,.$$

Show that Sf, Tf are in $C([a,b])$ and that S and T are bounded linear operators on $C([a,b])$.

2. Let S and T be as in problem 1.
(i) Is $\{Sf : \|f\|_\infty \le 1\}$ necessarily a totally bounded subset of $C([a,b])$?
(ii) Is $\{Tf : \|f\|_\infty \le 1\}$ necessarily a totally bounded subset of $C([a,b])$?
(iii) Let f_n be a sequence of continuous functions on $[a,b]$ satisfying $\sup_n \|f_n\|_\infty \le 1$. Does the sequence Sf_n necessarily converge? Does it have a convergent subsequence? Answer the same questions for Tf_n.

3. Let $K_n : [a,b] \times [a,b] \to \mathbb{R}$ be continuous functions, and assume that

$$\sup_n \max_{(x,t)\in[a,b]} |K_n(x,t)| \le 1.$$

For $f \in C([a,b])$ define

$$S_n f(x) = \int_a^b K_n(x,t)f(t)dt$$

and

$$T_n f(x) = \int_a^x K_n(x,t)f(t)dt\,.$$

Fix $f \in C[a,b]$. Are the sets $\{S_n f : n \in \mathbb{N}\}$, $\{T_n f : n \in \mathbb{N}\}$ necessarily totally bounded subsets of $C([a,b])$? Does the sequence $S_n f$ necessarily converge? Does it have a convergent subsequence? Answer the same questions for $T_n f$.