Mathematics 522
Homework assignment No. 4.

Due Friday, October 11.

1. Let X be a metric space, with metric $d : X \times X \to [0, \infty)$. Let Y be a subset of X. We can make Y a metric space by using the restriction of the metric d to $Y \times Y$.

Prove:

(i) A subset $G \subset Y$ is open in the metric space Y if and only if there exists a set $H \subset X$ which is open in the metric space X so that $G = H \cap Y$.

(ii) A subset $A \subset Y$ is closed in Y if and only if there exists a set $B \subset X$ which is closed in X so that $A = B \cap Y$.

2. A metric space is called *separable* if it contains a countable dense subset.

Prove that a totally bounded metric space is separable.

3. A collection $\{F_\alpha : \alpha \in A\}$ of closed sets has the *finite intersection property* if for every finite subset A_o of A the intersection $\cap_{\alpha \in A_o} F_\alpha$ is not empty.

Prove that the following statements (i), (ii) are equivalent.

(i) A metric space X, with metric d, is compact.

(ii) For every collection $\{F_\alpha\}_{\alpha \in A}$ of closed sets with the finite intersection property it follows that

$$\bigcap_{\alpha \in A} F_\alpha \neq \emptyset.$$

4. Let ℓ^∞ denote the space of all bounded real sequences with metric $d(a, b) = \sup_{n \in \mathbb{N}} |a_n - b_n|$.

Prove that the set of all sequences $\{a_n\}$ which satisfy $|a_n| \leq \frac{1}{n}$ for all $n \in \mathbb{N}$ is compact.

Remark: More generally, one can also prove that if $\{M_n\}$ is a fixed sequence of nonnegative terms with the property that $\lim_{n \to \infty} M_n = 0$ then the set of all sequences $a = (a_n)_{n=1}^\infty$ which satisfy $|a_n| \leq M_n$ for all n, is a compact subset of ℓ^∞.

5. Construct a compact subset of real numbers whose accumulation points form a countable set.

Turn the page.

6*. *Extra Credit:* Let $X = C([0,1])$ be the space of continuous functions on $[0,1]$ with the usual sup-norm $\|f\| = \max_{0\leq t\leq 1} |f(t)|$ (and metric $d(f,g) = \|f - g\|$).

Let $Y \subset X$ be the set of all functions $f : [0,1] \to \mathbb{R}$ with the additional properties that $|f(x) - f(\tilde{x})| \leq |x - \tilde{x}|$ for all $x, \tilde{x} \in [0,1]$ and $f(0) = 0$. Prove that Y is totally bounded.