Mathematics 522
Homework assignment No. 4
Due Wednesday, March 2

1. Let $C^1([0,1])$ be the space of differentiable functions on $[0,1]$ for which f' is continuous. Define

$$\|f\|_{C^1} = \max_{x\in[0,1]} |f(x)| + \max_{x\in[0,1]} |f'(x)|.$$

Show that C^1 is a complete normed vector space.

2. Let $\{f_n\}$ be a sequence of C^2 functions on $[0,1]$ for which $|f_n(0)| \leq 1$, $|f_n'(0)| \leq 2$ and $\max_{x\in[0,1]} |f_n''(x)| \leq 10$ for all $n \in \mathbb{N}$. Prove that this sequence has a subsequence which is convergent with respect to the C^1 norm (as defined in the previous problem).

3. Show that there is a constant C_n such that for all polynomials P of degree $\leq n$ we have

$$\max_{0\leq x\leq 20} |P(x)| \leq C_n \int_0^1 |P(t)| dt\,.$$

4. For $x \in \mathbb{R}^n$ set $\|x\|_p = (\sum_{j=1}^n |x_j|^p)^{1/p}$. Prove that if $1 < p < \infty$ then

$$\sup_{\|x\|_p\neq 0} \frac{\left|\sum_{j=1}^n x_j y_j\right|}{\|x\|_p} = \|y\|_{p'} \text{ where } p' = \frac{p}{p-1}\,.$$

5. Let $M(n,n)$ be the space of real $n \times n$ matrices. Specify a norm of your choice. Define $F : M(n,n) \to M(n,n)$ by $F(A) = 4A + 2A^3$, and, for every $A \in M(n,n)$, show that F is differentiable at A and compute the derivative DF_A.

6. Let V, W be normed vector spaces and let $T : V \to W$ be a bounded linear transformation. Show that T is differentiable everywhere and compute the derivative DT_v for all $v \in V$.