Mathematics 522
Homework assignment No. 5
Due Wednesday, March 9

1. For $k = 0, 1, 2, \ldots$ let $C^k([0,1])$ be the space of functions that are k-times differentiable on $[0,1]$ and for which $f^{(k)}$ is continuous. Define

$$\|f\|_{C^k[0,1]} = \sum_{j=0}^{k} \max_{x\in[0,1]} |f^{(j)}(x)|.$$

(i) Prove that $C^k([0,1])$ is a Banach space (i.e. complete normed vector space).

(ii) Let $\{f_n\}$ be a sequence of C^{k+1} functions on $[0,1]$ for which there is a constant B such that $\|f_n\|_{C^{k+1}} \le B$. Prove that this sequence has a subsequence which is convergent with respect to the $C^k([0,1])$ norm.

2. Let A be an $m \times n$ matrix. Recall the notation $\|x\|_1 = \sum_{j=1}^n |x_j|$, $\|x\|_\infty = \max_{j=1,\ldots,n} |x_j|$, and use similar notation for the corresponding norms for vectors in $\mathbb{R}^m$.

(i) Define $\|A\|_{1\to1} := \sup_{x\neq0} \frac{\|Ax\|_1}{\|x\|_1}$. Show that

$$\|A\|_{1\to1} = \max_{j=1,\ldots,n} \sum_{i=1}^{m} |a_{ij}|\,.$$

(ii) Define $\|A\|_{\infty\to\infty} := \sup_{x\neq0} \frac{\|Ax\|_\infty}{\|x\|_\infty}$. Show that

$$\|A\|_{\infty\to\infty} = \max_{i=1,\ldots,m} \sum_{j=1}^{n} |a_{ij}|\,.$$

(iii) Define $\|A\|_{1\to\infty} := \sup_{x\neq0} \frac{\|Ax\|_\infty}{\|x\|_1}$. Find and prove an explicit expression for $\|A\|_{1\to\infty}$.

3. Let $M(n,n)$ be the space of real $n \times n$ matrices. Specify a norm of your choice. Define $F : M(n,n) \to M(n,n)$ by $F(A) = 4A + 2A^3$, and, for every $A \in M(n,n)$, show that F is differentiable at A and compute the derivative DF_A.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{x_1x_2}{x_1^2+x_2^2} & \text{if } (x_1,x_2) \neq (0,0) \\ 0 & \text{if } (x_1,x_2) = (0,0) \end{cases}$$

Show that the partial derivatives $\frac{\partial f}{\partial x_1}$, $\frac{\partial f}{\partial x_2}$ exist everywhere on $\mathbb{R}^2$ but f is not continuous at $(0,0)$.