Mathematics 522
Fall 2013

Homework assignment No.1.

Due Friday, September 20. This is probably largely a review of Math 521 material.

1. What can you say about convergence or divergence of the following series? Prove or disprove!

$$\text{(i)} \sum_{n=1}^{\infty} 3^{-n}\Big(\frac{n+1}{n}\Big)^{n^2}, \qquad \text{(ii)} \sum_{n=2}^{\infty} \frac{1}{(\log n)^{100}}, \qquad \text{(iii)} \sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}}.$$

$$\text{(iv)} \sum_{n=1}^{\infty} \frac{n^n}{3^{n^2}}, \qquad \text{(v)} \sum_{n=1}^{\infty} \frac{1}{n(\log n)^p}, \qquad \text{(vi)} \sum_{n=1}^{\infty} (-1)^n\Big(\frac{1}{\log n} + \frac{\cos n}{n^2}\Big)$$

2. For which $x \in \mathbb{R}$ do the following series converge? (For series (i) compute the sum in case of convergence). On which sets do these series converge uniformly?

$$\text{(i)} \sum_{n=1}^{\infty} nx^{2n}, \qquad \text{(ii)} \sum_{n=1}^{\infty} (2^{1/n}-1)^n x^n \qquad \text{(iii)} \sum_{n=1}^{\infty} (\cos(\tfrac{2\pi}{n})-1)e^{nx}.$$

3. Consider the following sequences.
$a_n = \sqrt{n^3+n^2} - \sqrt{n^3}$
$b_n = \sqrt{n^3+n} - \sqrt{n^3}$
$c_n = \frac{1}{n!} + \frac{1}{(n+1)!} + \cdots + \frac{1}{(n^3-1)!} + \frac{1}{(n^3)!}$
$d_n = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n}$
$e_n = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n^2}$
For each one prove or disprove convergence of the sequence.

4. (i) Let a_n be a sequence for which $|a_n - a_{n+1}| \le (n \log n)^{-1}$ for all $n \ge 2$. Does a_n necessarily converge?
(ii) Let b_n be a sequence for which $|b_n - b_{n+1}| \le (3/2)^{-n}$. Does b_n necessarily converge?

5. Do the following sequences converge as $n \to \infty$? Prove or disprove (answers may depend on the parameters involved).

$$\text{(i)} \int_{1/n}^{1/2} \cos(x)x^a\Big(\log\frac{1}{x}\Big)^b dx, \qquad \text{(ii)} \int_2^n x^a\big(\log x\big)^b dx, \qquad \text{(iii)} \int_0^n e^{-\sqrt{x}}dx$$

$$\text{(iv)} \int_3^n \frac{\cos x}{x}dx, \qquad \text{(v)} \int_0^n x^m\cos(x^3)dx, \quad m = 0, 1, 2.$$