Mathematics 522
Problem set 6
Due Friday, October 25

1. Let $C^1([0,1])$ be the space of differentiable functions on $[0,1]$ for which f' is continuous. Define

$$\|f\|_{C_1} = \max_{x\in[0,1]} |f(x)| + \max_{x\in[0,1]} |f'(x)|.$$

Show that C^1 is a complete normed vector space.

2. Let $\{f_n\}$ be a sequence of C^2 functions on $[0,1]$ for which $|f_n(0)| \le 1$, $|f_n'(0)| \le 2$ and $\max_{x\in[0,1]} |f_n''(x)| \le 10$ for all $n \in \mathbb{N}$. Prove that this sequence has a subsequence which is convergent with respect to the C^1 norm (as defined in the previous problem).

3. Let $C_0(\mathbb{R})$ be the space of continuous functions on $\mathbb{R}$ with the property that $\lim_{|x|\to\infty} |f(x)| = 0$. The norm is the usual max-norm.

(i) Prove that $C_0(\mathbb{R})$ is complete.

(ii) Let $\mathcal{F} \subset C_0(\mathbb{R})$ be a family of functions satisfying the following assumptions:

(a) For every $R > 0$ the set $\mathcal{A}$ is (uniformly) equicontinuous on the interval $[-R, R]$.

(b) The family $\mathcal{F}$ is uniformly bounded.

(c) With $M_R := \sup_{f\in\mathcal{F}} \sup_{|t|\ge R} |f(t)|$ we have $\lim_{R\to\infty} M_R = 0$.

Prove that $\mathcal{F}$ is totally bounded. You may use the Arzela-Ascoli theorem.

4. Let f be a continuous function on $\mathbb{R}$ with the property that

$$\lim_{|x|\to\infty} f(x) = 0.$$

Let, for $n > 0$,

$$A_n f(x) = \frac{n}{2} \int_{-\infty}^{\infty} f(x-y) e^{-n|y|} dy\,.$$

Prove that $A_n f$ converges to f uniformly on $\mathbb{R}$, as $n \to \infty$.

5. Let f be a C^1 function on an interval $[a,b]$ (i.e. f is differentiable and f' is continuous on $[a,b]$).

Prove that there is a sequence of polynomials p_n so that p_n converges to f and p_n' converges to f', both uniformly on $[a,b]$.