CHARACTERIZATIONS OF COMPACTNESS FOR METRIC SPACES

Definition. Let X be a metric space with metric d.

(a) A collection $\{G_\alpha\}_{\alpha \in A}$ of open sets is called an *open cover* of X if every $x \in X$ belongs to at least one of the G_α, $\alpha \in A$. An open cover is finite if the index set A is finite.

(b) X is *compact* if every open cover of X contains a finite subcover.

Definition. Let X be a metric space with metric d and let $A \subset X$. We say that A is a compact subset if the metric space A with the inherited metric d is compact.

Examples: Any finite metric space is compact.

As an exercise show directly from the definition that the subset K of $\mathbb{R}$ consisting of 0 and the numbers $1/n$, $n = 1, 2, \dots$ is compact.

Definition. A subset A of X is *relatively compact* if the closure $\overline{A} \subset X$ is a compact subset of X.

Definition. A metric space is called *sequentially compact* if every sequence in X has a convergent subsequence.

Definition. A metric space is called *totally bounded* if for every $\epsilon > 0$ there is a finite cover of X consisting of balls of radius ϵ.

THEOREM. *Let X be a metric space, with metric d. Then the following properties are equivalent (i.e. each statement implies the others):*

(i) X is compact.

(ii) X has the Bolzano-Weierstrass property, namely that every infinite set has an accumulation point.[*]

(iii) X is sequentially compact, i.e. every sequence has a convergent subsequence.

(iv) X is totally bounded and complete.

[*]If A is a subset of X then p is called an *accumulation point* if every neighborhood of p contains a point $q \in A$ so that $q \neq p$. In Rudin's book the terminology 'limit point' is used for this.

Example: A closed bounded interval $I = [a, b]$ in $\mathbb{R}$ is totally bounded and complete, thus compact. For the proof that I is totally bounded note that we can cover I with $N(\varepsilon)$ intervals of length ε where $N(\varepsilon) \le 10\varepsilon^{-1}(b-a)$.

Example: Any closed bounded subset of $\mathbb{R}^n$ is totally bounded and complete. For the proof note that any ball of radius R (with respect to the usual Euclidean metric) can be covered that $N(\varepsilon)$ intervals where $N(\varepsilon) \le (10dR)^d\varepsilon^{-d}$; you can obtain of course a somewhat better constant.

Example: Let $\mathfrak{B}$ be the metric space of all bounded sequences on $\mathbb{N}$, with metric $d(a, b) = \sup_{n\in\mathbb{N}} |a_n - b_n|$. Let A be the closed ball of radius 1 centered at the zero sequence $(0, 0, \dots)$. Then A is bounded and closed but not sequentially compact (and by the theorem neither compact).

Indeed let $e^{(k)}$ be the member of $\mathfrak{B}$ with $e^{(k)}(n) = 0$ if $k \ne n$ and $e^{(n)}(n) = 1$. Then $d(e^{(k)}, e^{(l)}) = 1$ if $k \ne l$. Thus the $e^{(k)}$ form a bounded sequence in $\mathfrak{B}$ which does not have a convergent subsequence. The set A is complete (as a closed subset of a complete space) but it is *not totally bounded.* Show this directly from the definition!

We conclude that merely bounded and complete sets in an arbitrary metric space *may not be compact.* Thus the cases of $\mathbb{R}^n$ and $\mathbb{C}^n$ (where this characterization of compact subsets holds) are exceptional instances which do not have 'infinite dimensional' analogues. The condition of *total boundedness* is crucial.

Example: Let ℓ^1 denote the space of all absolutely summable sequences, i.e. the space of all sequences $\{a_n\}_{n=1,2,\dots}$ for which $\sum |a_n|$ converges. A metric in ℓ_1 is given by $d_1(a, b) = \sum_{n=1}^{\infty} |a_n - b_n|$. Let A be the closed ball of radius 1 centered at the zero sequence $(0, 0, \dots)$ (i.e. the set of all absolutely summable sequences for which $\sum_{n=1}^{\infty} |a_n| \le 1$.

Verify as in the previous example that A is bounded and closed but not sequentially compact (and therefore not compact).

However one can show that the set of sequences $\{a_n\}_{n=1}^{\infty}$ for which $|a_n| \le 2^{-n}$ for all $n \ge 1$ is a compact subset of A.

Example: Compact sets of continuous function (with respect to the sup metric) can be characterized by the Arzela-Ascoli theorem, see the section "equicontinuous families of functions" in ch. 7 of Rudin's book.

Lemma 1. *Any closed subset of a compact metric space is compact.*

The proof of the main theorem is contained in a sequence of lemmata which we now state. In the subsequent sections we discuss the proof of the lemmata.

Lemma 2. *A metric space is sequentially compact if and only if every infinite subset has an accumulation point.*

Lemma 3. *A compact metric space is sequentially compact.*

Lemma 4. *A sequentially compact subset of a metric space is bounded* [†] *and closed.*

Lemma 5. *A metric space which is sequentially compact is totally bounded and complete.*

Lemma 6. *A metric space which is totally bounded and complete is also sequentially compact.*

Lemma 7. *A sequentially compact space is compact.*

In what follows we shall always assume (without loss of generality) that the metric space X is not empty.

1. Proof of Lemma 1: Any closed subset of a compact metric space is compact.

Let F be a closed subset of X and let $\{G_\alpha\}_{\alpha\in A}$ be an open cover of F. Every G_α is of the form $U_\alpha \cap F$ where U_α is open in X (see Theorem 2.30 in Rudin's book). Since $X \setminus F$ is open in X as the complement of a closed set the collection $\mathfrak{W}$ of sets consisting of the U_α, $\alpha \in A$ together with $X \setminus F$ form an open cover of X. As X is compact there is a finite subcover which consists of $U_{\alpha_1}, \dots, U_{\alpha_N}$, for suitable indices $\alpha_1, \dots, \alpha_N$, and possibly also $X \setminus F$. But the latter set is disjoint from F and so the sets $U_{\alpha_1}, \dots, U_{\alpha_N}$ form a cover of F. Since $U_\alpha \cap F = G_\alpha$ the sets $G_{\alpha_1}, \dots, G_{\alpha_N}$ form a cover of F which is a subcover of the original collection. $\square$

2. Proof of Lemma2: A metric space is sequentially compact if and only if every infinite subset has an accumulation point.

Let Y be an infinite subset of X and let $\{p_n\}_{n\in\mathbb{N}}$ be a sequence of pairwise different points in Y. Since X is sequentially compact the sequence contains a convergent subsequence and the limit is an accumulation point of Y.

[†] A set A in a metric space is bounded if the diameter $\operatorname{diam}(A) = \sup\{d(x,\tilde{x}) : x \in A, \tilde{x} \in A\}$ is finite. This is the same as saying that A is contained in a fixed ball (of finite radius).

Vice versa let X be a metric space with the Bolzano-Weierstrass property, i.e. every infinite subset has an accumulation point. Now let $\{p_n\}$ be a sequence of points in X. If one point occurs an infinite number of times in the sequence then it has a 'constant' subsequence which of course converges. If every point in the sequence occurs only a finite number of times then we may choose a subsequence $\{p_{n_k}\}_{k\in\mathbb{N}}$ whose members are all pairwise different. But by assumption this set has an accumulation point and we can choose a subsequence which converges to this accumulation point. □

3. Proof of Lemma 3:
A compact metric space is sequentially compact.

By Lemma 2 we need to show the Bolzano-Weierstrass property, i.e. every infinite subset of X has an accumulation point.

Suppose not, so let Y be an infinite subset of X which does not have an accumulation point. Then for every $y \in Y$ there is an open ball B_y centered at y such that B_y contains no other points in y.

As Y has no accumulation points Y is closed in X and, by Lemma 1, Y is compact. For every $y \in Y$ the singleton set $\{y\} = B_y \cap Y$ is an open set in the metric space Y. Since Y is infinite they form an open cover from which we cannot select an open subcover, which gives a contradiction (since Y is compact). □

4. Proof of Lemma 4:
A sequentially compact subset of a metric space is bounded and closed.

Let K be a compact subset of X. We first show that K is bounded, i.e. the diameter of K

$$\operatorname{diam}(K) = \sup\{d(x, \tilde{x}) : x \in K, \tilde{x} \in K\}$$

is finite.

Suppose it is not finite. Then for a fixed y_1 we can choose y_2 so that $d(y_1, y_2) \geq 1$. Since the diameter is not finite we can choose a point y_3 so that $d(y_1, y_3) \geq 1 + d(y_1, y_2)$ and we continue this way so that if for $n \geq 3$ points $y_1, \dots, y_{n-1}$ so that $d(y_1, y_i) \geq 1 + d(y_1, y_{i-1})$ for $i \leq 3 \leq n$ we choose a point y_n so that $d(y_1, y_n) \geq 1 + d(y_1, y_{n-1})$.

It is easy to see that this implies

$$d(y_1, y_m) \geq 1 + d(y_1, y_n) \text{ for } m > n\ .$$

Thus by the triangle inequality

$$d(y_m, y_n) \geq |d(y_m, y_1) - d(y_n, y_1)| \geq 1$$

and therefore the sequence $\{y_n\}$ does not have a convergent subsequence. Thus the space is not sequentially compact and by Lemma 3 it is not compact, a contradiction to our hypothesis.

Thus we have shown that K is bounded. To prove that K is closed let $\{p_n\}$ be a convergent sequence of points in K; we have to show that the limit belongs to K. But since K is sequentially compact this sequence has a subsequence which converges to a limit in K. Thus the limit of the convergent sequence $\{p_n\}$ belongs to K. $\square$

5. Proof of Lemma 5: A sequentially compact metric space is totally bounded and complete

Definition: A set A is called an ε-net for X if A is finite and if the balls $B_\varepsilon(x)$ (with radius ε and center x) where $x \in A$, cover X.

We consider a sequentially compact space X and let $\varepsilon > 0$.

Claim: Let $\varepsilon > 0$ and let $\mathcal{A} \subset X$ be a set of points of mutual distance $\geq \varepsilon$ (i.e. if $p \in \mathcal{A}$ and $q \in \mathcal{A}$ and $p \neq q$ then $d(p,q) \geq \varepsilon$). Then $\mathcal{A}$ is finite.

Suppose the claim is not true, then we can construct a sequence $x_n \in X$ so that $d(x_n, x_m) \geq \varepsilon$ whenever $m \neq n$ and clearly this sequence does not have a convergent subsequence, in contradiction to the sequential compactness of X.

We now construct a finite ε-net.

Pick a point p_1. Then (if possible) pick a point p_2 with $d(p_1, p_2) \geq \varepsilon$, (if not possible, stop). Then (if possible) pick a point p_3 with $d(p_1, p_3) \geq \varepsilon$ and $d(p_2, p_3) \geq \varepsilon$, if not possible stop.

Continue (if possible) until points $p_1, p_2, \dots, p_m$ are chosen for which $d(p_i, p_j) \geq \varepsilon$ for $1 \leq i < j \leq m$. Then pick a point p_{m+1} so that $d(p_i, p_{m+1}) \geq \varepsilon$ for $i = 1, \dots, p_m$. If this is not possible then stop, and in this case every point in X is contained in an open ball of radius ε centered at one of the points $p_1, \dots, p_m$, so we have a finite ε-net of points.

By the claim above the construction *stops after a finite number of steps*, and the resulting set of points obtained before stopping form a finite ε-net for X. Thus we have shown that X is totally bounded.

‡

Next we observe that X is complete. Let $\{x_n\}$ be a Cauchy sequence in X. Since X is sequentially compact this sequence has a convergent subsequence whose limit is also the limit of the Cauchy-sequence (provide the details or refer to a previously done exercise!). $\square$

‡An alternative shorter formulation of this argument goes as follows: Let $\mathcal{P}$ be an ϵ-separated subset of X which is maximal with respect to inclusion. Then $\mathcal{P}$ is finite, by the avove claim. The balls $B_\varepsilon(p)$, $p \in \mathcal{P}$ form a finite ϵ-net.

6. Proof of Lemma 6: A totally bounded and complete metric space is sequentially compact.

We consider a sequence $\{a_n\}$ in X and assume that X is a totally bounded and complete metric space.

By the assumption 'X totally bounded' (applied with $\varepsilon = 1$) we can cover the space X with finitely many balls of radius 1; then one of them contains a_n's for infinitely many n's; i.e. there is a ball B_1 of radius 1 so that there is a subsequence of $\{a_n\}$ whose members all belong to B_1. We denote this subsequence by $\{a_n^{(1)}\}$ and thus all $a_n^{(1)}$ belong to B_1.

Similarly by the totally boundedness condition with $\varepsilon = 1/2$ we can find a subsequence $\{a_n^{(2)}\}$ of $\{a_n^{(1)}\}$ and a ball B_2 of radius $1/2$ so that all $a_n^{(2)}$ belong to B_2. Continuing in this way we obtain for any $k \geq 2$ a subsequence $\{a_n^{(k)}\}$ of $\{a_n^{(k-1)}\}$ and a ball B_k of radius 2^{-k} so that all $a_n^{(k)}$ belong to B_k.

Now consider the sequence $\{a_n^{(n)}\}$ which is a subsequence of the original sequence. We show that it is a Cauchy-sequence. Indeed if $m \geq n$ we have by the triangle inequality

$$d(a_m^{(m)}, a_n^{(n)}) \leq d(a_m^{(m)}, a_{m-1}^{(m-1)}) + \cdots + d(a_{n+1}^{(n+1)}, a_n^{(n)})$$

and since $a_j^{(j)}$ and $a_{j-1}^{(j-1)}$ are both in B_{j-1} their mutual distance is $\leq 2 \cdot 2^{1-j}$. Thus the previous displayed inequality implies that for $m > n$

$$d(a_m^{(m)}, a_n^{(n)}) \leq 2^{2-m} + \cdots + 2^{2-(n+1)} \leq 2^{2-n}$$

which shows that $\{a_n^{(n)}\}$ is a Cauchy sequence. Thus by the assumed completeness it converges, and we have found a convergent subsequence of $\{a_n\}$. $\square$

7. Proof of Lemma 7: A sequentially compact space is compact

To show this we first prove an auxiliary statement:

Sublemma. *Let X be a sequentially compact space. Suppose we are given an infinite open cover $\{G_\alpha\}_{\alpha \in A}$ of X. Then there exists an $\varepsilon > 0$ so that every ball of radius ε is contained in one of the (open) sets G_α.*

We argue by contradiction and assume that the statement does not hold. Then for every $n \in \mathbb{N}$ there is a ball B_n of radius $1/n$ which is *not contained in any* of the sets G_α. Let p_n be the center of B_n. Since we assume that X is sequentially compact the sequence of centers has a convergent subsequence $\{p_{n_k}\}$ whose limit we denote by p. Since the G_α is a cover there is an index α_o so that $p \in G_{\alpha_o}$. As p is an interior point of the (open) set G_{α_o} it contains an open ball of radius $\delta > 0$. Also there is an M so that for $k \geq M$ we have $d(p_{n_k}, p)) < \delta/2$. By the triangle inequality we see that the ball B_{n_k} is contained in G_{α_o}

provided that $k > M$. But this is a contradiction to the construction of the sequence $\{p_n\}$ (which implied that none of the balls B_n is contained in any of the sets G_α). $\square$

We now proceed to show that a sequentially compact is compact. We need to show that a given open cover $\{G_\alpha\}_{\alpha\in A}$ of X contains a finite subcover. By the sublemma there exists an $\varepsilon > 0$ so that every ball of radius ε is contained in one of the (open) sets G_α. We have shown in Lemma 5 that a sequentially compact space is totally bounded; thus there exist points $\{p_1, \dots, p_k\}$ so that X is contained in the union of the balls $B_\varepsilon(p_i)$, $i = 1, \dots, k$. As each $B_\varepsilon(p_i)$ is contained in one of the sets in the cover, say in G_{α_i}, the collection G_{α_i}, $i = 1, \dots, k$ is a finite subcover. $\square$

Exercises:

1. Prove that a totally bounded metric space is separable (i.e. contains a countable dense subset).

2. A collection $\{F_\alpha : \alpha \in A\}$ of closed sets has the *finite intersection property* if for every finite subset A_o of A the intersection $\cap_{\alpha\in A_o} F_\alpha$ is not empty.

Prove that the following statements (i), (ii) are equivalent.

(i) A metric space X, with metric d, is compact.

(ii) For every collection $\{F_\alpha\}_{\alpha\in A}$ of closed sets with the finite intersection property it follows that

$$\bigcap_{\alpha\in A} F_\alpha \neq \emptyset.$$

3. Let ℓ^1 denote the space of all absolutely summable sequences, i.e. the space of all sequences $\{a_n\}_{n=1,2,\dots}$ for which $\sum |a_n|$ converges, with the metric $d(a,b) = \sum_{n=1}^\infty |a_n - b_n|$.

(i) Prove that the set of all sequences $\{a_n\}$ which satisfy $|a_n| \le 2^{-n}$ for all $n \in \mathbb{N}$ is compact.

(ii) More generally, if $\{b_n\}$ is a fixed sequence of nonnegative terms with the property that $\sum b_n < \infty$ then the set of all sequences $\{a_n\}$ which satisfy $|a_n| \le b_n$ is compact.

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