The space $C(K)$

Let K be a compact metric space, with metric d_K. Let $\mathcal{B}(K)$ be the space of real valued bounded functions on K with the sup-norm

$$\|f\|_\infty = \sup_{x\in K} |f(x)|$$

Proposition : *$\mathcal{B}(K)$ is complete.*

Proof. Let f_n be a Cauchy sequence. By the definition of $\|\cdot\|_\infty$ this implies that for every $x \in K$ the numerical sequence $f_n(x)$ is a Cauchy sequence in $\mathbb{R}$, and thus convergent in $\mathbb{R}$. Define $f(x) = \lim_{n\to\infty} f_n(x)$.

We show that f is bounded, and that $\|f_n - f\|_\infty \to 0$. Let $\varepsilon > 0$. Then there is $N = N(\varepsilon)$ so that $\sup_x |f_n(x) - f_m(x)| < \varepsilon/2$ for $n, m \geq N$. This shows that $|f_n(x)-f(x)| \leq \varepsilon/2$ for $n, m \geq N$ and for all $x \in K$. In particular $|f(x)| \leq \sup |f_N(x)| + \varepsilon$ so that f is bounded; moreover $\|f_n - f\|_\infty \leq \varepsilon/2$ for $n \geq N$.

As $\varepsilon > 0$ was arbitrary this establishes that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. $\square$

Let $C(K)$ be the vector space of real valued *continuous* functions on K.

Proposition. *Every $f \in C(K)$ is bounded and uniformly continuous.*

Proof. Let $\varepsilon > 0$. We have to show that there is $\delta > 0$ so that $|f(x)-f(x')| < \varepsilon$ whenever $d_K(x, x') < \delta$; moreover there is M so that $\sup_{x\in K} |f(x)| \leq M$.

As f is continuous, for every x there is a $\delta_x > 0$ so that for all t with $d_K(t, x) < \delta_x$ we have $|f(x) - f(t)| < \varepsilon/2$. As K is compact there is a finite number of points x_i, $i = 1, \dots, L$ so that the balls $B(x_i, \delta_{x_i}/2)$ cover K.

Define $\delta = \min_{i=1,\dots,L} \delta_{x_i}/2$. Suppose $d_K(x, x') < \delta$. Then there is an x_i so that $d_K(x, x_i) < \delta_{x_i}/2$. Then also

$$d_K(x', x_i) \leq d_K(x', x) + d_K(x, x_i) < \delta + \delta_{x_i}/2 \leq \delta_{x_i}$$

so that

$$|f(x) - f(x')| \leq |f(x) - f(x_i)| + |f(x_i) - f(x')| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This already shows that f is uniformly continuous.

By the same argument we have $|f(x)| \leq |f(x) - f(x_i)| + |f(x_i)| \leq \varepsilon + |f(x_i)|$ for $x \in B(x_i, \delta_{x_i}/2)$. Thus, for all $x \in K$ it follows that $|f(x)| \leq \max_{i=1,\dots,L} |f(x_i)| + \varepsilon$ so that certainly f is bounded. $\square$

We have just seen that $C(K)$ is a subspace of $\mathcal{B}(K)$. We consider $C(K)$ as a normed space with the norm (and therefore with the metric) inherited from $\mathcal{B}(K)$. The following result shows that $C(K)$ is complete.

Proposition. *$C(K)$ is a closed subspace of $\mathcal{B}(K)$.*

Proof. Let f_n be a sequence of functions in $C(K)$ which converges to f (i.e. $\|f_n - f\|_\infty \to 0$). We will show that f is (uniformly) continuous, i.e. given $\varepsilon > 0$ there is $\delta > 0$ so that $|f(x) - f(x')| < \varepsilon$ if $d_K(x, x') < \delta$.

Let $\varepsilon > 0$. Let N be so that $\|f_n - f\|_\infty < \varepsilon/4$ for all $n \geq N$, that means $|f_n(t) - f(t)| < \varepsilon/4$ for all $t \in K$ and all $n \geq N$. Since f_N is (uniformly) continuous there is $\delta > 0$ so that $|f_N(x) - f_N(x')| < \varepsilon/2$ whenever $d_K(x, x') < \delta$. For those x, x' we also have

$$\begin{aligned} |f(x) - f(x')| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x')| + |f_N(x') - f(x')| \\ &\leq 2\|f - f_N\|_\infty + |f_N(x) - f_N(x')| < 2\varepsilon/4 + \varepsilon/2 = \varepsilon. \qquad \square \end{aligned}$$

The Arzelà-Ascoli theorem

Definition. Let $\mathcal{F}$ be a collection of functions in $C(K)$.

(i) The collection $\mathcal{F}$ is said to be *pointwise bounded* if for every $x \in K$ there is an $M(x) \geq 0$ so that $|f(x)| \leq M(x)$ for every $f \in \mathcal{F}$.

(ii) The collection $\mathcal{F}$ is said to be *equicontinuous* [1] if for every $\varepsilon > 0$ there exists $\delta > 0$ so that for all $f \in \mathcal{F}$ and x, x' with $d_K(x, x') < \delta$ we have $|f(x) - f(x')| < \varepsilon$.

Remark: It is not difficult to show that a collection $\mathcal{F}$ which is *both equicontinuous and pointwise bounded on a compact set* K is also *uniformly bounded on* K, meaning that there is an $M \geq 0$ so that $\|f\|_\infty = \sup_{x \in K} |f(x)| \leq M$ for all $f \in \mathcal{F}$. This implication is hidden in the proof of the theorem below where a stronger property is proved (namely that $\mathcal{F}$ is totally bounded).

Theorem. *A subset $\mathcal{F}$ of $C(K)$ is totally bounded if and only if it is pointwise bounded and equicontinuous.*

Proof of necessity: If $\mathcal{F}$ is a totally bounded subset of $C(K)$ then $\mathcal{F}$ is pointwise bounded and equicontinuous.

By the definition of $\mathcal{F}$ totally bounded there are functions $f_1, \dots, f_N$ in $\mathcal{F}$ so that for every $f \in \mathcal{F}$ there is an index $i \in \{1, \dots, N\}$ with $\|f_i - f\|_\infty < \varepsilon/4$. Clearly $\|f\| \leq \|f_i\|_\infty + \|f - f_i\|_\infty \leq \max_{i=1,\dots,N} \|f_i\|_\infty + \varepsilon/4$ so that $\mathcal{F}$ is bounded in norm (and clearly pointwise bounded).

Now we show the equicontinuity of the family $\mathcal{F}$. By a Lemma above each f_i is uniformly continuous. Thus for each i there exists a $\delta_i > 0$ such that $|f_i(x) - f_i(x')| < \varepsilon/2$ whenever $d_K(x, x') < \delta_i$. Let $\delta = \min\{\delta_1, \dots, \delta_N\}$. Then $\delta > 0$ and we have $|f_i(x) - f_i(x')| < \varepsilon/2$ for every i whenever $d_K(x, x') < \delta$.

Now pick any $f \in \mathcal{F}$, and let i be so that $\|f_i - f\|_\infty < \varepsilon/4$, and let x, x' be so that $d_K(x, x') < \delta$. Then

$$\begin{aligned} |f(x) - f(x')| &\leq |f(x) - f_i(x)| + |f_i(x) - f_i(x')| + |f_i(x') - f(x')| \\ &\leq 2\|f - f_i\|_\infty + |f_i(x) - f_i(x')| < 2\varepsilon/4 + \varepsilon/2 = \varepsilon\,. \qquad \square \end{aligned}$$

[1] Some authors also use the terminology “uniformly equicontinous”.

Proof of sufficiency: If $\mathcal{F} \subset C(K)$ is equicontinuous and pointwise bounded then $\mathcal{F}$ is totally bounded.

Fix $\varepsilon > 0$. We shall first find a finite collection $\mathcal{G}$ of functions in $\mathcal{B}(K)$ so that for every $f \in \mathcal{F}$ there exists a $g \in \mathcal{G}$ with $\|f - g\|_\infty < \varepsilon$.

Let $\delta > 0$ so that for all $f \in \mathcal{F}$ we have $|f(x) - f(x')| < \varepsilon/4$ whenever $|x - x'| < \delta$. Again we use the compactness of K and cover K with finitely many balls $B(x_i, \delta)$, $i = 1, \dots, L$. There is M_i so that $|f(x_i)| \le M_i$ for all $f \in \mathcal{F}$. Let $M = 1 + \max_{i=1,\dots,L} M_i$.

We now let $A_1 = B(x_1, \delta)$, and $A_i = B(x_i, \delta) \setminus \cup_{\nu=1}^{i-1} B(x_\nu, \delta)$, for $2 \le i \le L$. (Some of the A_i could be empty but that does not matter).

Let $\mathcal{Z}^L(M, \varepsilon)$ be the set of L-tuples $\vec{n}$ of integers $\vec{n} = (n_1, \dots, n_L)$ with the property that $|n_i|\varepsilon/4 \le M$ for $i = 1, \dots, L$. Note that $\mathcal{Z}^L(M, \varepsilon)$ is a finite set (indeed its cardinality is $\le (8M\varepsilon^{-1} + 1)^L$).

We now define a collection $\mathcal{G}$ of functions which are constant on the sets A_i (these are analogues of step functions). Namely given $\vec{n}$ in $\mathcal{Z}^L(M, \varepsilon)$ we let $g^{\vec{n}}$ be the unique function that takes the value $n_i\varepsilon/4$ on the set A_i (provided that that set is nonempty). Clearly the cardinality of $\mathcal{G}$ is not larger than the cardinality of $\mathcal{Z}^L(M, \varepsilon)$.

Let $f \in \mathcal{F}$. Consider an A_i which by construction is a subset of $B(x_i, \delta)$. Then $|f(x) - f(x_i)| < \varepsilon/4$ for all $x \in A_i$ (this condition is vacuous if A_i is empty). Now $|f(x_i)| \le M_i \le M$ and therefore there exists an integer n_i with the property that $-M \le n_i\varepsilon/4 \le M$ and $|f(x_i) - n_i\varepsilon/4| < \varepsilon/4$. Then we also have that for $i = 1, \dots, L$ and for every $x \in A_i$,

$$|f(x) - n_i\varepsilon/4| \le |f(x) - f(x_i)| + |f(x_i) - n_i\varepsilon/4| < \varepsilon/4 + \varepsilon/4 = \varepsilon/2.$$

This implies that for this choice of $\vec{n} = (n_1, \dots, n_L)$ we get $\|f - g^{\vec{n}}\|_\infty < \varepsilon/2$.

Finally, we need to find a finite cover of $\mathcal{F}$ with ε-balls centered at points in $\mathcal{F}$. Consider the subcollection $\widetilde{\mathcal{G}}$ of functions in $\mathcal{G}$ for which the ball of radius $\varepsilon/2$ centered at g contains a function in $\mathcal{F}$. Denote the functions in $\widetilde{\mathcal{G}}$ by $g_1, \dots, g_N$. The balls of radius $\varepsilon/2$ centered at $g_1, \dots g_N$ cover $\mathcal{F}$. For $i = 1, \dots, N$ pick $f_i \in \mathcal{F}$ so that $\|g_i - f_i\|_\infty < \varepsilon/2$. By the triangle inequality (for the norm in $\mathcal{B}(K)$ whose restriction to $C(K)$ is also the norm in $C(K)$) the ball of radius $\varepsilon/2$ centered at g_i is contained in the ball of radius ε centered at f_i. Thus the balls of radius ε centered at f_i, $i = 1, \dots, N$ cover the set $\mathcal{F}$. □

Corollary. *A closed subset $\mathcal{F}$ of $C(K)$ is compact if and only if it is pointwise bounded and equicontinuous.*

Proof. The space $\mathcal{B}(K)$ is complete and so is the closed subspace $C(K)$. Since we now assume that $\mathcal{F}$ is closed in $C(K)$ the metric space $\mathcal{F}$ is complete. Thus by the characterization of compactness ($\mathcal{F}$ compact $\iff$ $\mathcal{F}$ totally bounded and complete) the corollary follows from the theorem. □

Corollary. *An equicontinuous and bounded sequence $\{f_n\}$ of functions in $C(K)$ has a uniformly convergent subsequence.*

Proof. The closure of $\mathcal{F} := \{f_n : n \in \mathbb{N}\}$ is bounded, complete, and equicontinuous, thus compact. By a part of the theorem on the characterization of compactness it is also sequentially compact, therefore f_n has a convergent subsequence. $\square$