Mathematics 522

Homework assignment No. 10
Due Friday, April 29

1. Let

$$f(x) = \begin{cases} e^{-1/x} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

Show that f is a C^∞ function such that $f^{(n)}(0) = 0$ for $n = 0, 1, 2, \ldots$.

Hint: Prove first that $f^{(j)}(x) = P_j(1/x)e^{-1/x}$ for $x > 0$, where P_j is a polynomial.

2. Let f be as in problem 1.

(i) Does the Taylor series expanded at 0 represent the function $f(x)$ in an open interval containing 0?

(ii) Discuss the validity (true or false?) of the following statement: There is an open interval $(-r, r)$ and a positive constant C so that for every n the inequality $\sup_{x\in(-r,r)} |f^{(n)}(x)| \le C^n n!$ holds.

(iii) Let $g(x) = f(1 - x^2)$. Show that g is a C^∞-function on $\mathbb{R}$ with the property that $g(x) > 0$ for $x \in (-1, 1)$ and $g(x) = 0$ for $x \notin (-1, 1)$.

(iv) Given an interval (a, b) (with real numbers $a < b$) construct a C^∞-function h on $\mathbb{R}$ with the property that $h(x) > 0$ for $x \in (a, b)$ and $h(x) = 0$ for $x \notin (a, b)$ and $\int_a^b h(x)dx = 1$.

3. For complex numbers z define

$$\sinh z = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}$$

$$\cosh z = \sum_{\ell=0}^{\infty} \frac{z^{2\ell}}{(2\ell)!}$$

(i) Show that these series converge for all z and thus the functions sinh and cosh are well defined.

(ii) what is the relation between exp, cosh, sinh?

If we define $\cos z = \cosh(iz)$, $\sin z = \frac{1}{i}\sinh(iz)$ what is the relation between exp, cos, sin?

(iii) Show that (using the Cauchy product of series)

$$\sinh(z + w) = \sinh z \cosh w + \cosh z \sinh w$$

$$\cosh(z + w) = \cosh z \cosh w + \sinh z \sinh w$$

Derive similar formulas for $\sin(z + w)$, $\cos(z + w)$.

4.-5.-6. Problems 4,5,6 on p.165 in Rudin's book.