Equivalence of norms

Definition. Let $\mathbb{V}$ be a vector space over the real or complex numbers. Let $\|\cdot\|_a$, $\|\cdot\|_b$ be norms. We say that $\|\cdot\|_a$, $\|\cdot\|_b$ are *equivalent* if there exist positive constants c, C such that for all $x \in \mathbb{V}$

$$c\|x\|_a \leq \|x\|_b \leq C\|x\|_a.$$

Exercise: Check that this defines an equivalence relation on the set of norms on $\mathbb{V}$.

Exercise: Let $\|\cdot\|$, $\|\cdot\|_*$ be norms on $\mathbb{V}$. Let $B(x,r) = \{y : \|y-x\| < r\}$, $B_*(x,r) = \{y : \|y-x\|_* < r\}$. Show that $\|\cdot\|$, $\|\cdot\|_*$ are equivalent norms if and only if there exist two positive constants such that

$$B(x, c_1 r) \subset B_*(x,r) \subset B(x, c_2 r) \text{ for all } x \in \mathbb{V} \text{ and all } r > 0.$$

Exercise: Let $\|\cdot\|_a$, $\|\cdot\|_b$ be equivalent norms and let d_a, d_b the associated metrics defined by $d_a(x,y) = \|x-y\|_a$, $d_b(x,y) = \|x-y\|_b$.

(i) Show that a sequence converges in $(\mathbb{V}, d_a)$ if and only it converges in $(\mathbb{V}, d_b)$.

(ii) Show that $(\mathbb{V}, d_a)$ is complete if and only if $(\mathbb{V}, d_b)$ is complete.

(iiii) Show that a set $O \subset \mathbb{V}$ is open in $(\mathbb{V}, d_a)$ if and only it is open in $(\mathbb{V}, d_b)$.

(iiii) Show that a set $F \subset \mathbb{V}$ is closed in $(\mathbb{V}, d_a)$ if and only it is closed in $(\mathbb{V}, d_b)$.

Example. Let $\mathbb{V}$ be a finite dimensional vector space, and let $v_1, \dots, v_n$ be a basis of $\mathbb{V}$. Thus every $x \in \mathbb{V}$ has a unique representation as $x = \sum_{i=1}^n x_i v_i$ where x_i are the coordinates of x with respect to the basis $v_1, \dots, v_n$. We define

$$\|x\|_* = \max_{i=1,\dots,n} |x_i| \text{ for } x = \sum_{i=1}^n x_i v_i.$$

Then $\|\cdot\|_*$ is a norm on $\mathbb{V}$ and $\mathbb{V}$ with this norm is a complete vector space.

Exercise: Show this. Verify that a sequence $x^{(m)}$ of vectors converges in $\mathbb{V}$ if and only if for $i = 1, \dots, n$ the coordinate sequences $x_i^{(m)}$ converge in $\mathbb{R}$ (or $\mathbb{C}$).

Theorem. *Let $\mathbb{V}$ be finite-dimensional. Then all norms are equivalent.*

Proof. Let $v_1, \dots, v_n$ be a basis of $\mathbb{V}$. We define $\|x\|_* := \max_{i=1,\dots,n} |x_i|$ for $x = \sum_{i=1}^n x_i v_i$ as in the above example.

Let $\|\cdot\|$ be any norm. Our goal is to prove that there are constants $c, C > 0$ such that $c\|x\|_* \le \|x\| \le C\|x\|_*$. For the second inequality we use

$$\|x\| = \Big\|\sum_{i=1}^n x_i v_i\Big\| \le \sum_{i=1}^n \|x_i v_i\| = \sum_{i=1}^n |x_i|\|v_i\| \le n \max_{i=1,\dots,n} \|v_i\| \max_{i=1,\dots,n} |x_i|$$

and so we find

$$\|x\| \le C\|x\|_*, \text{ with } C = n \max_{i=1,\dots,n} \|v_i\|\,.$$

For the opposite inequality we use that $S = \{x : \|x\|_* = 1\}$ is compact. To show this we verify that S is complete and totally bounded. Note that the function $x \mapsto \|x\|_*$ is continuous since $|\|x\|_* - \|y\|_*| \le \|x - y\|_*$, by the triangle inequality. Thus S is a closed subset of $\mathbb{V}$ (as the inverse image of $\{1\}$ under this function) and hence S is complete as a closed subset of the complete space $\mathbb{V}$. To show that S is totally bounded let $\varepsilon > 0$ and choose $m > 1/\varepsilon$. Then S is covered by the finite collection of balls (of radius $1/m$ with respect to the $\|\cdot\|_*$−norm)

$$B_{j_1,\dots,j_n} := \Big\{\sum_{i=1}^n x_i v_i : \frac{j_i}{m} - \varepsilon < x_i < \frac{j_i}{m} + \varepsilon,\, i = 1, \dots, n\Big\}$$

where $j_i = -m, \dots, m$ for $i = 1, \dots, n$. [1] Hence S is totally bounded.

Now consider the function $f(x) = \|x\|$ which has values in $[0, \infty)$ and is also continuous, since

$$|f(x) - f(y)| \le \big|\|x\| - \|y\|\big| \le \|x - y\| \le C\|x - y\|_*.$$

Then f has a minimum on the compact set S (see Theorem 4.16 in Rudin). Since $x \neq 0$ for $x \in S$ we have that $c := \min_{x\in S} f(x) > 0$. Now let $x \neq 0$, then $x/\|x\|_* \in S$ and hence

$$\Big\|\frac{x}{\|x\|_*}\Big\| \equiv f(x/\|x\|_*) \ge c$$

which implies

$$\|x\| \ge c\|x\|_*.$$

[1] These are $O(m^d)$ many balls. Verify that for the covering of S we only need $O(m^{d-1})$ of those balls.