

Notes about $C(K)$, Arzelà-Ascoli and applications to differential equations

I. The space $C(K)$

Let K be a compact metric space, with metric d_K . Let $\mathcal{B}(K)$ be the space of real valued bounded functions on K with the sup-norm

$$\|f\|_\infty = \sup_{x \in K} |f(x)|$$

Proposition : $\mathcal{B}(K)$ is complete.

Proof. Let f_n be a Cauchy sequence. By the definition of $\|\cdot\|_\infty$ this implies that for every $x \in K$ the numerical sequence $f_n(x)$ is a Cauchy sequence in $\mathbb{R}$, and thus convergent in $\mathbb{R}$. Define $f(x) = \lim_{n \rightarrow \infty} f_n(x)$.

We show that f is bounded, and that $\|f_n - f\|_\infty \rightarrow 0$. Let $\varepsilon > 0$. Then there is $N = N(\varepsilon)$ so that $\sup_x |f_n(x) - f_m(x)| < \varepsilon/2$ for $n, m \geq N$. This shows that $|f_n(x) - f(x)| \leq \varepsilon/2$ for $n, m \geq N$ and for all $x \in K$. In particular $|f(x)| \leq \sup |f_N(x)| + \varepsilon$ so that f is bounded; moreover $\|f_n - f\|_\infty \leq \varepsilon/2$ for $n \geq N$.

As $\varepsilon > 0$ was arbitrary this establishes that $\|f_n - f\|_\infty \rightarrow 0$ as $n \rightarrow \infty$. $\square$

Let $C(K)$ be the vector space of real valued *continuous* functions on K .

Proposition. Every $f \in C(K)$ is bounded and uniformly continuous.

Proof. Let $\varepsilon > 0$. We have to show that there is $\delta > 0$ so that $|f(x) - f(x')| < \varepsilon$ whenever $d_K(x, x') < \delta$; moreover there is M so that $\sup_{x \in K} |f(x)| \leq M$.

As f is continuous, for every x there is a $\delta_x > 0$ so that for all t with $d_K(t, x) < \delta_x$ we have $|f(x) - f(t)| < \varepsilon/2$. As K is compact there is a finite number of points $x_i, i = 1, \dots, L$ so that the balls $B(x_i, \delta_{x_i}/2)$ cover K .

Define $\delta = \min_{i=1, \dots, L} \delta_{x_i}/2$. Suppose $d_K(x, x') < \delta$. Then there is an x_i so that $d_K(x, x_i) < \delta_{x_i}/2$. Then also

$$d_K(x', x_i) \leq d_K(x', x) + d_K(x, x_i) < \delta + \delta_{x_i}/2 \leq \delta_{x_i}$$

so that

$$|f(x) - f(x')| \leq |f(x) - f(x_i)| + |f(x_i) - f(x')| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This already shows that f is uniformly continuous.

By the same argument we have $|f(x)| \leq |f(x) - f(x_i)| + |f(x_i)| \leq \varepsilon + |f(x_i)|$ for $x \in B(x_i, \delta_{x_i}/2)$. Thus, for all $x \in K$ it follows that $|f(x)| \leq \max_{i=1, \dots, L} |f(x_i)| + \varepsilon$ so that certainly f is bounded. $\square$

We have just seen that $C(K)$ is a subspace of $\mathcal{B}(K)$. We consider $C(K)$ as a normed space with the norm (and therefore with the metric) inherited from $\mathcal{B}(K)$. The following result shows that $C(K)$ is complete.

Proposition. $C(K)$ is a closed subspace of $\mathcal{B}(K)$.

Proof. Let f_n be a sequence of functions in $C(K)$ which converges to f (i.e. $\|f_n - f\|_\infty \rightarrow 0$). We will show that f is (uniformly) continuous, i.e. given $\varepsilon > 0$ there is $\delta > 0$ so that $|f(x) - f(x')| < \varepsilon$ if $d_K(x, x') < \delta$.

Let $\varepsilon > 0$. Let N be so that $\|f_n - f\|_\infty < \varepsilon/4$ for all $n \geq N$, that means $|f_n(t) - f(t)| < \varepsilon/4$ for all $t \in K$ and all $n \geq N$. Since f_N is (uniformly) continuous there is $\delta > 0$ so that $|f_N(x) - f_N(x')| < \varepsilon/2$ whenever $d_K(x, x') < \delta$. For those x, x' we also have

$$\begin{aligned} |f(x) - f(x')| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x')| + |f_N(x') - f(x')| \\ &\leq 2\|f - f_N\|_\infty + |f_N(x) - f_N(x')| < 2\varepsilon/4 + \varepsilon/2 = \varepsilon. \quad \square \end{aligned}$$

II. The Arzelà-Ascoli theorem

Definition. Let $\mathcal{F}$ be a collection of functions in $C(K)$.

(i) The collection $\mathcal{F}$ is said to be *pointwise bounded* if for every $x \in K$ there is an $M(x) \geq 0$ so that $|f(x)| \leq M(x)$ for every $f \in \mathcal{F}$.

(ii) The collection $\mathcal{F}$ is said to be *equicontinuous*¹ if for every $\varepsilon > 0$ there exists $\delta > 0$ so that for all $f \in \mathcal{F}$ and x, x' with $d_K(x, x') < \delta$ we have $|f(x) - f(x')| < \varepsilon$.

Remark: It is not difficult to show that a collection $\mathcal{F}$ which is *both equicontinuous and pointwise bounded on a compact set K* is also *uniformly bounded on K* , meaning that there is an $M \geq 0$ so that $\|f\|_\infty = \sup_{x \in K} |f(x)| \leq M$ for all $f \in \mathcal{F}$. This implication is hidden in the proof of the theorem below where a stronger property is proved (namely that $\mathcal{F}$ is totally bounded).

Theorem. *A subset $\mathcal{F}$ of $C(K)$ is totally bounded if and only if it is pointwise bounded and equicontinuous.*

Proof of necessity: If $\mathcal{F}$ is a totally bounded subset of $C(K)$ then $\mathcal{F}$ is pointwise bounded and equicontinuous.

By the definition of $\mathcal{F}$ totally bounded there are functions $f_1, \dots, f_N$ in $\mathcal{F}$ so that for every $f \in \mathcal{F}$ there is an index $i \in \{1, \dots, N\}$ with $\|f_i - f\|_\infty < \varepsilon/4$. Clearly $\|f\| \leq \|f_i\|_\infty + \|f - f_i\|_\infty \leq \max_{i=1, \dots, N} \|f_i\|_\infty + \varepsilon/4$ so that $\mathcal{F}$ is bounded in norm (and clearly pointwise bounded).

Now we show the equicontinuity of the family $\mathcal{F}$. By a Lemma above each f_i is uniformly continuous. Thus for each i there exists a $\delta_i > 0$ such that $|f_i(x) - f_i(x')| < \varepsilon/2$ whenever $d_K(x, x') < \delta_i$. Let $\delta = \min\{\delta_1, \dots, \delta_N\}$. Then $\delta > 0$ and we have $|f_i(x) - f_i(x')| < \varepsilon/2$ for every i whenever $d_K(x, x') < \delta$.

¹Some authors also use the terminology “uniformly equicontinuous”.

Now pick any $f \in \mathcal{F}$, and let i be so that $\|f_i - f\|_\infty < \varepsilon/4$, and let x, x' be so that $d_K(x, x') < \delta$. Then

$$\begin{aligned} |f(x) - f(x')| &\leq |f(x) - f_i(x)| + |f_i(x) - f_i(x')| + |f_i(x') - f(x')| \\ &\leq 2\|f - f_i\|_\infty + |f_i(x) - f_i(x')| < 2\varepsilon/4 + \varepsilon/2 = \varepsilon. \end{aligned} \quad \square$$

Proof of sufficiency: If $\mathcal{F} \subset C(K)$ is equicontinuous and pointwise bounded then $\mathcal{F}$ is totally bounded.

Fix $\varepsilon > 0$. We shall first find a finite collection $\mathcal{G}$ of functions in $\mathcal{B}(K)$ so that for every $f \in \mathcal{F}$ there exists a $g \in \mathcal{G}$ with $\|f - g\|_\infty < \varepsilon$.

Let $\delta > 0$ so that for all $f \in \mathcal{F}$ we have $|f(x) - f(x')| < \varepsilon/4$ whenever $|x - x'| < \delta$. Again we use the compactness of K and cover K with finitely many balls $B(x_i, \delta)$, $i = 1, \dots, L$. There is M_i so that $|f(x_i)| \leq M_i$ for all $f \in \mathcal{F}$. Let $M = 1 + \max_{i=1, \dots, L} M_i$.

We now let $A_1 = B(x_1, \delta)$, and $A_i = B(x_i, \delta) \setminus \bigcup_{\nu=1}^{i-1} B(x_\nu, \delta)$, for $2 \leq i \leq L$. (Some of the A_i could be empty but that does not matter).

Let $\mathcal{Z}^L(M, \varepsilon)$ be the set of L -tuples $\vec{n}$ of integers $\vec{n} = (n_1, \dots, n_L)$ with the property that $|n_i|\varepsilon/4 \leq M$ for $i = 1, \dots, L$. Note that $\mathcal{Z}^L(M, \varepsilon)$ is a finite set (indeed its cardinality is $\leq (8M\varepsilon^{-1} + 1)^L$).

We now define a collection $\mathcal{G}$ of functions which are constant on the sets A_i (these are analogues of step functions). Namely given $\vec{n}$ in $\mathcal{Z}^L(M, \varepsilon)$ we let $g^{\vec{n}}$ be the unique function that takes the value $n_i\varepsilon/4$ on the set A_i (provided that that set is nonempty). Clearly the cardinality of $\mathcal{G}$ is not larger than the cardinality of $\mathcal{Z}^L(M, \varepsilon)$.

Let $f \in \mathcal{F}$. Consider an A_i which by construction is a subset of $B(x_i, \delta)$. Then $|f(x) - f(x_i)| < \varepsilon/4$ for all $x \in A_i$ (this condition is vacuous if A_i is empty). Now $|f(x_i)| \leq M_i \leq M$ and therefore there exists an integer n_i with the property that $-M \leq n_i\varepsilon/4 \leq M$ and $|f(x_i) - n_i\varepsilon/4| < \varepsilon/4$. Then we also have that for $i = 1, \dots, L$ and for every $x \in A_i$,

$$|f(x) - n_i\varepsilon/4| \leq |f(x) - f(x_i)| + |f(x_i) - n_i\varepsilon/4| < \varepsilon/4 + \varepsilon/4 = \varepsilon/2.$$

This implies that for this choice of $\vec{n} = (n_1, \dots, n_L)$ we get $\|f - g^{\vec{n}}\|_\infty < \varepsilon/2$.

Finally, we need to find a finite cover of $\mathcal{F}$ with ε -balls centered at points in $\mathcal{F}$. Consider the subcollection $\tilde{\mathcal{G}}$ of functions in $\mathcal{G}$ for which the ball of radius $\varepsilon/2$ centered at g contains a function in $\mathcal{F}$. Denote the functions in $\tilde{\mathcal{G}}$ by $g_1, \dots, g_N$. The balls of radius $\varepsilon/2$ centered at $g_1, \dots, g_N$ cover $\mathcal{F}$. For $i = 1, \dots, N$ pick $f_i \in \mathcal{F}$ so that $\|g_i - f_i\|_\infty < \varepsilon/2$. By the triangle inequality (for the norm in $\mathcal{B}(K)$ whose restriction to $C(K)$ is also the norm in $C(K)$) the ball of radius $\varepsilon/2$ centered at g_i is contained in the ball of radius ε centered at f_i . Thus the balls of radius ε centered at f_i , $i = 1, \dots, N$ cover the set $\mathcal{F}$. $\square$

Corollary. *A closed subset $\mathcal{F}$ of $C(K)$ is compact if and only if it is pointwise bounded and equicontinuous.*

Proof. The space $\mathcal{B}(K)$ is complete and so is the closed subspace $C(K)$. Since we now assume that $\mathcal{F}$ is closed in $C(K)$ the metric space $\mathcal{F}$ is complete. Thus by the characterization of compactness ($\mathcal{F}$ compact $\iff \mathcal{F}$ totally bounded and complete) the corollary follows from the theorem. $\square$

Corollary. *An equicontinuous and bounded sequence $\{f_n\}$ of functions in $C(K)$ has a uniformly convergent subsequence.*

Proof. The closure of $\mathcal{F} := \{f_n : n \in \mathbb{N}\}$ is bounded, complete, and equicontinuous, thus compact. By a part of the theorem on the characterization of compactness it is also sequentially compact, therefore f_n has a convergent subsequence. $\square$

III. The Peano Existence theorem for differential equations

We are concerned with the initial value problem for a differential equation

$$y'(t) = F(t, y(t)), \quad y(t_0) = y_0.$$

Here $(t, y) \mapsto F(t, y)$ is a continuous function of two variables defined near a point $(t_0, y_0) \in \mathbb{R}^2$ and we wish to find a function $t \mapsto y(t)$ for which the derivative $y'(t)$ equals the value of $F(t, y)$ for $y = y(t)$, and which has the value y_0 at the “initial” time t_0 .

This problem has a solution, as stated in the following existence theorem.

Theorem. *Let Ω be a nonempty open set in $\mathbb{R} \times \mathbb{R}$, let $(t_0, y_0) \in \Omega$ and let $F : \Omega \rightarrow \mathbb{R}$ be continuous. Let $\mathcal{R}$ be a compact rectangle of the form*

$$\mathcal{R} = \{(t, y) : |t - t_0| \leq a, \quad |y - y_0| \leq b\}$$

contained in Ω . Suppose that $|F(t, y)| \leq M$ for $(t, y) \in \mathcal{R}$ and let $\tilde{a} = \min\{a, b/M\}$. Then there exists a function $t \mapsto y(t)$ defined on $(t_0 - \tilde{a}, t_0 + \tilde{a})$ which satisfies the initial value problem

$$\begin{aligned} y'(t) &= F(t, y(t)) \quad \text{for } |t - t_0| < \tilde{a} \\ y(t_0) &= y_0. \end{aligned}$$

Proof. ²

The strategy is to prove the existence of a continuous function y on $[t_0 - \tilde{a}, t_0 + \tilde{a}]$ which satisfies the integral equation

$$y(t) = y_0 + \int_{t_0}^t F(s, y(s)) \, ds.$$

²The proof becomes less technical if one makes the more restrictive assumption that $|F(t, y)| \leq M$ for all (t, y_0) with $|t - t_0| \leq a$ and $y \in \mathbb{R}$. We then have $\tilde{a} = a$ and much of the discussion in step 1 is then superfluous. We will first discuss this special case in class.

Once we have found this function we observe that the integrand $F(s, y(s))$ is also a continuous function. Thus the integral represents a differentiable function (by the fundamental theorem of calculus) and its derivative is $F(t, y(t))$. Thus y is also differentiable and we have $y' = F(t, y(t))$ for $t \in (t_0 - \tilde{a}, t_0 + \tilde{a})$, i.e. y is a solution of the differential equation. Also by the last display $y(t_0) = y_0$ and a solution is found.

We write up the proof of the existence of y only for the interval $[t_0, t_0 + \tilde{a}]$ and leave the notational changes for the interval $[t_0 - \tilde{a}, t_0]$ to the reader. We will split the proof into five steps.

1. We shall construct functions with polygonal graphs which are candidates to approximate the solutions.

F is uniformly continuous on the compact set $\mathcal{R}$. Let $\varepsilon > 0$ a small number and let $\delta = \delta(\varepsilon)$ be as in the definition of uniform continuity, i.e. we have

$$|F(t, y) - F(t', y')| < \varepsilon$$

whenever $|(t, y) - (t', y')| \leq \delta$ and $(t, y) \in \mathcal{R}, (t', y') \in \mathcal{R}$.

Let

$$t_0 < t_1 < \cdots < t_N = t_0 + \tilde{a}$$

be a partition of $[t_0, t_0 + \tilde{a}]$ so that $t_{k+1} - t_k < \frac{1}{2} \min\{\delta, \delta/M\}$ for $k = 0, \dots, N-1$.

We now construct a function $Y \equiv Y_\varepsilon$ on $[t_0, \tilde{a}]$; this definition depends on ε, δ and the partition chosen, however keeping this dependence in mind we will omit the subscript ε in steps 1-4 to avoid cluttered notation.

To define $Y \equiv Y_\varepsilon$ we set $Y(t_0) = y_0$. On the first partition interval $[t_0, t_1]$ the graph of Y will be a line with initial point $(t_0, Y(t_0))$ and slope $F(t_0, Y(t_0)) = F(t_0, y_0)$. The value of this function at t_1 is $Y(t_1) = Y(t_0) + F(t_0, Y(t_0))(t_1 - t_0)$. On the interval $(t_1, t_2]$ we wish to define Y as the graph of a line starting at $(t_1, Y(t_1))$ with readjusted slope $F(t_1, Y(t_1))$. In order for this construction to work we need to make sure that $F(t_1, Y(t_1))$ is still well defined, meaning that the point $(t_1, Y(t_1))$ belongs to the rectangle $\mathcal{R}$. For this we have to check $|Y(t_1) - y_0| \leq b$. Indeed we have that $|Y(t_1) - y_0| = |F(t_0, Y(t_0))(t_1 - t_0)| \leq M(t_1 - t_0) \leq M\tilde{a} \leq b$ by definition of $\tilde{a}$. A similar calculation has to be made at every step.

To be rigorous we formulate the following

Claim. For $k = 0, \dots, N$ there are numbers y_k so that

$$|y_k - y_0| \leq M(t_k - t_0) \leq b \text{ and}$$

$$y_k = y_{k-1} + F(t_{k-1}, y_{k-1})(t_k - t_{k-1}).$$

For $k = 0$ the statement is clear. We argue by induction. Above we have just verified this claim for $k = 1$, and in a similar way we do the induction step.

If $k \in \{1, \dots, N-1\}$ we prove the claim for $k+1$, i.e. the existence of y_{k+1} with the required properties, under the induction hypothesis, that $y_1, \dots, y_k$ have been found. Since by the induction hypothesis $|y_k - y_0| \leq M(t_k - t_0)$ which is $\leq b$ the expression $F(t_k, y_k)$ is well defined and thus $y_{k+1} = y_k + F(t_k, y_k)(t_{k+1} - t_k)$ is well defined. To check that $|y_{k+1} - y_0| \leq M(t_{k+1} - t_0)$ we observe that

$$\begin{aligned} |y_{k+1} - y_0| &= |y_{k+1} - y_k + y_k - y_0| \leq |y_{k+1} - y_k| + |y_k - y_0| \\ &= |F(t_k, y_k)|(t_{k+1} - t_k) + |y_k - y_0| \leq M(t_{k+1} - t_k) + |y_k - y_0| \\ &\leq M(t_{k+1} - t_k) + M(t_k - t_0) = M(t_{k+1} - t_0) \end{aligned}$$

where we have in the second to last step used the induction hypothesis. Of course $M(t_{k+1} - t_0) \leq M\tilde{a} \leq b$. The claim follows by induction.

Now that the claim is verified we can define $Y(t)$ on $[t_0, \tilde{a}]$ by $Y(t_k) = y_k$ and

$$Y(t) = y_k + F(t_k, y_k)(t - t_k), \quad t_k < t < t_{k+1}, \quad k = 0, 1, \dots, N-1.$$

Observe that this definition is also valid for $t_k \leq t \leq t_{k+1}$. The function Y is continuous, piecewise linear, and the absolute values of all slopes are bounded by M .

2. *The function $Y \equiv Y_\varepsilon$ constructed in part 1 satisfies the inequality*

$$|Y(t) - Y(t')| \leq M|t - t'| \text{ whenever } t, t' \text{ are both in } [t_0, t_0 + \tilde{a}].$$

Proof: W.l.o.g $t' < t$. If t', t lie in the same partition interval $[t_k, t_{k+1}]$ then this is immediate since

$$|Y(t) - Y(t')| = |F(t_k, y_k)(t - t')| \leq M|t - t'|.$$

If t', t lie in different partition intervals, $t' \in [t_k, t_{k+1}]$, $t \in [t_l, t_{l+1}]$ with $k < l$, then

$$\begin{aligned} |Y(t) - Y(t')| &= \left| Y(t) - Y(t_l) + \sum_{k < \nu < l} Y(t_{\nu+1}) - Y(t_\nu) + Y(t_{k+1}) - Y(t') \right| \\ &\leq |Y(t) - Y(t_l)| + \sum_{k < \nu < l} |Y(t_{\nu+1}) - Y(t_\nu)| + |Y(t_{k+1}) - Y(t')| \\ &= |F(t_l, y_l)|(t - t_l) + \sum_{k < \nu < l} |F(t_\nu, y_\nu)|(t_{\nu+1} - t_\nu) + |F(t_k, y_k)|(t_{k+1} - t') \\ &\leq M(t - t_l) + \sum_{k < \nu < l} M(t_{\nu+1} - t_\nu) + M(t_{k+1} - t') = M(t - t'); \end{aligned}$$

here the middle terms with the sum $\sum_{k < \nu < l}$ are only present when $k+1 < l$. The claim 2 is proved.

3. Note that if we define $g(t) = F(t_{k-1}, Y(t_{k-1}))$ if $t_{k-1} \leq t < t_k$ then g is a step function and Y is differentiable in the open intervals (t_{k-1}, t_k) with derivative $Y'(t) = g(t)$.

Claim: For $t_0 \leq t \leq t_0 + \tilde{a}$ we have

$$Y(t) = y_0 + \int_{t_0}^t g(s) ds$$

and

$$|g(s) - F(s, Y(s))| \leq \varepsilon \text{ if } t_{k-1} < s < t_k.$$

We first verify the first formula for $t = t_k$. Then

$$\begin{aligned} Y(t_k) - y_0 &= Y(t_k) - Y(t_0) = \sum_{\nu=1}^k [Y(t_\nu) - Y(t_{\nu-1})] \\ &= \sum_{\nu=1}^k F(t_{\nu-1}, Y(t_{\nu-1}))(t_\nu - t_{\nu-1}) = \sum_{\nu=1}^k \int_{t_{\nu-1}}^{t_\nu} g(s) ds = \int_{t_0}^{t_k} g(s) ds \end{aligned}$$

Similarly for $t_k < t < t_{k+1}$,

$$Y(t) - Y(t_k) = F(t_k, Y(t_k))(t - t_k) = \int_{t_k}^t g(s) ds.$$

We put the two formulas together and get

$$Y(t) = Y(t_k) + \int_{t_k}^t g(s) ds = y_0 + \int_{t_0}^{t_k} g(s) ds + \int_{t_k}^t g(s) ds = y_0 + \int_{t_0}^t g(s) ds.$$

For the second assertion let $t_{k-1} < s < t_k$ and observe

$$|g(s) - F(s, Y(s))| = |F(t_{k-1}, Y(t_{k-1})) - F(s, Y(s))|$$

and $|Y(t_{k-1}) - Y(s)| \leq M|t_{k-1} - s| \leq M(t_k - t_{k-1}) \leq \delta/2$ (since the maximal width of the partition is $< \delta/2M$). Thus the distance of the points $(t_{k-1}, Y(t_{k-1}))$ and $(s, Y(s))$ is no more than δ . It follows that

$$|F(t_{k-1}, Y(t_{k-1})) - F(s, Y(s))| < \varepsilon.$$

4. Claim: For $t_0 \leq s \leq t_0 + \tilde{a}$,

$$\left| Y(t) - \left(y_0 + \int_{t_0}^t F(s, Y(s)) ds \right) \right| \leq \varepsilon \tilde{a}.$$

This follows from part 3, since the right hand side is

$$\begin{aligned} &\left| y_0 + \int_{t_0}^t g(s) ds - \left(y_0 + \int_{t_0}^t F(s, Y(s)) ds \right) \right| \\ &= \left| \int_{t_0}^t [g(s) - F(s, Y(s))] ds \right| \end{aligned}$$

and this is estimated by

$$\int_{[t_0, t]} |g(s) - F(s, Y(s))| ds \leq \varepsilon(t - t_0) \leq \varepsilon \tilde{a}.$$

5. So far we have only considered a fixed function $Y \equiv Y_\varepsilon$. Now consider a sequence of such functions $Y_{\varepsilon(n)}$ where $\varepsilon(n) \rightarrow 0$ (for example $\varepsilon(n) = 2^{-n}$). These functions satisfy the properties in part 1-4 with the parameter $\varepsilon = \varepsilon(n)$ and with $\delta = \delta(\varepsilon(n))$.

By part 2 the family of functions $Y_{\varepsilon(n)}$ is uniformly bounded and uniformly equicontinuous. Indeed for all n and all $t \in [t_0, t_0 + \tilde{a}]$ we have shown

$$|Y_{\varepsilon(n)}(t) - Y_{\varepsilon(n)}(t')| \leq M|t - t'|$$

(which implies the uniform equicontinuity) and it follows also that

$$|Y_{\varepsilon(n)}(t)| \leq |Y_{\varepsilon(n)}(t_0)| + |Y_{\varepsilon(n)}(t) - Y_{\varepsilon(n)}(t_0)| \leq |y_0| + M|t - t_0| \leq |y_0| + M\tilde{a}.$$

Thus the Arzelà-Ascoli theorem allows us to choose an increasing sequence of integers m_n such that the subsequence $Y_{\varepsilon(m_n)}(t)$ converges uniformly on $[t_0, t_0 + \tilde{a}]$ to a limit $y(t)$. As a uniform limit of continuous functions y is continuous on $[t_0, t_0 + \tilde{a}]$. Now by the uniform continuity of F and the uniform convergence of $Y_{\varepsilon(m_n)}$ we see³ that $F(t, Y_{\varepsilon(m_n)}(t))$ converges to $F(t, y(t))$, uniformly on $[t_0, t_0 + \tilde{a}]$. This implies that $y_0 + \int_{t_0}^t F(s, Y_{\varepsilon(m_n)}(s)) ds$ converges to $y_0 + \int_{t_0}^t F(s, y(s)) ds$. But by part (4) we also have

$$\left| Y_{\varepsilon(m_n)}(t) - \left(y_0 + \int_{t_0}^t F(s, Y_{\varepsilon(m_n)}(s)) ds \right) \right| \leq \varepsilon(m_n)\tilde{a} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Thus we obtain for the limit (as $m \rightarrow \infty$)

$$y(t) = y_0 + \int_{t_0}^t F(s, y(s)) ds$$

which is the desired integral equation for y . □

IV. Possible failure of uniqueness

The Peano theorem does not guarantee uniqueness.

We give an example. Here $F(t, y) = \sqrt{|y|}$ which is continuous on $\mathbb{R}^2$.

Consider the initial value problem

$$y'(t) = \sqrt{|y(t)|}, \quad y(0) = 0.$$

Verify that the four functions

$$\begin{aligned} Y_1(t) &= 0 & Y_2(t) &= \begin{cases} t^2/4 & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases} \\ Y_3(t) &= \begin{cases} 0 & \text{if } t > 0 \\ -t^2/4 & \text{if } t \leq 0 \end{cases} & Y_4(t) &= \begin{cases} t^2/4 & \text{if } t > 0 \\ -t^2/4 & \text{if } t \leq 0 \end{cases} \end{aligned}$$

are solutions of the initial value problem.

³Provide the details of this argument.

V. An existence and uniqueness theorem

It turns out that if the function F has additional regularity properties then one can prove uniqueness (and an example below shows that some additional hypothesis is needed). This section can be read independently of sections II-III.

The Picard-Lindelöf Theorem. *Let Ω be a nonempty open set in $\mathbb{R} \times \mathbb{R}$, let $(t_0, y_0) \in \Omega$ and let $F : \Omega \rightarrow \mathbb{R}$ be continuous. Let $\mathcal{R}$ be a compact rectangle of the form*

$$\mathcal{R} = \{(t, y) : |t - t_0| \leq a, \quad |y - y_0| \leq b\}$$

contained in Ω . Suppose that $|F(t, y)| \leq M$ for $(t, y) \in \mathcal{R}$ and let

$$\tilde{a} = \min\{a, b/M\}.$$

In addition assume that there is a constant C so that

$$|F(t, y) - F(t, u)| \leq C|y - u| \text{ whenever } (t, y) \in \mathcal{R}, (t, u) \in \mathcal{R}.$$

Then there exists a unique function $t \mapsto y(t)$ defined on $(t_0 - \tilde{a}, t_0 + \tilde{a})$ which satisfies the initial value problem

$$\begin{aligned} y'(t) &= F(t, y(t)) \quad \text{for } |t - t_0| < \tilde{a} \\ y(t_0) &= y_0. \end{aligned}$$

Remark: The additional hypothesis says that on $\mathcal{R}$ the function F satisfies a Lipschitz-condition with respect to the variable y . In the case where F is differentiable with respect to y and the partial derivative $\frac{\partial F}{\partial y}$ satisfies the bound $|\frac{\partial F}{\partial y}| \leq C$ on $\mathcal{R}$, the hypothesis is satisfied. This is a consequence of the mean value theorem applied to $y \mapsto F(t, y)$, for any fixed t .

Equivalence of the initial value problem and an integral equation. Suppose that on an interval I there is a C^1 function y with values in $[y_0 - b, y_0 + b]$ so that $y'(t) = F(t, y(t))$. Then by integrating we also get that

$$y(t) = y_0 + \int_{t_0}^t F(s, y(s)) ds$$

for $t \in I$. Vice versa if there is a continuous function y satisfying the displayed integral equation, then $y(t_0) = y_0$ and the integrand $s \mapsto F(s, y(s))$ is continuous. By the fundamental theorem of calculus the right hand side defines a differentiable function. We differentiate and get $y'(t) = F(t, y(t))$.

Proof of the Picard-Lindelöf Theorem. Let $I := [t_0 - \tilde{a}, t_0 + \tilde{a}]$. Define $\mathcal{M}$ be the subspace of all continuous functions g on I for which $g(t) \in$

$[y_0 - b, y_0 + b]$ for all $t \in I$. Then $\mathcal{M}$ is a closed subspace of $C(I)$, with the usual maximum's metric

$$d(g_1, g_2) = \sup_{t \in I} |g_1(t) - g_2(t)|$$

However we will prefer to work with ⁴

$$d_C(g_1, g_2) = \sup_{x \in I} e^{-2C|t-t_0|} |g_1(t) - g_2(t)|$$

which is an equivalent metric on $C(I)$ in the sense that

$$d_C(g_1, g_2) \leq d(g_1, g_2) \leq e^{2C\tilde{a}} d_C(g_1, g_2)$$

for all g_1, g_2 . It is immediate that $\mathcal{M}$ with the metric d_C is also complete. Define an operator $\mathcal{T}$ acting on functions in $\mathcal{M}$ by

$$\mathcal{T}g(t) = y_0 + \int_{t_0}^t F(s, g(s)) ds$$

We show that $\mathcal{T} : \mathcal{M} \rightarrow \mathcal{M}$ and that the contraction principle can be applied.

Clearly $\mathcal{T}$ is a continuous function, and for $t \in I$ we also have the estimate

$$|\mathcal{T}g(t) - y_0| \leq M|t_0 - t| \leq M\tilde{a}$$

and since we assume $\tilde{a} \leq b/M$ we see that $\mathcal{M}$ is mapped into $\mathcal{M}$.

We need to show the inequality $d_C(\mathcal{T}g_1, \mathcal{T}g_2) \leq \alpha d_C(g_1, g_2)$ for some $\alpha < 1$.

First observe that

$$\mathcal{T}g_2(t) - \mathcal{T}g_1(t) = \int_{t_0}^t [F(s, g_2(s)) - F(s, g_1(s))] ds.$$

By the Lipschitz condition,

$$|\mathcal{T}g_2(t) - \mathcal{T}g_1(t)| \leq \begin{cases} C \int_{t_0}^t |g_2(s) - g_1(s)| ds & \text{if } t \geq t_0 \\ C \int_t^{t_0} |g_2(s) - g_1(s)| ds & \text{if } t \leq t_0 \end{cases}$$

or, with shorter notation,

$$|\mathcal{T}g_2(t) - \mathcal{T}g_1(t)| \leq C \int_{I(t_0, t)} |g_2(s) - g_1(s)| ds$$

where $I(t_0, t)$ denotes the interval $[t_0, t]$ if $t_0 \leq t$ and the interval $[t, t_0]$ if $t \leq t_0$.

⁴One can prove a similar theorem without making this change, working with the usual max-metric; however in the conclusion of the theorem the number $\tilde{a}$ in the theorem would have to be replaced by a smaller number depending on C .

Now for $|t - t_0| \leq \tilde{a}$

$$\begin{aligned}
|\mathcal{T}g_2(t) - \mathcal{T}g_1(t)| &\leq \int_{I(t_0, t)} C|g_2(s) - g_1(s)| ds \\
&= C \int_{I(t_0, t)} e^{2C|s-t_0|} e^{-2C|s-t_0|} |g_2(s) - g_1(s)| ds \\
&\leq C \int_{I(t_0, t)} e^{2C|s-t_0|} d_C(g_1, g_2) ds = C d_C(g_1, g_2) \int_{I(t_0, t)} e^{2C|s-t_0|} ds \\
&= C d_C(g_1, g_2) \frac{e^{2C|t-t_0|} - 1}{2C} \leq \frac{1}{2} e^{2C|t-t_0|} d_C(g_1, g_2).
\end{aligned}$$

This implies

$$e^{-2C|t-t_0|} |\mathcal{T}g_2(t) - \mathcal{T}g_1(t)| \leq \frac{1}{2} d_C(g_1, g_2) \text{ whenever } |t - t_0| \leq \tilde{a}$$

and after taking the sup in t ,

$$d_C(\mathcal{T}g_1, \mathcal{T}g_2) \leq \frac{1}{2} d_C(g_1, g_2).$$

□