I. The space $C(K)$

Let K be a compact metric space, with metric d_K. Let $\mathcal{B}(K)$ be the space of real valued bounded functions on K with the sup-norm

$$\|f\|_\infty = \sup_{x\in K} |f(x)|$$

Proposition : *$\mathcal{B}(K)$ is complete.*

Proof. Let f_n be a Cauchy sequence. By the definition of $\|\cdot\|_\infty$ this implies that for every $x \in K$ the n umerical sequence $f_n(x)$ is a Cauchy sequence in $\mathbb{R}$, and thus convergent in $\mathbb{R}$. Define $f(x) = \lim_{n\to\infty} f_n(x)$.

We show that f is bounded, and that $\|f_n - f\|_\infty \to 0$. Let $\varepsilon > 0$. Then there is $N = N(\varepsilon)$ so that $\sup_x |f_n(x) - f_m(x)| < \varepsilon/2$ for $n, m \geq N$. This shows that $|f_n(x)-f(x)| \leq \varepsilon/2$ for $n, m \geq N$ and for all $x \in K$. In particular $|f(x)| \leq \sup |f_N(x)| + \varepsilon$ so that f is bounded; moreover $\|f_n - f\|_\infty \leq \varepsilon/2$ for $n \geq N$.

As $\varepsilon > 0$ was arbitrary this establishes that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. $\square$

Let $C(K)$ be the vector space of real valued *continuous* functions on K.

Proposition. *Every $f \in C(K)$ is bounded and uniformly continuous.*

Proof. Let $\varepsilon > 0$. We have to show that there is $\delta > 0$ so that $|f(x)-f(x')| < \varepsilon$ whenever $d_K(x, x') < \delta$; moreover there is M so that $\sup_{x\in K} |f(x)| \leq M$.

As f is continuous, for every x there is a $\delta_x > 0$ so that for all t with $d_K(t, x) < \delta_x$ we have $|f(x) - f(t)| < \varepsilon/2$. As K is compact there is a finite number of points x_i, $i = 1, \dots, L$ so that the balls $B(x_i, \delta_{x_i}/2)$ cover K.

Define $\delta = \min_{i=1,\dots,L} \delta_{x_i}/2$. Suppose $d_K(x, x') < \delta$. Then there is an x_i so that $d_K(x, x_i) < \delta_{x_i}/2$. Then also

$$d_K(x', x_i) \leq d_K(x', x) + d_K(x, x_i) < \delta + \delta_{x_i}/2 \leq \delta_{x_i}$$

so that

$$|f(x) - f(x')| \leq |f(x) - f(x_i)| + |f(x_i) - f(x')| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This already shows that f is uniformly continuous.

By the same argument we have $|f(x)| \leq |f(x) - f(x_i)| + |f(x_i)| \leq \varepsilon + |f(x_i)|$ for $x \in B(x_i, \delta_{x_i}/2)$. Thus, for all $x \in K$ it follows that $|f(x)| \leq \max_{i=1,\dots,L} |f(x_i)| + \varepsilon$ so that certainly f is bounded. $\square$

We have just seen that $C(K)$ is a subspace of $\mathcal{B}(K)$. We consider $C(K)$ as a normed space with the norm (and therefore with the metric) inherited from $\mathcal{B}(K)$. The following result shows that $C(K)$ is complete.

Proposition. *$C(K)$ is a closed subspace of $\mathcal{B}(K)$.*

Proof. Let f_n be a sequence of functions in $C(K)$ which converges to f (i.e. $\|f_n - f\|_\infty \to 0$). We will show that f is (uniformly) continuous, i.e. given $\varepsilon > 0$ there is $\delta > 0$ so that $|f(x) - f(x')| < \varepsilon$ if $d_K(x, x') < \delta$.

Let $\varepsilon > 0$. Let N be so that $\|f_n - f\|_\infty < \varepsilon/4$ for all $n \geq N$, that means $|f_n(t) - f(t)| < \varepsilon/4$ for all $t \in K$ and all $n \geq N$. Since f_N is (uniformly) continuous there is $\delta > 0$ so that $|f_N(x) - f_N(x')| < \varepsilon/2$ whenever $d_K(x, x') < \delta$. For those x, x' we also have

$$\begin{aligned}|f(x) - f(x')| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x')| + |f_N(x') - f(x')| \\ &\leq 2\|f - f_N\|_\infty + |f_N(x) - f_N(x')| < 2\varepsilon/4 + \varepsilon/2 = \varepsilon. \qquad \square\end{aligned}$$

II. The Arzelà-Ascoli theorem

Definition. Let $\mathcal{F}$ be a collection of functions in $C(K)$.

(i) The collection $\mathcal{F}$ is said to be *pointwise bounded* if for every $x \in K$ there is an $M(x) \geq 0$ so that $|f(x)| \leq M(x)$ for every $f \in \mathcal{F}$.

(ii) The collection $\mathcal{F}$ is said to be *equicontinuous* [1] if for every $\varepsilon > 0$ there exists $\delta > 0$ so that for all $f \in \mathcal{F}$ and x, x' with $d_K(x, x') < \delta$ we have $|f(x) - f(x')| < \varepsilon$.

Remark: It is not difficult to show that a collection $\mathcal{F}$ which is *both equicontinuous and pointwise bounded on a compact set* K is also *uniformly bounded on* K, meaning that there is an $M \geq 0$ so that $\|f\|_\infty = \sup_{x \in K} |f(x)| \leq M$ for all $f \in \mathcal{F}$. This implication is hidden in the proof of the theorem below where a stronger property is proved (namely that $\mathcal{F}$ is totally bounded).

Theorem. *A subset $\mathcal{F}$ of $C(K)$ is totally bounded if and only if it is pointwise bounded and equicontinuous.*

Proof of necessity: If $\mathcal{F}$ is a totally bounded subset of $C(K)$ then $\mathcal{F}$ is pointwise bounded and equicontinuous.

By the definition of $\mathcal{F}$ totally bounded there are functions $f_1, \dots, f_N$ in $\mathcal{F}$ so that for every $f \in \mathcal{F}$ there is an index $i \in \{1, \dots, N\}$ with $\|f_i - f\|_\infty < \varepsilon/4$. Clearly $\|f\| \leq \|f_i\|_\infty + \|f - f_i\|_\infty \leq \max_{i=1,\dots,N} \|f_i\|_\infty + \varepsilon/4$ so that $\mathcal{F}$ is bounded in norm (and clearly pointwise bounded).

Now we show the equicontinuity of the family $\mathcal{F}$. By a Lemma above each f_i is uniformly continuous. Thus for each i there exists a $\delta_i > 0$ such that $|f_i(x) - f_i(x')| < \varepsilon/2$ whenever $d_K(x, x') < \delta_i$. Let $\delta = \min\{\delta_1, \dots, \delta_N\}$. Then $\delta > 0$ and we have $|f_i(x) - f_i(x')| < \varepsilon/2$ for every i whenever $d_K(x, x') < \delta$.

Now pick any $f \in \mathcal{F}$, and let i be so that $\|f_i - f\|_\infty < \varepsilon/4$, and let x, x' be so that $d_K(x, x') < \delta$. Then

$$\begin{aligned}|f(x) - f(x')| &\leq |f(x) - f_i(x)| + |f_i(x) - f_i(x')| + |f_i(x') - f(x')| \\ &\leq 2\|f - f_i\|_\infty + |f_i(x) - f_i(x')| < 2\varepsilon/4 + \varepsilon/2 = \varepsilon\,. \qquad \square\end{aligned}$$

[1] Some authors also use the terminology "uniformly equicontinous".

Proof of sufficiency: If $\mathcal{F} \subset C(K)$ is equicontinuous and pointwise bounded then $\mathcal{F}$ is totally bounded.

Fix $\varepsilon > 0$. We shall first find a finite collection $\mathcal{G}$ of functions in $\mathcal{B}(K)$ so that for every $f \in \mathcal{F}$ there exists a $g \in \mathcal{G}$ with $\|f - g\|_\infty < \varepsilon$.

Let $\delta > 0$ so that for all $f \in \mathcal{F}$ we have $|f(x) - f(x')| < \varepsilon/4$ whenever $|x - x'| < \delta$. Again we use the compactness of K and cover K with finitely many balls $B(x_i, \delta)$, $i = 1, \dots, L$. There is M_i so that $|f(x_i)| \le M_i$ for all $f \in \mathcal{F}$. Let $M = 1 + \max_{i=1,\dots,L} M_i$.

We now let $A_1 = B(x_1, \delta)$, and $A_i = B(x_i, \delta) \backslash \cup_{\nu=1}^{i-1} B(x_\nu, \delta)$, for $2 \le i \le L$. (Some of the A_i could be empty but that does not matter).

Let $\mathcal{Z}^L(M, \varepsilon)$ be the set of L-tuples $\vec{n}$ of integers $\vec{n} = (n_1, \dots, n_L)$ with the property that $|n_i|\varepsilon/4 \le M$ for $i = 1, \dots, L$. Note that $\mathcal{Z}^L(M, \varepsilon)$ is a finite set (indeed its cardinality is $\le (8M\varepsilon^{-1} + 1)^L$).

We now define a collection $\mathcal{G}$ of functions which are constant on the sets A_i (these are analogues of step functions). Namely given $\vec{n}$ in $\mathcal{Z}^L(M, \varepsilon)$ we let $g^{\vec{n}}$ be the unique function that takes the value $n_i\varepsilon/4$ on the set A_i (provided that that set is nonempty). Clearly the cardinality of $\mathcal{G}$ is not larger than the cardinality of $\mathcal{Z}^L(M, \varepsilon)$.

Let $f \in \mathcal{F}$. Consider an A_i which by construction is a subset of $B(x_i, \delta)$. Then $|f(x) - f(x_i)| < \varepsilon/4$ for all $x \in A_i$ (this condition is vacuous if A_i is empty). Now $|f(x_i)| \le M_i \le M$ and therefore there exists an integer n_i with the property that $-M \le n_i\varepsilon/4 \le M$ and $|f(x_i) - n_i\varepsilon/4| < \varepsilon/4$. Then we also have that for $i = 1, \dots, L$ and for every $x \in A_i$,

$$|f(x) - n_i\varepsilon/4| \le |f(x) - f(x_i)| + |f(x_i) - n_i\varepsilon/4| < \varepsilon/4 + \varepsilon/4 = \varepsilon/2.$$

This implies that for this choice of $\vec{n} = (n_1, \dots, n_L)$ we get $\|f - g^{\vec{n}}\|_\infty < \varepsilon/2$.

Finally, we need to find a finite cover of $\mathcal{F}$ with ε-balls centered at points in $\mathcal{F}$. Consider the subcollection $\widetilde{\mathcal{G}}$ of functions in $\mathcal{G}$ for which the ball of radius $\varepsilon/2$ centered at g contains a function in $\mathcal{F}$. Denote the functions in $\widetilde{\mathcal{G}}$ by $g_1, \dots, g_N$. The balls of radius $\varepsilon/2$ centered at $g_1, \dots g_N$ cover $\mathcal{F}$. For $i = 1, \dots, N$ pick $f_i \in \mathcal{F}$ so that $\|g_i - f_i\|_\infty < \varepsilon/2$. By the triangle inequality (for the norm in $\mathcal{B}(K)$ whose restriction to $C(K)$ is also the norm in $C(K)$) the ball of radius $\varepsilon/2$ centered at g_i is contained in the ball of radius ε centered at f_i. Thus the balls of radius ε centered at f_i, $i = 1, \dots, N$ cover the set $\mathcal{F}$. $\square$

Corollary. *A closed subset $\mathcal{F}$ of $C(K)$ is compact if and only if it is pointwise bounded and equicontinuous.*

Proof. The space $\mathcal{B}(K)$ is complete and so is the closed subspace $C(K)$. Since we now assume that $\mathcal{F}$ is closed in $C(K)$ the metric space $\mathcal{F}$ is complete. Thus by the characterization of compactness ($\mathcal{F}$ compact $\iff$ $\mathcal{F}$ totally bounded and complete) the corollary follows from the theorem. $\square$

Corollary. *An equicontinuous and bounded sequence $\{f_n\}$ of functions in $C(K)$ has a uniformly convergent subsequence.*

Proof. The closure of $\mathcal{F} := \{f_n : n \in \mathbb{N}\}$ is bounded, complete, and equicontinuous, thus compact. By a part of the theorem on the characterization of compactness it is also sequentially compact, therefore f_n has a convergent subsequence. $\square$

III. The Peano Existence theorem for differential equations

We are concerned with the initial value problem for a differential equation

$$y'(t) = F(t, y(t)), \quad y(t_0) = y_0\,.$$

Here $(t, y) \mapsto F(t, y)$ is a continuous function of two variables defined near a point $(t_0, y_0) \in \mathbb{R}^2$ and we wish to find a function $t \mapsto y(t)$ for which the derivative $y'(t)$ equals the value of $F(t, y)$ for $y = y(t)$, and which has the value y_0 at the "initial" time t_0.

This problem has a solution, as stated in the following existence theorem.

Theorem. *Let Ω be a nonempty open set in $\mathbb{R} \times \mathbb{R}$, let $(t_0, y_0) \in \Omega$ and let $F : \Omega \to \mathbb{R}$ be continuous. Let $\mathcal{R}$ be a compact rectangle of the form*

$$\mathcal{R} = \{(t, y) : \ |t - t_0| \le a, \qquad |y - y_0| \le b\}$$

contained in Ω. Suppose that $|F(t, y)| \le M$ for $(t, y) \in \mathcal{R}$ and let $\tilde{a} = \min\{a, b/M\}$. Then there exists a function $t \mapsto y(t)$ defined on $(t_0 - \tilde{a}, t_0 + \tilde{a})$ which satisfies the initial value problem

$$\begin{aligned} &y'(t) = F(t, y(t)) \quad \text{for } |t - t_0| < \tilde{a} \\ &y(t_0) = y_0. \end{aligned}$$

Proof. [2]

The strategy is to prove the existence of a continuous function y on $[t_0 - \tilde{a}, t_0 + \tilde{a}]$ which satisfies the integral equation

$$y(t) = y_0 + \int_{t_0}^{t} F(s, y(s))\, ds.$$

Once we have found this function we observe that the integrand $F(s, y(s))$ is also a continuous function. Thus the integral represents a differentiable function (by the fundamental theorem of calculus) and its derivative is $F(t, y(t))$. Thus y is also differentiable and we have $y' = F(t, y(t))$ for $t \in (t_0 - \tilde{a}, t_0 + \tilde{a})$, i.e. y is a solution of the differential equation. Also by the last display $y(t_0) = y_0$ and a solution is found.

[2] The proof becomes less technical if one makes the more restrictive assumption that $|F(t, y)| \le M$ for all (t, y_0) with $|t - t_0| \le a$ and $y \in \mathbb{R}$. We then have $\tilde{a} = a$ and much of the discussion in step 1 is then superfluous. We will first discuss this special case in class.

We write up the proof of the existence of y only for the interval $[t_0, t_0+\tilde{a}]$ and leave the notational changes for the interval $[t_0 - \tilde{a}, t_0]$ to the reader. We will split the proof into five steps.

1. We shall construct functions with polygonal graphs which are candidates to approximate the solutions.

F is uniformly continuous on the compact set $\mathcal{R}$. Let $\varepsilon > 0$ a small number and let $\delta = \delta(\varepsilon)$ be as in the definition of uniform continuity, i.e. we have

$$|F(t,y) - F(t',y')| < \varepsilon \qquad \text{whenever } |(t,y) - (t',y')| \le \delta \text{ and } (t,y) \in \mathcal{R},\ (t',y') \in \mathcal{R}.$$

Let

$$t_0 < t_1 < \cdots < t_N = t_0 + \tilde{a}$$

be a partition of $[t_0, t_0 + \tilde{a}]$ so that $t_{k+1} - t_k < \frac{1}{2}\min\{\delta, \delta/M\}$ for $k = 0, \dots, N-1$.

We now construct a function $Y \equiv Y_\varepsilon$ on $[t_0, \tilde{a}]$; this definition depends on ε, δ and the partition chosen, however keeping this dependence in mind we will omit the subscript ϵ in steps 1-4 to avoid cluttered notation.

To define $Y \equiv Y_\varepsilon$ we set $Y(t_0) = y_0$. On the first partition interval $[t_0, t_1]$ the graph of Y will be a line with initial point $(t_0, Y(t_0))$ and slope $F(t, y_0) = F(t, Y(t_0))$. The value of this function at t_1 is $Y(t_1) = Y(t_0) + F(t_0, Y(t_0))(t_1 - t_0)$. On the interval $(t_1, t_2]$ we wish to define Y as the graph of a line starting at $(t_1, Y(t_1))$ with readjusted slope $F(t_1, Y(t_1))$. In order for this construction to work we need to make sure that $F(t_1, Y(t_1))$ is still well defined, meaning that the point $(t, Y(t_1))$ belongs to the rectangle $\mathcal{R}$. For this we have to check $|Y(t_1) - y_0| \le b$. Indeed we have that $|Y(t_1) - y_0| = |F(t_0, Y(t_0))(t_1 - t_0)| \le M(t_1 - t_0) \le M\tilde{a} \le b$ by definition of $\tilde{a}$. A similar calculation has to be made at every step.

To be rigorous we formulate the following

Claim. *For $k = 0, \dots, N$ there are numbers y_k so that*

$$\begin{aligned} &|y_k - y_0| \le M(t_k - t_0) \le b \text{ and} \\ &y_k = y_{k-1} + F(t_{k-1}, y_{k-1})(t_k - t_{k-1}). \end{aligned}$$

For $k = 0$ the statement is clear. We argue by induction. Above we have just verified this claim for $k = 1$, and in a similar way we do the induction step.

If $k \in \{1, \dots, N-1\}$ we prove the claim for $k+1$, i.e. the existence of y_{k+1} with the required properties, under the induction hypothesis, that $y_1, \dots, y_k$ have been found. Since by the induction hypothesis $|y_k - y_0| \le M(t_k - t_0)$ which is $\le b$ the expression $F(t_k, y_k)$ is well defined and thus $y_{k+1} = y_k + F(t_k, y_k)(t_{k+1} - t_k)$ is well defined. To check that $|y_{k+1} - y_0| \le M(t_{k+1} - t_0)$

we observe that

$$\begin{aligned}
&|y_{k+1} - y_0| = |y_{k+1} - y_k + y_k - y_0| \le |y_{k+1} - y_k| + |y_k - y_0| \\
&= |F(t_k, y_k)|(t_{k+1} - t_k) + |y_k - y_0| \le M(t_{k+1} - t_k) + |y_k - y_0| \\
&\le M(t_{k+1} - t_k) + M(t_k - t_0) = M(t_{k+1} - t_0)
\end{aligned}$$

where we have in the second to last step used the induction hypothesis. Of course $M(t_{k+1} - t_0) \le M\tilde{a} \le b$. The claim follows by induction.

Now that the claim is verified we can define $Y(t)$ on $[t_0, \tilde{a}]$ by $Y(t_k) = y_k$ and

$$Y(t) = y_k + F(t_k, y_k)(t - t_k), \quad t_k < t < t_{k+1}, \quad k = 0, 1, \dots, N-1.$$

Observe that this definition is also valid for $t_k \le t \le t_{k+1}$. The function Y is continuous, piecewise linear, and the absolute values of all slopes are bounded by M.

2. *The function $Y \equiv Y_\varepsilon$ constructed in part 1 satisfies the inequality*

$$|Y(t) - Y(t')| \le M|t - t'| \text{ whenever } t,\ t' \text{ are both in } [t_0, t_0 + \tilde{a}].$$

Proof: W.l.o.g $t' < t$. If t', t lie in the same partition interval $[t_k, t_{k+1}]$ then this is immediate since

$$|Y(t) - Y(t')| = |F(t_k, y_k)(t - t')| \le M|t - t'|.$$

If t', t lie in different partition intervals, $t' \in [t_k, t_{k+1}]$, $t \in [t_l, t_{l+1}]$ with $k < l$, then

$$\begin{aligned}
|Y(t) - Y(t')| &= \Big|Y(t) - Y(t_l) + \sum_{k<\nu<l} Y(t_{\nu+1}) - Y(t_\nu) + Y(t_{k+1}) - Y(t')\Big| \\
&\le |Y(t) - Y(t_l)| + \sum_{k<\nu<l} |Y(t_{\nu+1}) - Y(t_\nu)| + |Y(t_{k+1}) - Y(t')| \\
&= |F(t_l, y_l)|(t - t_l) + \sum_{k<\nu<l} |F(t_\nu, y_\nu)|(t_{\nu+1} - t_\nu) + |F(t_k, y_k)|(t_{k+1} - t') \\
&\le M(t - t_l) + \sum_{k<\nu<l} M(t_{\nu+1} - t_\nu) + M(t_{k+1} - t') = M(t - t')\,;
\end{aligned}$$

here the middle terms with the sum $\sum_{k<\nu<l}$ are only present when when $k + 1 < l$. The claim 2 is proved.

3. Note that if we define $g(t) = F(t_{k-1}, Y(t_{k-1}))$ if $t_{k-1} \le t < t$ then g is a step function and Y is differentiable in the open intervals (t_{k-1}, t_k) with derivative $Y'(t) = g(t)$.

Claim: *For $t_0 \le t \le t_0 + \widetilde{a}$ we have*

$$Y(t) = y_0 + \int_{t_0}^{t} g(s)ds$$

and

$$\big|g(s) - F(s, Y(s))\big| \le \varepsilon \text{ if } t_{k-1} < s < t_k\,.$$

We first verify the first formula for $t = t_k$. Then

$$Y(t_k) - y_0 = Y(t_k) - Y(t_0) = \sum_{\nu=1}^{k} [Y(t_\nu) - Y(t_{\nu-1})]$$

$$= \sum_{\nu=1}^{k} F(t_{\nu-1}, Y(t_{\nu-1}))(t_\nu - t_{\nu-1}) = \sum_{\nu=1}^{k} \int_{t_{\nu-1}}^{t_\nu} g(s)ds = \int_{t_0}^{t_k} g(s)ds$$

Similarly for $t_k < t < t_{k+1}$,

$$Y(t) - Y(t_k) = F(t_k, Y(t_k))(t - t_k) = \int_{t_k}^{t} g(s)ds\,.$$

We put the two formulas together and get

$$Y(t) = Y(t_k) + \int_{t_k}^{t} g(s)ds \ = y_0 + \int_{t_0}^{t_k} g(s)ds + \int_{t_k}^{t} g(s)ds = y_0 + \int_{t_0}^{t} g(s)ds\,.$$

For the second assertion let $t_{k-1} < s < t_k$ and observe

$$|g(s) - F(s, Y(s))| = |F(t_{k-1}, Y(t_{k-1})) - F(s, Y(s))|$$

and $|Y(t_{k-1}) - Y(s)| \le M|t_{k-1} - s| \le M(t_k - t_{k-1}) \le \delta/2$ (since the maximal width of the partition is $< \delta/2M$). Thus the distance of the points $(t_{k-1}, Y(t_{k-1})$ and $(s, Y(s))$ is no more than δ. It follows that

$$|F(t_{k-1}, Y(t_{k-1})) - F(s, Y(s))| < \varepsilon.$$

4. Claim: *For $t_0 \le s \le t_0 + \tilde{a}$,*

$$\Big|Y(t) - \Big(y_0 + \int_{t_0}^{t} F(s, Y(s))ds\Big)\Big| \le \varepsilon\tilde{a}\,.$$

This follows from part 3, since the right hand side is

$$\Big|y_0 + \int_{t_0}^{t} g(s)ds - \Big(y_0 + \int_{t_0}^{t} F(s, Y(s))ds\Big)\Big|$$
$$= \Big|\int_{t_0}^{t} [g(s) - F(s, Y(s))]ds\Big|$$

and this is estimated by

$$\int_{[t_0,t]} |g(s) - F(s, Y(s))|ds \le \varepsilon(t - t_0) \le \varepsilon\tilde{a}\,.$$

5. So far we have only considered a fixed function $Y \equiv Y_\varepsilon$. Now consider a sequence of such functions $Y_{\varepsilon(n)}$ where $\varepsilon(n) \to 0$ (for example $\varepsilon(n) = 2^{-n}$). These functions satisfy the properties in part 1-4 with the parameter $\varepsilon = \varepsilon(n)$ and with $\delta = \delta(\varepsilon(n))$.

By part 2 the family of functions $Y_{\varepsilon(n)}$ is uniformly bounded and uniformly equicontinuous. Indeed for all n and all $t \in [t_0, t_0 + \tilde{a}]$ we have shown

$$|Y_{\varepsilon(n)}(t) - Y_{\varepsilon(n)}(t')| \le M|t - t'|$$

(which implies the uniform equicontinuity) and it follows also that

$$|Y_{\varepsilon(n)}(t)| \le |Y_{\varepsilon(n)}(t_0)| + |Y_{\varepsilon(n)}(t) - Y_{\varepsilon(n)}(t_0)| \le |y_0| + M|t - t_0| \le |y_0| + M\tilde{a}\,.$$

Thus the Arzelà-Ascoli theorem allows us to choose an increasing sequence of integers m_n such that the subsequence $Y_{\varepsilon(m_n)}(t)$ converges uniformly on $[t_0, t_0+\tilde{a}]$ to a limit $y(t)$. As a uniform limit of continuous functions y is continuous on $[t_0, t_0 + \tilde{a}]$. Now by the uniform continuity of F and the uniform convergence of $Y_{\varepsilon(m_n)}$ we see[3] that $F(t, Y_{\varepsilon(m_n)}(t))$ converges to $F(t, y(t))$, uniformly on $[t_0, t_0 + \tilde{a}]$. This implies that $y_0 + \int_{t_0}^t F(s, Y_{\varepsilon(m_n)}(s))ds$ converges to $y_0 + \int_{t_0}^t F(s, y(s))ds$. But by part (4) we also have

$$\Big|Y_{\varepsilon(m_n)}(t) - \Big(y_0 + \int_{t_0}^t F(s, Y_{\varepsilon(m_n)}(s))\,ds\Big)\Big| \le \varepsilon(m_n)\tilde{a} \;\to 0 \text{ as } n \to \infty.$$

Thus we obtain for the limit (as $m \to \infty$)

$$y(t) = y_0 + \int_{t_0}^t F(s, y(s))\,ds$$

which is the desired integral equation for y. □

Remark. The proof of the Peano existence theorem is not constructive (and, given the nonuniqueness examples below, one should perhaps not expect such a proof). In the next section will provide an alternative constructive proof based on a concrete iterative scheme, that works under the additional Lipschitz assumption on F. The metric d_C in the uniqueness proof will again play an important role in that proof.

IV. Possible failure of uniqueness

The Peano theorem does not guarantee uniqueness.

We give an example. Here $F(t, y) = \sqrt{|y|}$ which is continuous (but near near the x-axis it does not satisfy the Lipschitz condition with respect to y in the above uniqueness theorem). Consider the initial value problem

$$y'(t) = \sqrt{|y(t)|}\,, \qquad y(0) = 0\,.$$

[3]Provide the details of this argument.

Verify that the four functions

$$Y_1(t) = 0 \qquad\qquad Y_2(t) = \begin{cases} t^2/4 \text{ if } t > 0 \\ 0 \text{ if } t \le 0 \end{cases}$$

$$Y_3(t) = \begin{cases} 0 \text{ if } t > 0 \\ -t^2/4 \text{ if } t \le 0 \end{cases} \qquad\qquad Y_4(t) = \begin{cases} t^2/4 \text{ if } t > 0 \\ -t^2/4 \text{ if } t \le 0 \end{cases}$$

are solutions of the initial value problem.

V. An existence and uniqueness theorem

It turns out that if the function F has additional regularity properties then one can prove uniqueness (and the example in §IV shows that some additional hypothesis is needed). This section does not rely on §III.

Theorem. *Let Ω be a nonempty open set in $\mathbb{R} \times \mathbb{R}$, let $(t_0, y_0) \in \Omega$ and let $F : \Omega \to \mathbb{R}$ be continuous. Let $\mathcal{R}$ be a compact rectangle of the form*

$$\mathcal{R} = \{(t, y) : \ |t - t_0| \le a, \qquad |y - y_0| \le b\}$$

contained in Ω. Suppose that $|F(t, y)| \le M$ for $(t, y) \in \mathcal{R}$ and let $\tilde{a} = \min\{a, b/M\}$. In addition assume that there is a constant C so that

$$|F(t, y) - F(t, u)| \le C|y - u| \text{ whenever } (t, y) \in \mathcal{R},\ (t, u) \in \mathcal{R}.$$

Then there exists a unique function $t \mapsto y(t)$ defined on $(t_0 - \tilde{a}, t_0 + \tilde{a})$ which satisfies the initial value problem

$$\begin{aligned} &y'(t) = F(t, y(t)) \quad \text{for } |t - t_0| < \tilde{a} \\ &y(t_0) = y_0. \end{aligned}$$

Remark: The additional hypothesis says that on $\mathcal{R}$ the function F satisfies a Lipschitz-condition with respect to the variable y. In the case where F is differentiable with respect to y and the partial derivative $\frac{\partial F}{\partial y}$ satisfies the bound $|\frac{\partial F}{\partial y}| \le C$ on $\mathcal{R}$, the hypothesis is satisfied. This is a consequence of the mean value theorem applied to $y \mapsto F(t, y)$, for any fixed t.

Proof of a weaker version of the theorem. The existence of a solution in $[t_0 - \tilde{a}, t_0 + \tilde{a}]$ follows from the previous theorem and we shall now establish uniqueness. In this weaker version we will be content to establish uniqueness on a smaller interval $[t_0 - \gamma, t_0 + \gamma]$ where $\gamma = \min\{\tilde{a}, (2C)^{-1}\}$.

Let Y_1, Y_2 are solutions; we will prove that $Y_1 = Y_2$ on $[t_0 - \gamma, t_0 + \gamma]$.

Y_1 and Y_2 will then satisfy the above integral equation, i.e.

$$Y_1(t) = y_0 + \int_{t_0}^{t} F(s, Y_1(s))\, ds\,,$$
$$Y_2(t) = y_0 + \int_{t_0}^{t} F(s, Y_2(s))\, ds\,.$$

Hence

$$Y_2(t) - Y_1(t) = \int_{t_0}^{t} \big[F(s, Y_2(s)) - F(s, Y_1(s))\big]\, ds$$

and therefore by our Lipschitz condition

$$|Y_2(t) - Y_1(t)| \le \int_{t_0-\gamma}^{t_0+\gamma} C|Y_2(s) - Y_1(s)|\, ds.$$

We take the maximum over $t \in [t_0 - \gamma, t_0 + \gamma]$ on the right hand side and get

$$\max_{|t-t_0|\le\gamma} |Y_2(t) - Y_1(t)| \le 2\gamma C \max_{|t-t_0|\le\gamma} |Y_2(t) - Y_1(t)|.$$

Since we assumed $2\gamma C < 1$ this forces $\max_{|t-t_0|\le\gamma} |Y_2(t) - Y_1(t)| = 0$, that is $Y_1(t) = Y_2(t)$ for $t \in [t_0 - \gamma, t_0 + \gamma]$. □

Proof of the stronger claimed version of the theorem. It remains to show that uniqueness holds on the $[t_0 - \tilde{a}, t_0 + \tilde{a}]$ (which does not depend on C). This requires a refined estimate.

We start as above with solutions Y_1 and Y_2 and get

$$Y_2(t) - Y_1(t) = \int_{t_0}^{t} \big[F(s, Y_2(s)) - F(s, Y_1(s))\big]\, ds.$$

By the Lipschitz condition,

$$|Y_2(t) - Y_1(t)| \le \begin{cases} C\int_{t_0}^{t} |Y_2(s) - Y_1(s)|\, ds & \text{if } t \ge t_0 \\ C\int_{t}^{t_0} |Y_2(s) - Y_1(s)|\, ds & \text{if } t \le t_0 \end{cases}.$$

It will now be advantageous to replace on the space of functions which are continuous on $[t_0 - \tilde{a}, t_0 + \tilde{a}]$, the standard sup norm by

$$\|f\|_{\infty;C} = \max_{|t-t_0|\le\tilde{a}} e^{-2C|t-t_0|}|f(t)|;$$

this is a sup-norm with a damping factor.

The associated metric is then

$$d_C(f, g) = \max_{|t-t_0|\le\tilde{a}} e^{-2C|t-t_0|}|f(t) - g(t)|.$$

We are aiming to show the inequality $d_C(Y_1, Y_2) \le \frac{1}{2} d_C(Y_1, Y_2)$ which forces $d_c(Y_1, Y_2) = 0$ and hence $Y_1 = Y_2$ on $[t_0 - \tilde{a}, t_0 + \tilde{a}]$.

To show this inequality we estimate for $t \in [t_0, t_0 + \tilde{a}]$

$$\begin{aligned}
|Y_2(t) - Y_1(t)| &\le \int_{t_0}^{t} C|Y_2(s) - Y_1(s)|\,ds \\
&= C\int_{t_0}^{t} e^{2C(s-t_0)}e^{-2C(s-t_0)}|Y_2(s) - Y_1(s)|\,ds \\
&\le C\int_{t_0}^{t} e^{2C(s-t_0)}d_C(Y_1,Y_2)\,ds = Cd_C(Y_1,Y_2)\int_{t_0}^{t} e^{2C(s-t_0)}\,ds \\
&= Cd_C(Y_1,Y_2)\frac{e^{2C(t-t_0)}-1}{2C} \;\le\; \frac{1}{2}e^{2C|t-t_0|}d_C(Y_1,Y_2)\,.
\end{aligned}$$

Similarly, for $t \in [t_0 - \tilde{a}, t_0]$

$$\begin{aligned}
|Y_2(t) - Y_1(t)| &\le \int_{t}^{t_0} C|Y_2(s) - Y_1(s)|\,ds \\
&= C\int_{t}^{t_0} e^{2C(t_0-t)}e^{-2C(t_0-s)}|Y_2(s) - Y_1(s)|\,ds \\
&\le C\int_{t}^{t_0} e^{2C(t_0-s)}d_C(Y_1,Y_2)\,ds \\
&= Cd_C(Y_1,Y_2)\frac{e^{2C(t_0-s)}-1}{2C} \le \frac{1}{2}e^{2C|t_0-t|}d_C(Y_1,Y_2)\,.
\end{aligned}$$

Combining both cases we see that

$$e^{-2C|t-t_0|}|Y_2(t) - Y_1(t)| \le \frac{1}{2}d_C(Y_1,Y_2) \text{ whenever } |t - t_0| \le \tilde{a}$$

which means $d_C(Y_1,Y_2) \le \frac{1}{2}d_C(Y_1,Y_2)$. □