Some analysis problems [1]

1. Let f be a continuous function on $\mathbb{R}$ and let for $n = 1, 2, \ldots,$

$$F_n(x) = \int_0^x (x-t)^{n-1} f(t)dt.$$

Prove that F_n is n times differentiable, and prove a simple formula for its n-th derivative.

2. Let

$$f(x,y) = \sum_{n=1}^{\infty} \frac{x}{x^2 + yn^2}, \quad y > 0.$$

(i) Show that for each $y > 0$, the limit

$$g(y) := \lim_{x \to +\infty} f(x,y)$$

exists. Evaluate the limit function g.

(ii) Determine if $f(x,y)$ converges to $g(y)$ uniformly for $y \in (0, \infty)$ as $x \to +\infty$.

3. (i) Find an explicit value of $\epsilon > 0$ such that for every $x \in [0\ ,\ 1]$

$$|\sqrt{x} - \sqrt{x+\epsilon}| \leqslant \frac{1}{200}.$$

(ii) Find an explicit integer N such that there exists a polynomial P of degree at most N such that for every $x \in [0\ ,\ 1]$

$$|\sqrt{x} - P(x)| \leqslant \frac{1}{100}.$$

Hint: Although this may not be clever, you can use the expansion of $\sqrt{x+\epsilon}$ in power series in $(x-1)$.

4. Determine all complex-valued functions f, which are continuous in $[0, 2]$ and satisfy the condition $\int_0^2 f(x)x^n dx = 0$ for $n = 0, 1, 2, 3, \ldots$.

5. Does the series

$$\sum_{n=1}^{\infty} e^{-x/n} \frac{(-1)^n}{n}$$

converge uniformly on $[0, \infty)$?

Prove or disprove.

[1] Mostly from previous qualifying exams, some slightly modified

6. Let $\{a_n\}_{n=1}^{\infty}$ be a numerical sequence and let

$$b_n = \frac{1}{n^6} \sum_{k=1}^{n} k^5 a_k.$$

(i) Does the limit exist when $a_k = a$ is constant?
(ii) Prove or disprove: If a_n converges then b_n converges.
(iii) Prove or disprove: If b_n converges then a_n converges.

7. Given a sequence $\{c_n\}_{n=0}^{\infty}$ of complex numbers, we let $s_n = \sum_{k=0}^{n} c_k$ denote the partial sums and $\sigma_N = \frac{s_0+\ldots+s_N}{N+1}$ their arithmetic means. We say that the series $\sum_{n\geqslant 0} c_n$ is *Cesáro summable* to σ if $\lim_{N\to\infty} \sigma_N = \sigma$.

Show that if $\sum_{n\geqslant 0} c_n$ is Cesáro summable to σ and $\lim_{n\to\infty} nc_n = 0$ then the series $\sum c_n$ converges and $\sum_{n=0}^{\infty} c_n = \sigma$.

8. Let

$$s_N(x) = \sum_{n=1}^{N} (-1)^n \frac{x^{3n}}{n^{2/3}}.$$

Prove that $s_N(x)$ converges to a limit $s(x)$ on $[0,1]$ and that there is a constant C so that for all $N \geqslant 1$ the inequality

$$\sup_{x\in[0,1]} |s_N(x) - s(x)| \leqslant CN^{-2/3}$$

holds.

9. Let $\alpha \in \mathbb{R}$, and let u be the function defined on $(1,\infty)$, by $u(x) = x^{\alpha}$. For which values of α does

$$\frac{u(x+h) - u(x)}{h} \to u'(x) ,$$

uniformly on $(1,\infty)$, as $h \to 0$ $(h \neq 0)$.

10. Let f be a real-valued differentiable function defined on the entire real line. Assume that $\frac{f(x+h)-f(x)}{h} \to f'(x)$, uniformly as $h \to 0$. Show that f' is uniformly continuous. Must f itself be uniformly continuous?

11. Recall the following definition: a function $g : (0,1) \to \mathbb{R}$ has *bounded variation* if

$$\sup_{N \geqslant 2} \sup_{0<x_N<\ldots<x_1<1} |g(x_1) - g(x_2)| + \ldots + |g(x_{N-1}) - g(x_N)| < \infty,$$

where the second sup is taken over all strictly decreasing sequences $x_N < \cdots < x_1$ with $x_i \in (0,1)$.

Find the exponents p for which the function $f : (0,1) \to \mathbb{R}$,

$$f(x) = x^p \sin(1/x)$$

has bounded variation.

12. Let λ_1, λ_2,..., λ_n, ... be real numbers. Argue that

$$f(x) = \sum_{n=1}^{\infty} \frac{e^{i\lambda_n x}}{n^2}$$

defines a continuous bounded function on $\mathbb{R}$ and then show that the limit

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T}^{T} f(x)dx$$

exists.

13. For every $b \in \mathbb{R}$ *prove or disprove* that the improper integral

$$\int_0^\infty x^b \cos(e^x)dx$$

converges.

14. (i) Let $\{f_n\}$ be a sequence of C^1 functions on a compact interval I such that $|f_n(x)| + |f_n'(x)| \leqslant M$ for all $x \in I$. Show that there is a subsequence $\{f_{n_k}\}$ which converges uniformly on I.

(ii) Is the preceding statement still true if we drop the assumption that I is compact? (Proof or counterexample)

(iii) Can one also show that under the assumptions in (i) the sequence f_n has a subsequence whose derivatives converge uniformly? (Proof or counterexample)

15. Let $\sum_{n=1}^\infty a_n$ be the rearrangement of $\sum_{n=1}^\infty \frac{(-1)^n}{n}$ where M positive terms are followed by N negative terms, *i.e.*

$$\sum_{n=1}^{\infty} a_n = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2M} - 1 - \frac{1}{3} - \cdots - \frac{1}{2N-1} + \frac{1}{2(M+1)} + \ldots$$

Compute $\sum_{n=1}^\infty a_n$.

16. (i) Determine for which $\eta \in \mathbb{R}$ the indefinite integral

$$\int_0^\infty t^{i\eta-1} \sin t \, dt$$

exists.

(ii) Determine whether the limit

$$\lim_{\eta \to 0} \int_0^\infty t^{i\eta-1} \sin t \, dt$$

exists.

(iii) Determine whether the limit

$$\lim_{\eta \to \infty} \int_1^\infty t^{i\eta-1} \sin t \, dt$$

exists.

17.
Let $1 \leqslant p < q < \infty$.
Prove or disprove the following statements.
(i) $L^p(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$.
(ii) $L^q(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$.
(iii) $L^q([0,1]) \subset L^p([0,1])$.
(iv) $L^p([0,1]) \subset L^q([0,1])$.
(v) $\ell^p(\mathbb{Z}) \subset \ell^q(\mathbb{Z})$.
(vii) $\ell^q(\mathbb{Z}) \subset \ell^p(\mathbb{Z})$.
(viii) $\ell^p(\mathbb{Z}) \subset \ell^q(\mathbb{Z})$.
(ix) $L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n) \subset L^s(\mathbb{R}^n)$ for all $s \in [p,q]$.

18. Suppose $\alpha \geqslant 0$, and let f be a bounded function on the real line with the property that

$$|f(x+h) - f(x)| \leqslant A|h|^\alpha$$

for all $h \in \mathbb{R}$ and almost all $x \in \mathbb{R}$.

Show that there is a constant C and for each $t > 0$ a C^1-function g_t such that

$$\sup_x |f(x) - g_t(x)| \leqslant Ct^\alpha$$

and

$$\sup_x |g_t'(x)| \leqslant Ct^{\alpha-1}.$$

Hint: Use an approximation of the identity.

19. Let X, Y be normed spaces, and let Ω be an open subset of X. A function $f : \Omega \to Y$ is differentiable at $a \in \Omega$ if there is a bounded linear transformation $L : X \to Y$ so that

$$\lim_{\|h\| \to 0} \frac{\|f(a+h) - f(a) - L(h)\|_Y}{\|h\|_X} = 0.$$

(i) Show that this linear transformation is unique. We call it the derivative of f at a, also denoted by $f'(a)$.

(ii) Let M_n be the set of $n \times n$ matrices. Specify the norm on M_n that you work with. Define $g : M_n \to M_n$ by $g(A) = A^3$. Show that g is differentiable everywhere and determine the derivative $g'(A)$.

(iii) Let $\Omega \subset M_n$ be the subset of invertible matrices. Show that Ω is open Define $f : \Omega \to \Omega$ by $f(A) = A^{-1}$. Show that f is differentiable on Ω and determine $f'(A)$.

(iv) Let M_n be the set of $n \times n$ matrices and define $f : M_n \to \mathbb{R}$ by $f(A) = \det(A)$. Show that f is differentiable everywhere and determine $f'(A)$.

20. Let $b \geqslant 1$. The sequence $\{a_n\}_{n=0}^{\infty}$ satisfies

$$a_{n+1} = \frac{a_n}{2} + \frac{b}{2a_n} \text{ for } n \geqslant 0.$$

(i) Suppose that a_n converges to L. Then determine L.

(ii) Does a_n converge?

21. Let f be a continuous function on $[0, 1]$. Prove that

$$\lim_{p \to \infty} \left(\int_0^1 |f(x)|^p dx \right)^{1/p}$$

exists and identify this limit.

22. For which exponent $p > 0$ is the function

$$f(x, y) = \frac{1}{|x|^p + y^2}$$

integrable in a neighborhood of 0 in $\mathbb{R}^2$?

Hint: you can decompose the domain of integration in regions defined by $2^{-m-1} \leqslant |x|^p + y^2 < 2^{-m}$.

23. Let $a > 0$ and $b > 0$. Prove that there is a unique differentiable function f defined on $(-\infty, \infty)$ satisfying $f(0) = 0$ and

$$f'(x) = a - b|f(x)|^{3/2}$$

for all x. Also show that $\lim_{x \to \infty} f(x)$ exists and determine this limit.

24. Let g be the solution on some interval $[0\ ,\ T)$ to the problem:

$$g(0) = 1$$
$$g'(t) = [g(t)]^2\big(2 + \sin(e^t + g(t))\big)].$$

Show that $T < 1$.

25. Let $f \in L^1(\mathbb{R})$ (meaning that $\int_{-\infty}^{\infty} |f(x)|dx < \infty$). Let

$$G(\lambda) = \int_{\mathbb{R}} e^{i\lambda t^2} f(t)dt.$$

Prove that G is a continuous function and that $\lim_{\lambda\to\infty} G(\lambda) = 0$.

26.
(i) Does $p_N = \prod_{n=2}^{N}(1 + \frac{(-1)^n}{n})$ tend to a nonzero limit as $N \to \infty$?
(ii) Does $q_N = \prod_{n=2}^{N}(1 + \frac{(-1)^n}{\sqrt{n}})$ tend to a nonzero limit as $N \to \infty$?

27. Let a be a decreasing C^1-function in $[0, \infty)$ such that $\lim_{t\to\infty} a(t) = 0$.
(i) Show that $\lim_{N\to\infty} \int_0^N a(t)\sin(tx)dt$ exists for all $x > 0$.
(ii) For $\epsilon > 0$ show that $\lim_{N\to\infty} \int_0^N a(t)\sin(tx)dt$ converges uniformly for $x \in [\epsilon, \infty)$.
(iii) Show that uniform convergence fails in $(0, \infty)$, for a suitable choice of a.

28. For $n \geqslant 0$ let $a_n = [\log(2 + n)]^{-1}$.
(i) For which complex numbers z does the series $\sum_{n=0}^{\infty} a_n\, z^n$ converge?
(ii) For which complex numbers z does the series $\sum_{n=0}^{\infty} a_n\, z^n$ converge absolutely?
(iii) On which compact sets of the complex plane does the series $\sum_{n=0}^{\infty} a_n\, z^n$ converge uniformly?

29. Give an example of a Riemann integrable function $f\colon [0, 1] \to [0, 1]$ which has a dense set of discontinuities. Verify all conclusions.

30. Let

$$s_n(x) = \sum_{k=1}^{n} \sin(kx).$$

Show that there exists a constant C, independent of N, x, such that

$$\sum_{n=1}^{N} \frac{|s_n(x)|}{n^2} < C, \quad 0 < x < \pi, \quad N = 1, 2, 3, \ldots.$$

(*Hint:* Estimate $s_n(x)$ for $n \leqslant \frac{1}{x}$ and for $n > \frac{1}{x}$ separately.)

31. For $\lambda > 1$, define

$$H(\lambda) = \int_0^{+\infty} e^{-\lambda(x^3+x^5)}\ dx\ .$$

Prove that, for some constant $C > 0$,

$$H(\lambda) = C\lambda^{-1/3} + O(\lambda^{-1}).$$

Hint : Evaluate $\int_0^{+\infty} e^{-\lambda x^3}\ dx$ in terms of λ and $\int_0^{+\infty} e^{-x^3}\ dx$. Use the same change of variables, and estimate the difference $\int_0^{+\infty} e^{-\lambda(x^3+x^5)}\ dx - \int_0^{+\infty} e^{-\lambda x^3}\ dx$, dealing separately with 'large' and 'small' values of x.

32. Let $M(n,\mathbb{R})$ be the vector space of $n \times n$ matrices with real entries. Denote by $\|\cdot\|$ a norm on $M(n,\mathbb{R})$. For $A \in M(n,\mathbb{R})$ let $tr(A)$ be the trace of A (that is the sum over all entries on the diagonal). Show that there are neighborhoods U, V of the identity matrix I such that for every $A \in V$ there is a unique $B \in U$ with $n^{-1}tr(B)B^3 = A$.

33. Let $u : \mathbb{R}^3 \to \mathbb{R}$ denote a smooth function and let $\Delta u = \partial_x^2 u + \partial_y^2 u + \partial_z^2 u$ be the Laplacian of u.

Suppose that $\Delta u = 1$ on $\mathbb{R}^3$ and $u(x,y,z) = x^3y^3$ on the sphere of radius R centered at the origin. Find $u(0,0,0)$.

34. Calculate

$$\oint_{\mathcal{C}} \frac{-y^3dx + xy^2dy}{(x^2+y^2)^2}$$

where $\mathcal{C}$ is the plane curve given by the equation $10x^{12} + 22y^8 = 240$, with the positive orientation.

35. Let $\mathcal{D} \subset \mathbb{R}^d$, $d \geqslant 2$ be a compact convex set with smooth boundary $\partial\mathcal{D}$ so that the origin belongs to the interior of $\mathcal{D}$. For every $y \in \partial\mathcal{D}$ let $\alpha(x) \in [0,\pi)$ be the angle between the position vector x and the outer normal vector $\mathfrak{n}(x)$. Let ω_d be the surface area of the unit sphere in $\mathbb{R}^d$. Compute

$$\frac{1}{\omega_d}\int_{\partial\mathcal{D}} \frac{\cos(\alpha(x))}{|x|^{d-1}} d\sigma(x)$$

where $d\sigma$ denotes surface measure on $\partial\mathcal{D}$.

Does (a reasonable interpretation of) your result hold true if $d = 1$?

36. Let f be a function in $C^1(\mathbb{R})$ with compact support and let $b > 0$. Show that the limit

$$A_b(x) = \lim_{\varepsilon\to 0+} \int_{\mathbb{R}\setminus[-\epsilon, b\epsilon]} \frac{f(x-y)}{y} dy$$

exists for all $x \in \mathbb{R}$.

How do $A_b(x)$ and $A_c(x)$ differ for $b \neq c$?

37. Prove or disprove the following statement:

$$\lim_{\varepsilon\to 0} \iint_{x^2+y^2\geqslant\varepsilon^2} \frac{f(x,y)}{(x+iy)^3}\,dx\,dy$$

exists for every function $f \in C^2(\mathbb{R}^2)$ with compact support.

Hint: For $0 < a < b$, what are the values of

$$\iint_{a^2<x^2+y^2<b^2} \frac{x}{(x+iy)^3}\,dx\,dy \text{ and } \iint_{a^2<x^2+y^2<b^2} \frac{y}{(x+iy)^3}\,dx\,dy?$$

38. Let X be a metric space with metric d.
(i) Define $\rho : X \times X \to \mathbb{R}$ by

$$\rho(x,y) = \frac{d(x,y)}{1+d(x,y)}.$$

Prove that ρ is a metric on X.

(ii) Show that a subset U of X is open with respect to the metric d if and only if it is open with respect to the metric ρ.

39. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with $f(x) \geqslant 0$ for all $x \geqslant 0$. Consider the two statements

$$\int_0^\infty f(x)\,dx \quad \text{converges,} \tag{1}$$

$$\sum_{n=0}^{\infty} f(n) \quad \text{converges.} \tag{2}$$

(i) Discuss the truth of the implications (1) $\implies$ (2) and (2) $\implies$ (1).

(ii) Assume f is continuously differentiable and satisfies $|f'(x)| \leqslant A$ for some constant $A < \infty$; again discuss the truth of the implications (1) $\implies$ (2) and (2) $\implies$ (1).

(iii) Finally, assume $|f'(x)| \leqslant A|f(x)|$ for some constant $A < \infty$ and once more discuss the truth of the implications (1) $\implies$ (2) and (2) $\implies$ (1).

40. Assume that $a_k > 0$ and $\sum_{k=0}^{+\infty} a_k = +\infty$. Assume that (b_k) is a bounded sequence. Show that one can choose an increasing sequence of integers $k(n)$ such that

$$\sum_{n=0}^{+\infty} a_{k(n)} = +\infty$$

and the sequence $\big(b_{k(n)}\big)$ has a limit.

41. (i) What is the volume of the region Ω in $\mathbb{R}^n$, defined by

$$\Omega = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_j > 0,\ 0 < x_1 + x_2 + \cdots + x_n < 1\}?$$

(ii) What is the area of the parallelogram spanned by the vectors $(1, 1, -1, 1)$ and $(2, 1, 2, 1)$ in $\mathbb{R}^4$?

(iii) What is the (3 dimensional) volume of the box (parallelepiped) spanned by the vectors $(1, 1, 0, 0, 0)$, $(0, 1, 1, 1, 0)$, and $(0, 0, 1, -1, 1)$ in $\mathbb{R}^5$?

42. (i) Let $\psi \in C_0(\mathbb{R})$ be a compactly supported continuous function. Show that

$$\lim_{N\to\infty} \frac{1}{N} \int_0^\infty \frac{\psi(x/N)}{\sqrt{1+x}}\, dx = 0.$$

Let

$$J_N = \int_0^N e^{i\,x}\sqrt{1+x}\, dx.$$

Does $\lim\limits_{N\to+\infty} J_N$ exist?

(ii) Let $\chi \in C_0^2(\mathbb{R})$. Prove that

$$\lim_{N\to+\infty} \int_0^{+\infty} \chi(\frac{x}{N})\, e^{i\,x}\sqrt{1+x}\, dx$$

exists.

(iii) To what extent does the limit in part (3) depend on the choice of the function χ?

43. Let f be a positive decreasing function defined on $(0, \infty)$. This means that if $0 < a < b < \infty$, then $f(a) \geqslant f(b) > 0$. Let $\epsilon > 0$ be a fixed positive number.

(i) Suppose that for all $0 < x < \infty$, $f(2x) \leqslant 2^{-1-\epsilon} f(x)$. Prove that there is a constant C depending only on ϵ so that for $a > 0$,

$$\int_a^\infty f(x)\, dx \leqslant C\, a\, f(a).$$

(ii) Suppose that for all $0 < x < \infty$, $f(x) \leqslant 2^{+1-\epsilon} f(2x)$. Prove that there is a constant C depending only on ϵ so that for $a > 0$,

$$\int_0^a f(x)\, dx \leqslant C\, a\, f(a).$$

(iii) Suppose that for all $0 < x < \infty$, $f(2x) \geqslant 2^{-1} f(x)$. Prove that the improper integral $\displaystyle\int_1^\infty f(x)\, dx$ diverges.

44. For $a, b > 0$, let

$$F(a,b) = \int_{-\infty}^{+\infty} \frac{dx}{x^4 + (x-a)^4 + (x-b)^4}.$$

For which $p > 0$ is it true that

$$\int_0^1 \int_0^1 F(a,b)^p \, da \, db < +\infty?$$

Hint: Do not try to evaluate the integral defining $F(a,b)$ directly. Instead, first suppose $a \leqslant b$ and show that there are positive constants C_1 and C_2 so that

$$C_1 \leqslant b^3 F(a,b) \leqslant C_2.$$

45.
(i) State and prove the Baire category theorem for complete metric spaces.
(ii) Let $\{f_n\}_{n \geqslant 1}$ be a sequence of real valued continuous functions on the interval $[0,1]$, and let E be the set of $x \in [0,1]$ for which $\sup_n |f_n(x)| = \infty$.
Show that E cannot be $[0,1] \cap \mathbb{Q}$.

46. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions on $[0,1]$ and assume that $\sup_n |f_n(x)| < \infty$ for every $x \in [0,1]$. Show that there exists an interval $(a,b) \subset [0,1]$ and an $M \in \mathbb{R}$ so that $|f_n(x)| \leqslant M$ for all $x \in (a,b)$ and all $n = 1, 2, \ldots$.

47. Consider a function $f : \mathbb{R} \to \mathbb{R}$.
(i) Prove: If the second derivative $f''(x_0)$ exists then

$$\lim_{h \to 0} \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} = f''(x_0).$$

(ii) Suppose that

$$\lim_{h \to 0} \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2}$$

exists. Is it true that the second derivative of f exists at $f''(x_0)$?
Give a proof or a counterexample!

48. Let f be a function defined in the interval $[-2, 2]$ satisfying

$$\frac{f(b) - f(a)}{b - a} \leqslant \frac{f(c) - f(b)}{c - b} \leqslant A$$

whenever $-2 \leqslant a < b < c \leqslant 2$ (i.e., f is a convex function).

Show that there is $C \geqslant 0$ such that for $|h| \leqslant 1$

$$\int_{-1}^{1} |f(x + h) + f(x - h) - 2f(x)| dx \leqslant Ch^2.$$

Hint: Show first that if $x_0 < x_1 < \cdots < x_N$ and $x_i - x_{i-1} = h$ then

$$\sum_{1}^{N-1} |f(x_{i+1}) - 2f(x_i) + f(x_{i-1})| \leqslant C'|h|$$

49. Let K be a continuous function on the unit square $Q = [0, 1] \times [0, 1]$ with the property that $|K(x, y)| < 1$ for all $(x, y) \in Q$. Show that there is a continuous function g defined on $[0, 1]$ so that

$$g(x) + \int_0^1 K(x, y)g(y)dy = \frac{e^x}{1 + x^2}, \quad 0 \leqslant x \leqslant 1.$$

50. Prove that there is a unique C^∞ function f defined on $[0, 1]$ which satisfies the integral equation

$$f(x) + \int_0^x \frac{t \cos(tx) f(t)}{1 + f(t)^2} dt = 0$$

for all $x \in [0, 1]$.