Mathematics 522
Handout and Problem set 5

Metric entropy and covering numbers. Let E be a totally bounded subset of a metric space X, i.e. for every $\delta > 0$ it is contained in a finite collection of δ-balls.

For $\delta > 0$ let $\mathcal{N}(E,\delta)$ be the minimal number of δ-balls needed to cover E (the centers of these balls are not required to belong to E). This number is called the δ-covering number of E; note that it depends not only on E but also on the underlying metric space X and the given metric d. The function $\delta \mapsto \log \mathcal{N}(E,\delta)$ is called the *metric entropy function* of E.

One is interested in the behavior of $\mathcal{N}(E,\delta)$ for small δ. For compact E this serves as a quantitative measure of compactness.

We also set $\mathcal{N}(E,\delta) = \infty$ if E is not totally bounded.

The number

$$\overline{\dim}(E) = \limsup_{\delta\to 0+} \frac{\log \mathcal{N}(E,\delta)}{\log(\frac{1}{\delta})}$$

is called the *upper Minkowski dimension* [1] or upper metric dimension of E. The analogous expression $\underline{\dim}(E)$ where the lim sup is replaced by a lim inf is called *lower Minkowski dimension* or lower metric dimension of E. If $\underline{\dim}(K) = \overline{\dim}(E) = \alpha$ we say that E has Minkowski dimension α.

1. (i) Show that if we replace the natural log in the above definitions by another $\log_b$ with base $b > 1$ then the definitions of the dimensions do not change.

(ii) Let $\alpha > 0$. Suppose that for every $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ and a positive constant $C_\varepsilon \geq 1$ such that $C_\varepsilon^{-1}\delta^{-\alpha+\varepsilon} \leq \mathcal{N}(E,\delta) \leq C_\varepsilon \delta^{-\alpha-\varepsilon}$ for $0 < \delta < \delta(\varepsilon)$. Show that E has Minkowski dimension α.

(iii) Let $E \subset X$ be totally bounded and let $\overline{E}$ be the closure of E. Then $\overline{E}$ is totally bounded and we have

$$\mathcal{N}(\delta, E) \leq \mathcal{N}(\overline{E},\delta) \leq \mathcal{N}(E,\delta') \text{ if } 0 < \delta' < \delta.$$

(iv) Define $N^{\text{cent}}(E,\delta)$ to be the minimal number of δ-balls with center in E needed to cover E. Show that

$$\mathcal{N}(E,\delta) \leq N^{\text{cent}}(E,\delta) \leq \mathcal{N}(E,\delta/2).$$

(v) Let $B_1,\dots,B_M$ be balls of radius δ in X, so that each ball has nonempty intersection with the set E. For each $i = 1,\dots,M$ denote by B_i^* the ball with same center as B_i and radius 3δ. Assume that the balls $B_1^*,\dots,B_M^*$ are disjoint. Prove that $M \leq \mathcal{N}(E,\delta)$.

Remark: This can be an effective tool to prove lower bounds for the covering numbers.

[1] Minkowski is pronounced as `Minkoffski`

2. Consider the following norms in $\mathbb{R}^n$

$$\|x\|_1 = \sum_{i=1}^n |x_i|, \quad \|x\|_2 = \Big(\sum_{i=1}^n |x_i|^2\Big)^{1/2}, \quad \|x\|_\infty = \max_{i=1,\dots,n} |x_i|.$$

with associated metrics d_1, d_2, d_∞.

(i) Show that $\|x\|_\infty \le \|x\|_1 \le \sqrt{n}\|x\|_2 \le n\|x\|_\infty$ for all $x \in \mathbb{R}^n$.

(ii) Let $E \subset \mathbb{R}^n$ and let $\mathcal{N}_1(E,\delta)$, $\mathcal{N}_2(E,\delta)$, $\mathcal{N}_\infty(E,\delta)$ be the metric entropy numbers of E associated with to the metrics d_1, d_2, d_∞, respectively. Show that

$$\mathcal{N}_\infty(E,\delta) \le \mathcal{N}_2(E,\delta) \le \mathcal{N}_1(E,\delta) \le \mathcal{N}_2(E,\delta/\sqrt{n}) \le \mathcal{N}_\infty(E,\delta/n).$$

(iii) Let $Q = [0,1]^n$ be the unit cube in $\mathbb{R}^n$. Show that Q has Minkowski dimension n (with respect to any of the metrics d_1, d_2, d_3).

(iv) Let f be a differentiable function on $[0,1]$ with bounded derivative. Let E be the set of all $x = (x_1, x_2) \in \mathbb{R}^2$ for which $0 \le x_1 \le 1$ and $x_2 = f(x_1)$. What is the Minkowski dimension of E?

(v)* Let E be the set of all $x = (x_1, x_2) \in \mathbb{R}^2$ for which $0 \le x_1 \le 1$ and $x_2 = \sqrt{x_1}$. What is the Minkowski dimension of E?

3. (i) Let $\beta > 0$. Consider the subset E of $\mathbb{R}$ consisting of the numbers $n^{-\beta}$, for $n = 1, 2, \dots$. Show that E has a Minkowski dimension and determine it.

Hint: It might help to try this first for the sequence $1/n$ which, perhaps counterintuitively, turns out to have Minkowski dimension $\frac{1}{2}$.

(ii)* *Recommended only* exercise for those of you who know the Cantor middle third set: its Minkowski dimension is equal to $\frac{\log 2}{\log 3}$.

4. Let A be the space of functions $f : \mathbb{N} \to \mathbb{R}$ (aka sequences) so that $|f(n)| \le 2^{-n}$ for all $n \in \mathbb{N}$. It is a subset of the space of bounded sequences with norm $\|f\|_\infty = \sup_{n\in\mathbb{N}} |f(n)|$ and associated metric d_∞. Show that for $\delta < 1/2$ the covering numbers $\mathcal{N}(A,\delta)$ satisfy the bounds

$$\mathcal{N}(A,\delta) \le \Big(\frac{1}{\delta}\Big)^{C + \frac{1}{2}\log_2 \frac{1}{\delta}}$$

where C is independent of δ. *Hint:* It helps to work with $\delta = 2^{-M}$ where $M \in \mathbb{N}$.

Also provide a lower bound which shows that A does not have finite lower Minkowski dimension.

Equivalence of norms

Definition. Two norms $\|\cdot\|_1$, $\|\cdot\|_2$ on a vector space V are said to be equivalent if there exist two positive constants c, C so that

$$c\|x\|_1 \le \|x\|_2 \le C\|x\|_1 \text{ for all } x \in V.$$

5. (i) Show that the above definition yields an equivalence relation.

(ii) If V is a finite dimensional vector space, show that all norms are equivalent.

Remark. The relies on the fact that a continuous real-valued function on a compact space has a minimum and a maximum.

6. Let V be the space of continuous functions on $[0,1]$ and set

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$$

and for $1 \le p < \infty$ let

$$\|f\|_p = \Big(\int_0^1 |f(x)|^p dx\Big)^{1/p}.$$

(i) Show that these expressions defines norms on V.

(ii) Show that $\|\cdot\|_{p_1}$ and $\|\cdot\|_{p_2}$ are not equivelent when $p_1 \neq p_2$.

(iii) Show that V is a complete space with respect to the norm $\|\cdot\|_\infty$.

(iv) Show that V is not a complete space with respect to the norm $\|\cdot\|_p$ when $p < \infty$.

(v) Show that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$ for all $f \in V$.