Math 222 – Review problems.

1. Determine

$$\lim_{x\to 0}\frac{\sin x}{e^x-1}$$

by using the Taylor expansions of e^x, e^{-x} and $\sin x$. Same for

$$\lim_{x\to 0}\frac{e^x+e^{-x}-2}{\sin^2 x}.$$

Use $\sin x = x + o(x^2)$, $e^x - 1 = x + o(x)$ to see that

$$\lim_{x\to 0}\frac{\sin x}{e^x-1}=1$$

Use that $e^x = 1+x+\frac{x^2}{2}+o(x^2)$, $e^{-x} = 1-x+\frac{x^2}{2}+o(x^2)$ to see that $e^x+e^{-x}-2 = x^2+o(x^2)$. Since $\sin^2 x = (x+o(x))^2$ hence $\sin^2 x = x^2 + o(x)$ we see that $\lim_{x\to 0}\frac{e^x+e^{-x}-2}{\sin^2 x} = 1$.

2. (i) Show that

$$0 < e^x - 1 - x < \frac{1}{100} \text{ for } 0 < x < 1/10\ .$$

(ii) Show that for all $x > 0$

$$0 < \int_0^x \arctan(t)\,dt < \frac{x^2}{2}.$$

Taylor's formula shows that $e^x - 1 - x = e^c x^2/2$ where c is between 0 and $1/10$. Thus e^c is between 1 and $e^{1/10}$. Clearly $e^c x^2/2 > 0$ for $x > 0$ and $e^c x^2/2 \le e^{1/10}\frac{x^2}{200}$.

You can check that $e^{1/10} < 2$. In fact $0 < e^{1/10} - 1 < e/10$ by another Taylor expansion (or mean value theorem), thus $e^{1/10} < 1 + e/10 < 2$.

4. Prove that the Taylor series of $\sin(2x)$ converges to $\sin(2x)$ for all x.

Compute the derivatives of $f(x) = \sin(2x)$ and find that

$$f^{(n)}(x) = \begin{cases} 2^n \sin 2x & \text{if } n = 4k \\ 2^n \cos 2x & \text{if } n = 4k+1 \\ -2^n \sin 2x & \text{if } n = 4k+2 \\ -2^n \cos 2x & \text{if } n = 4k+3 \end{cases}$$

(k any nonnegative integer).

Now $|\sin(2x)| \le 1$, $|\cos(2x)| \le 1$ for all x. Thus $|f^{(n+1)}(x)| \le 2^{n+1}$ for all x and hence the remainder term $R_n(x)$ satisfies $|R_n(x)| \le 2^{n+1}|x|^{n+1}/(n+1)!$ which tends to 0 as $n\to\infty$. See Example 18.11.

5. (i) Find the Taylor series for $\frac{x}{x+4}$ (in terms of powers of x). Explain why it converges for $|x| < 4$.

Write $\frac{x}{x+4} = \frac{x}{4}\frac{1}{1-(-x/4)}$ and use the geometric series representation $\frac{1}{1-(-x/4)} = \sum_{n=0}^{\infty}(-x/4)^n$ which converges for $|x/4| < 1$. Thus $\frac{x}{x+4} = \sum_{n=0}^{\infty}(-1)^n 4^{-n-1}x^{n+1}$ which converges for $|x| < 4$.

(ii) Find the Taylor series of $\frac{1}{(x+4)(x-2)}$ (in terms of powers of x). Explain why it converges for $|x| < 2$.

Write $\frac{1}{(x+4)(x-2)} = \frac{A}{x+4} + \frac{B}{x-2}$ (use partial fractions and determine A and B). Then write

$$\frac{A}{x+4} + \frac{B}{x-2} = \frac{A}{4}\frac{1}{1-\frac{-x}{4}} - \frac{B}{2}\frac{1}{1-\frac{x}{2}}$$

and then use the geometric sum formula.

6. Suppose $f(x) = o(x^2)$ as $x \to 0$. Why is it true that $f(x) = o(x)$ as $x \to 0$?
$f(x) = o(x^2)$ means $\lim_{x\to 0} x^{-2} f(x) = 0$. This implies that $\lim_{x\to 0} x \cdot x^{-2} f(x) = 0$ which is the definition of $f(x) = o(x)$ as $x \to 0$.

7. Each of the following expressions $f(x)$ satisfies $\lim_{x\to 0} f(x) = 0$. Find the *largest* non-negative integer n so that the expression is $o(x^n)$ as $x \to 0$.
(i)

$$\cos x - \cosh x.$$

(ii)

$$\cos(x^2) + \cosh(x^2) - 2.$$

(iii)

$$\int_0^x \frac{\sin t}{t} dt.$$

(iv)

$$\int_0^x \frac{\cos(3t^2) - 1}{t^3} dt.$$

(v)

$$\frac{1}{1-x} - 1.$$

Also, for each expression $f(x)$ above find a number m for which the limit $\lim_{x\to 0} \frac{f(x)}{x^m}$ exists (as a number) and is not equal to 0. Then determine this limit.

For these we use Taylor expansions.
(i) Expand $\cos x - \cosh x = 1 - \frac{x^2}{2} - (1 + \frac{x^2}{2}) + o(x^3) = -x^2 + o(x^3)$.
We get that $\cos x - \cosh x = o(x)$ but not $o(x^2)$, and $\lim_{x\to 0} x^{-2}(\cos x - \cosh x) = -1$.
(ii)

$$\cos(x^2) + \cosh(x^2) - 2.$$

Find that $\cos(t) + \cosh(t) - 2 = \frac{t^4}{12} + o(t^4)$. Thus $\cos(x^2) + \cosh(x^2) - 2 = \frac{x^8}{12} + o(x^8)$. Thus $\lim_{x\to 0} \frac{\cos(x^2)+\cosh(x^2)-2}{x^8} = 1/12$ and $\cos(x^2) + \cosh(x^2) - 2 = o(x^7)$ (but not $o(x^8)$).
(iii)

$$\int_0^x \frac{\sin t}{t} dt.$$

Expand $\sin t = t - \frac{t^3}{6} + o(t^4)$ and thus $\frac{\sin t}{t} = 1 - \frac{t^2}{6} + o(t^3)$. Therefore $\int_0^x \frac{\sin t}{t} dt = x - \frac{x^3}{18} + o(x^4)$. The expression is $o(1)$ but not $o(x)$. Also $\frac{1}{x}\int_0^x \frac{\sin t}{t} dt$ tends to 1 as $x \to 0$.
Additional problem: Answer the same questions for $\int_0^x \frac{\sin t}{t} dt - x$.

(iv)

$$g(x) := \int_0^x \frac{\cos(3t^2) - 1}{t^3} dt.$$

Expand $\cos(3t^2) - 1 = -\frac{1}{2}(3t^2)^2 + \frac{1}{24}(3t^2)^4 + o(t^8)$, thus $\frac{\cos(3t^2)-1}{t^3} = -\frac{9}{2}t + \frac{27}{8}t^4 + o(t^4)$ and then

$$\int_0^x \frac{\cos(3t^2) - 1}{t^3} dt = -\frac{9}{4}x^2 + o(x^4).$$

Thus $g(x) = o(x)$ and not $o(x^2)$. Also $\lim_{x\to 0} g(x)/x^2 = -9/4$.

8. Determine the Taylor polynomial of degree 11 for the following functions (i) $\sin(x^2)$, (ii) $1 + x^2 + e^{3x^2}$, (iii) $\int_0^x (1 + t^2 + e^{3t^2})dt$

Which theorem do you use that supports your calculation? *If $f(x) = P(x) + o(x^n)$ where P is a polynomial of degree $\le n$ then P is the Taylor polynomial $T_n(x)$ for f (expanded in powers of x).*

Answer for (i): Expand $\sin(t) = t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 + o(t^7)$ hence

$$\sin(x^2) = x^2 - \frac{1}{3!}x^6 + \frac{1}{5!}x^{10} + o(x^{13})$$

Thus $x^2 - \frac{1}{3!}x^6 + \frac{1}{5!}x^{10}$ is the Taylor polynomial $T_{11}(x)$.

9. Approximate e^x by its third order Taylor polynomial (in powers of x) to find an approximate value of

$$\int_0^1 \frac{e^x - 1}{x} dx\,.$$

Estimate the error.

$e^x - 1 = x + \frac{x^2}{2} + \frac{x^3}{6} + R_3(x)$ where $R_3(x) = \frac{x^4}{24}e^c$; here c is some number between 0 and 1 depending on x. Note that $|R_3(x)| \le \frac{e}{24}x^4$. Now divide by x and integrate to see that

$$\begin{aligned}\int_0^1 \frac{e^x - 1}{x} dx &= \int_0^1 \Big[1 + \frac{x}{2} + \frac{x^2}{6}\Big] dx + \int_0^1 x^{-1} R_3(x) dx \\ &= 1 + \frac{1}{4} + \frac{1}{18} + \int_0^1 x^{-1} R_3(x) dx\end{aligned}$$

The main term (the approximate value of the integral) is equal to 47/36 and the error term can be estimated as

$$\Big| \int_0^1 x^{-1} R_3(x) dx \Big| \le \frac{e}{24} \int_0^1 x^3\, dx = \frac{e}{96} < \frac{3}{96} = 1/32.$$

10. We expand $3x\cos(x^2)$ in its Taylor series in powers of x. It converges for all x (why?). Write $3x\cos(x^2) = \sum_{n=0}^{\infty} a_n x^n$ and give a formula for a_n. You may have to distinguish several cases.

You get $\cos t = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!}$ and then $3x\cos(x^2) = 3\sum_{k=0}^{\infty} (-1)^k \frac{x^{4k+1}}{(2k)!}$.

Thus $a_{4k+1} = 3(-1)^k \frac{1}{(2k)!}$, and $a_n = 0$ whenever $n - 1$ is not divisible by four. What are a_{36}, a_{53}, a_{101} and a_{102}?

11. (i) Compute the integrals

$$\int_0^x \frac{3}{\sqrt{5+t^2}}\,dt$$

and

$$\int_0^x t\cosh(t)\,dt\,.$$

For the first one use that the derivative of $\sinh^{-1}(x) = \ln(x+\sqrt{x^2+1})$ is equal to $\frac{1}{\sqrt{x^2+1}}$, see the handout on hyperbolic functions. Reduce to this case by a substitution $t = \sqrt{5}\,s$, $dt = \sqrt{5}\,ds$. A calculation shows

$$\int_0^x \frac{3}{\sqrt{5+t^2}}\,dt = 3\int_0^{x/\sqrt{5}} \frac{1}{\sqrt{1+s^2}}\,ds = 3\ln\left(\tfrac{x}{\sqrt{5}} + \sqrt{\tfrac{x^2}{5}+1}\right).$$

For the second one use integration by parts to write $\int_0^x t\cosh(t)\,dt \;= x\sinh x - \int_0^x \sinh(t)dt = x\sinh x - \cosh x + 1$.

(ii) Approximate for small x both integrals by a cubic polynomial and estimate the error (depending on x).

We assume $|x| \le 1$ (the problem just mentions "small x" without further specifications, so we are interested in what happens as $x \to 0$).

Here is one possible procedure for the first part (there are other possible approaches).

A Taylor expansion for $f(s) = (5+s)^{-1/2}$ gives

$$(5+s)^{-1/2} = 5^{-1/2} - \frac{5^{-3/2}}{2}s + \frac{3}{8}5^{-5/2}s^2 + E(s)$$

where the error term satisfies $E(s) = \frac{1}{3!}(-\frac{15}{8})(5+c)^{-7/2}s^3$ with c between 0 and 1. Thus $|E(s)| \le \frac{15}{3!\cdot 8}5^{-7/2}|s|^3$ for $|s| \le |x|$ which gives $|E(s)| \le \frac{1}{400\sqrt{5}}|s|^3$

This implies $(5+t^2)^{-1/2} = 5^{-1/2} - \frac{5^{-3/2}}{2}t^2 + \frac{3}{8}5^{-5/2}t^4 + \widetilde{E}(t)$ with $|\widetilde{E}(t)| \le \frac{1}{400\sqrt{5}}t^6$ for t between 0 and x.

Note: to go on we could now omit the term $\frac{3}{8}5^{-5/2}t^4$ since after integration it becomes *constant* $\times\, x^5$ which is not part of a cubic polynomial. We would then have to incorporate it in the error time (this discussion also shows that we would have gotten away with just expanding $(5+s)^{-1/2}$ without the quadratic term). I however choose to keep $\frac{3}{8}5^{-5/2}t^4$, and compute the integral explicitly since we have already done the work for it. We may then get a more precise bound for the error.

Now

$$\int_0^x \frac{3}{\sqrt{5+t^2}}\,dt = 3\int_0^x \left[5^{-1/2} - \frac{5^{-3/2}}{2}t^2 + \frac{3}{8}5^{-5/2}t^4\right]dt \;+\; E_*(x)$$

where the main term is

$$3\times\left[5^{-1/2}x - \frac{5^{-3/2}}{2}x^3/3 + \frac{3}{8}5^{-5/2}x^5/5\right] = \frac{3}{\sqrt{5}}x - \frac{x^3}{10\sqrt{5}} + \frac{9}{1000\sqrt{5}}x^5$$

and $|E_*(x)| \le \frac{3}{400\sqrt{5}}\left|\int_0^x t^6\,dt\right|$. Thus

$$|E_*(x)| \le \frac{3}{2800\sqrt{5}}|x|^7.$$

Conclusion (for the approximation by a cubic polynomial): We can write

$$\int_0^x \frac{3}{\sqrt{5+t^2}}\,dt = \frac{3}{\sqrt{5}}x - \frac{x^3}{10\sqrt{5}} + \text{Error}(x)$$

where $|\text{Error}(x)| \le \frac{9}{1000\sqrt{5}}|x|^5 + \frac{3}{2800\sqrt{5}}|x|^7$ which is $\le 5\times 10^{-3}|x|^5$.

If we wanted approximation by a fifth order polynomial we would get

$$\int_0^x \frac{3}{\sqrt{5+t^2}}\,dt = \frac{3}{\sqrt{5}}x - \frac{x^3}{10\sqrt{5}} + \frac{9}{1000\sqrt{5}}x^5 + E^*(x)$$

with the estimate of E^* as above.

Now to the second integral. Notice that $t\cosh t = t(1+\frac{t^2}{2}) + t\cosh(c)\frac{t^4}{4!}$ where c is between 0 and x. Then we compute the main term and make an estimate for the error (fill in the details). We get

$$\int_0^x t\cosh(t)\,dt = \int_0^x t(1+\frac{t^2}{2})dt + \text{Error}(x)$$

where $\int_0^x t(1+\frac{t^2}{2})dt = x^2/2 + x^4/8$ and $\text{Error}(x) \le \cosh(1)\frac{|x|^5}{120}$.

12. (i) $(3+i)(4-2i) = 14-2i$,
(ii) $\frac{2+i}{2-i} = \frac{3}{5} + \frac{4}{5}i$,
(iii) $\frac{3+i}{2+5i} = \frac{11}{29} - \frac{13}{29}i$,
(iv) $(\cos\alpha + i\sin\alpha)^3 = \cos(3\alpha) + i\sin(3\alpha)$,
(v) For $(1+i)^5$ we use the polar coordinate form. Note that $|1+i| = \sqrt{2}$ and that $(1+i) = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$. Thus $(1+i)^5 = (\sqrt{2})^5(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4})$. Now $\cos\frac{5\pi}{4} = -\sqrt{2}/2$, $\sin\frac{5\pi}{4} = -\sqrt{2}/2$. This yields $(1+i)^5 = -4-4i$.