Mathematics 522
Homework assignment No. 2.

Due Friday, February 12.

1. Let $1 \le p < \infty$ and let $\ell^p(\mathbb{N})$ be the vector space of real-valued sequences $\{a(j)\}_{j=1}^{\infty}$ such

$$\|a\|_p := \Big(\sum_{j=1}^{\infty} |a(j)|^p\Big)^{1/p}$$

is finite.

Prove that $\ell^p(\mathbb{N})$ is a normed vector space which is complete.

2. A collection $\{F_\alpha : \alpha \in A\}$ of closed sets has the *finite intersection property* if for every finite subset A_o of A the intersection $\cap_{\alpha\in A_o} F_\alpha$ is not empty.

Prove that the following statements (i), (ii) are equivalent.

(i) A metric space X, with metric d, is compact.

(ii) For every collection $\{F_\alpha\}_{\alpha\in A}$ of closed sets with the finite intersection property it follows that

$$\bigcap_{\alpha\in A} F_\alpha \neq \emptyset.$$

3. A metric space is called *separable* if it contains a countable dense subset.

Prove that a totally bounded metric space is separable.

4. Let ℓ^∞ denote the space of all bounded real sequences $a = \{a(j)\}_{j=1}^{\infty}$ with metric $d(a,b) = \sup_{j\in\mathbb{N}} |a(j) - b(j)|$.

Let $\mathcal{A}$ be the set of all sequences a which satisfy

$$|a(j)| \le \frac{1}{j} \text{ for all } j \in \mathbb{N}.$$

Prove that $\mathcal{A}$ is compact.

Remark: More generally, one can also prove that if $\{m_j\}_{j=1}^{\infty}$ is a fixed sequence of nonnegative terms with the property that $\lim_{j\to\infty} m_j = 0$ then the set of all sequences a which satisfy $|a(j)| \le m_j$ for all j, is a compact subset of ℓ^∞.

5. Construct a compact subset of real numbers whose accumulation points form a countable set.