MATH 222
SECOND SEMESTER
CALCULUS

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Math 222 – 2nd Semester Calculus
Lecture notes version 1.6(Fall 2009)

This is a self contained set of lecture notes for Math 222. The notes were written by **Sigurd Angenent**, starting from an extensive collection of notes and problems compiled by **Joel Robbin.** A chapter on Taylor's theorem with integral remainder has been included for the current version (Fall 2009, taught by A. Seeger).

The LaTeX files, as well as the XFIG and OCTAVE files which were used to produce these notes are available at the following web site

www.math.wisc.edu/~angenent/Free-Lecture-Notes

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Methods of Integration

1. The indefinite integral

We recall some facts about integration from first semester calculus.

Definition 1.1. *A function $y = F(x)$ is called an* ***antiderivative*** *of another function $y = f(x)$ if $F'(x) = f(x)$ for all x.*

◀ **1.2 Example.** $F_1(x) = x^2$ is an antiderivative of $f(x) = 2x$.

$F_2(x) = x^2 + 2004$ is also an antiderivative of $f(x) = 2x$.

$G(t) = \frac{1}{2}\sin(2t + 1)$ is an antiderivative of $g(t) = \cos(2t + 1)$. ▶

The Fundamental Theorem of Calculus states that if a function $y = f(x)$ is continuous on an interval $a \le x \le b$, then there always exists an antiderivative $F(x)$ of f, and one has

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a). \tag{1}$$

The best way of computing an integral is often to find an antiderivative F of the given function f, and then to use the Fundamental Theorem (1). *How you go about finding an antiderivative F for some given function f is the subject of this chapter.*

The following notation is commonly used for antiderivates:

$$F(x) = \int f(x)\mathrm{d}x. \tag{2}$$

The integral which appears here does not have the integration bounds a and b. It is called an ***indefinite integral***, as opposed to the integral in (1) which is called a ***definite integral***. It's important to distinguish between the two kinds of integrals. Here is a list of differences:

INDEFINITE INTEGRAL	DEFINITE INTEGRAL
$\int f(x)\mathrm{d}x$ is a function of x.	$\int_a^b f(x)\mathrm{d}x$ is a number.
By definition $\int f(x)\mathrm{d}x$ is *any function of x whose derivative is $f(x)$.*	$\int_a^b f(x)\mathrm{d}x$ was defined in terms of Riemann sums and can be interpreted as "area under the graph of $y = f(x)$", at least when $f(x) > 0$.
x is not a dummy variable, for example, $\int 2x\mathrm{d}x = x^2 + C$ and $\int 2t\mathrm{d}t = t^2 + C$ are functions of diffferent variables, so they are not equal.	x is a dummy variable, for example, $\int_0^1 2x\mathrm{d}x = 1$, and $\int_0^1 2t\mathrm{d}t = 1$, so $\int_0^1 2x\mathrm{d}x = \int_0^1 2t\mathrm{d}t$.

2. You can always check the answer

Suppose you want to find an antiderivative of a given function $f(x)$ and after a long and messy computation which you don't really trust you get an "answer", $F(x)$. You can then throw away the dubious computation and differentiate the $F(x)$ you had found. If $F'(x)$ turns out to be equal to $f(x)$, then your $F(x)$ is indeed an antiderivative and your computation isn't important anymore.

◀ **2.1 Example.** Suppose we want to find $\int \ln x \, dx$. My cousin Bruce says it might be $F(x) = x \ln x - x$. Let's see if he's right:

$$\frac{d}{dx}(x \ln x - x) = x \cdot \frac{1}{x} + 1 \cdot \ln x - 1 = \ln x.$$

Who knows how Bruce thought of this[1], but he's right! We now know that $\int \ln x dx = x \ln x - x + C$. ▶

3. About "$+C$"

Let $f(x)$ be a function defined on some interval $a \le x \le b$. If $F(x)$ is an antiderivative of $f(x)$ on this interval, then for any constant C the function $\tilde{F}(x) = F(x) + C$ will also be an antiderivative of $f(x)$. So one given function $f(x)$ has many different antiderivatives, obtained by adding different constants to one given antiderivative.

Theorem 3.1. *If $F_1(x)$ and $F_2(x)$ are antiderivatives of the same function $f(x)$ on some interval $a \le x \le b$, then there is a constant C such that $F_1(x) = F_2(x) + C$.*

Proof. Consider the difference $G(x) = F_1(x) - F_2(x)$. Then $G'(x) = F_1'(x) - F_2'(x) = f(x) - f(x) = 0$, so that $G(x)$ must be constant. Hence $F_1(x) - F_2(x) = C$ for some constant. □

It follows that there is some ambiguity in the notation $\int f(x) \, dx$. Two functions $F_1(x)$ and $F_2(x)$ can both equal $\int f(x) \, dx$ without equaling each other. When this happens, they (F_1 and F_2) differ by a constant. This can sometimes lead to confusing situations, e.g. you can check that

$$\int 2 \sin x \cos x \, dx = \sin^2 x$$

$$\int 2 \sin x \cos x \, dx = -\cos^2 x$$

are both correct. (Just differentiate the two functions $\sin^2 x$ and $-\cos^2 x$!) These two answers look different until you realize that because of the trig identity $\sin^2 x + \cos^2 x = 1$ they really only differ by a constant: $\sin^2 x = -\cos^2 x + 1$.

> To avoid this kind of confusion we will from now on never forget to include the "arbitrary constant $+C$" in our answer when we compute an antiderivative.

[1] He integrated by parts.

4. **Standard Integrals**

Here is a list of the standard derivatives and hence the standard integrals everyone should know.

$$\int f(x)\,\mathrm{d}x = F(x) + C$$

$$\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + C \qquad \text{for all } n \neq -1$$

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C$$

$$\int \sin x\,\mathrm{d}x = -\cos x + C$$

$$\int \cos x\,\mathrm{d}x = \sin x + C$$

$$\int \tan x\,\mathrm{d}x = -\ln|\cos x| + C$$

$$\int \frac{1}{1+x^2}\,\mathrm{d}x = \arctan x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \arcsin x + C \qquad \left(= \frac{\pi}{2} - \arccos x + C\right)$$

$$\int \frac{\mathrm{d}x}{\cos x} = \frac{1}{2}\ln\frac{1+\sin x}{1-\sin x} + C \qquad \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

All of these integrals are familiar from first semester calculus (like Math 221), except for the last one. You can check the last one by differentiation (using $\ln\frac{a}{b} = \ln a - \ln b$ simplifies things a bit).

5. **Method of substitution**

5.1. *Substitution in a definite integral.*

Theorem 5.1. *Let G be differentiable on the interval $[a,b]$ and G' is continuous and let f be function which is continuous on the range of f. Then*

$$\int_a^b f(G(t))G'(t)\,\mathrm{d}t = \int_{G(a)}^{G(b)} f(u)\,\mathrm{d}u. \tag{3}$$

Proof. Let F be an antiderivative for f, thus $F' = f$. By the chain rule we have

$$\frac{\mathrm{d}F(G(t))}{\mathrm{d}t} = F'(G(t))\cdot G'(t) = f(G(t))\cdot G'(t).$$

We can integrate both sides from a to b and then evaluate the integral $\int_a^b \frac{\mathrm{d}F(G(t))}{\mathrm{d}t}\,\mathrm{d}t$ as $F(G(b)) - F(G(a))$, by the fundamental theorem of calculus. But $F(G(b)) - F(G(a))$ is also equal to $\int_{G(a)}^{G(b)} F'(u)\,\mathrm{d}u$, again by the fundamental theorem of calculus. Bus as $F' = f$ we see that

$$\int_{G(a)}^{G(b)} f(u)\,\mathrm{d}u = F(G(b)) - F(G(a)) = \int_a^b f(G(t))G'(t)\,\mathrm{d}t$$

□

◀ **5.2 Example.** As an example we compute the integral

$$\int_2^3 3t^2 \cos(t^3)\,\mathrm{d}t$$

the integral $t^2 \cos(t^3)$ does not appear in the list of standard integrals we know by heart. but the integral can be computed by the substitution formula above. Set $G(t) = t^3$ and $f(u) = \cos u$. Then $G'(t) = 3t^2$ and we may apply the formula and get

$$\int_2^3 \cos(t^3)3t^2\,\mathrm{d}t = \int_{G(2)}^{G(3)} \cos u\,\mathrm{d}u = \sin(G(3)) - \sin G(2)) = \sin 27 - \sin 8.$$

▶

◀ **5.3 Example.** now let's compute

$$\int_0^1 \frac{t}{1+t^2}\,\mathrm{d}t,$$

We would like to set $G(t) = 1 + t^2$ and then $G'(t) = 2t$. There is no factor 2 in the integral, so we "create" it and write

$$\int_0^1 \frac{t}{1+t^2}\,\mathrm{d}t, = \frac{1}{2}\int_0^1 \frac{1}{1+t^2}\,2t\,\mathrm{d}t.$$

Observe that $G(0) = 1$, $G(1) = 2$ and that the integrand on the right hand side can be written as $\frac{1}{1+t^2}\,2t = \frac{1}{G(t)}G'(t)$. Thus we apply our formula to get

$$\int_0^1 \frac{1}{1+t^2}\,2t\,\mathrm{d}t = \int_{G(0)}^{G(1)} \frac{1}{u}du = \Big[\ln G(1) - \ln G(0)\Big] = \Big[\ln 2 - \ln 1\Big] = \ln 2.$$

Therefore (taking into account the factor 1/2 above) we get

$$\int_0^1 \frac{t}{1+t^2}\,\mathrm{d}t \;=\; \frac{\ln 2}{2}.$$

▶

5.2. *The use of substitution for the computation of antiderivatives*

The chain rule says

$$\frac{\mathrm{d}F(G(x))}{\mathrm{d}x} = F'(G(x))\cdot G'(x),$$

so that

$$\int F'(G(x))\cdot G'(x)\,\mathrm{d}x = F(G(x)) + C.$$

◀ **5.4 Example.** Consider the function $f(x) = 2x\sin(x^2+3)$. Again this function does not appear in the list of standard integrals we know by heart. But we do notice[2] that $2x = \frac{\mathrm{d}}{\mathrm{d}x}(x^2+3)$. So let's call $G(x) = x^2 + 3$, and $F(u) = -\cos u$, then

$$F(G(x)) = -\cos(x^2+3)$$

[2] You *will* start noticing things like this after doing several examples.

and

$$\frac{\mathrm{d}F(G(x))}{\mathrm{d}x} = \underbrace{\sin(x^2+3)}_{F'(G(x))} \cdot \underbrace{2x}_{G'(x)} = f(x),$$

so that

$$\int 2x \sin(x^2+3)\,\mathrm{d}x = -\cos(x^2+3) + C.$$

▶

The most transparent way of computing an integral by substitution is by introducing new variables. Thus to do the integral

$$\int f(G(x))G'(x)\,\mathrm{d}x$$

where $f(u) = F'(u)$, we introduce the substitution $u = G(x)$, and agree to write $\mathrm{d}u = \mathrm{d}G(x) = G'(x)\,\mathrm{d}x$. Then we get

$$\int f(G(x))G'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u = F(u) + C.$$

At the end of the integration we must remember that u really stands for $G(x)$, so that

$$\int f(G(x))G'(x)\,\mathrm{d}x = F(u) + C = F(G(x)) + C.$$

◀ **5.5 Example.** We may also use indefinite integrals to compute definite integrals. As an example we return to the integral already computed above, namely

$$\int_0^1 \frac{x}{1+x^2}\,\mathrm{d}x,$$

(observe that x here is a dummy variable and we can write t or another variable for it). We use the substitution $u = G(x) = 1 + x^2$. Since $\mathrm{d}u = 2x\,\mathrm{d}x$, the associated ***indefinite*** integral is

$$\int \underbrace{\frac{1}{1+x^2}}_{\frac{1}{u}} \underbrace{x\,\mathrm{d}x}_{\frac{1}{2}\mathrm{d}u} = \tfrac{1}{2}\int \frac{1}{u}\,\mathrm{d}u.$$

To find the definite integral you must compute the new integration bounds $G(0)$ and $G(1)$ (see equation (3).) If x runs between $x = 0$ and $x = 1$, then $u = G(x) = 1 + x^2$ runs between $u = 1 + 0^2 = 1$ and $u = 1 + 1^2 = 2$, so the definite integral we must compute is

$$\int_0^1 \frac{x}{1+x^2}\,\mathrm{d}x = \tfrac{1}{2}\int_1^2 \frac{1}{u}\,\mathrm{d}u,$$

which is in our list of memorable integrals. So we find

$$\int_0^1 \frac{x}{1+x^2}\,\mathrm{d}x = \tfrac{1}{2}\int_1^2 \frac{1}{u}\,\mathrm{d}u = \tfrac{1}{2}\big[\ln u\big]_1^2 = \tfrac{1}{2}\ln 2.$$

▶

6. The double angle trick

If an integral contains $\sin^2 x$ or $\cos^2 x$, then you can remove the squares by using the double angle formulas from trigonometry.

Recall that

$$\cos^2\alpha - \sin^2\alpha = \cos 2\alpha \quad \text{and} \quad \cos^2\alpha + \sin^2\alpha = 1,$$

Adding these two equations gives

$$\cos^2\alpha = \frac{1}{2}\left(\cos 2\alpha + 1\right)$$

while substracting them gives

$$\sin^2\alpha = \frac{1}{2}\left(1 - \cos 2\alpha\right).$$

◀ **6.1 Example.** The following integral shows up in many contexts, so it is worth knowing:

$$\begin{aligned}\int \cos^2 x\,\mathrm{d}x &= \frac{1}{2}\int(1+\cos 2x)\mathrm{d}x \\ &= \frac{1}{2}\left\{x + \frac{1}{2}\sin 2x\right\} + C \\ &= \frac{x}{2} + \frac{1}{4}\sin 2x + C.\end{aligned}$$

Since $\sin 2x = 2\sin x\cos x$ this result can also be written as

$$\int \cos^2 x\,\mathrm{d}x = \frac{x}{2} + \frac{1}{2}\sin x\cos x + C.$$

▶

If you don't want to memorize the double angle formulas, then you can use "Complex Exponentials" to do these and many similar integrals. However, you will have to wait until we are in §27 where this is explained.

7. Integration by Parts

7.1. *Integration by parts in a definite integral*

Theorem 7.1. *If u and v are differentiable on the interval $[a, b]$, and the derivatives are continuos, then*

$$\int_a^b u(t)v'(t)\,\mathrm{d}t = u(b)v(b) - u(a)v(a) - \int_a^b u'(t)v(t)\,\mathrm{d}t \tag{4}$$

Proof. For the proof we use the product rule $(uv)' = u'v + uv'$ and thus $uv' = (uv)' - u'v$. We integrate both sides of the equation from a to b and by the fundamental theorem of calculus

$$\int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\big(u(t)v(t)\big)\,\mathrm{d}t = u(b)v(b) - u(a)v(a).$$

This yields formula (4). □

Remark: Often the formula (4) is rewritten in the following form:

$$\int_a^b u(t)v'(t)\,\mathrm{d}t = uv\Big|_a^b - \int_a^b u'(t)v(t)\,\mathrm{d}t$$

7.2. *Computing antiderivatives using integration by parts*

Similarly we can apply integration by parts to compute antiderivatives. Again the product rule states

$$\frac{\mathrm{d}}{\mathrm{d}x}(F(x)G(x)) = \frac{\mathrm{d}F(x)}{\mathrm{d}x}G(x) + F(x)\frac{\mathrm{d}G(x)}{\mathrm{d}x}$$

and therefore, after rearranging terms,

$$F(x)\frac{\mathrm{d}G(x)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(F(x)G(x)) - \frac{\mathrm{d}F(x)}{\mathrm{d}x}G(x).$$

This implies the formula for ***integration by parts***

$$\int F(x)\frac{\mathrm{d}G(x)}{\mathrm{d}x}\,\mathrm{d}x = F(x)G(x) - \int \frac{\mathrm{d}F(x)}{\mathrm{d}x}G(x)\,\mathrm{d}x\,(+C).$$

◀ **7.2 Example – Integrating by parts once.**

$$\int \underbrace{x}_{F(x)}\underbrace{e^x}_{G'(x)}\,\mathrm{d}x = \underbrace{x}_{F(x)}\underbrace{e^x}_{G(x)} - \int \underbrace{e^x}_{G(x)}\underbrace{1}_{F'(x)}\,\mathrm{d}x = xe^x - e^x + C.$$

Observe that in this example e^x was easy to integrate, while the factor x becomes an easier function when you differentiate it. This is the usual state of affairs when integration by parts works: differentiating one of the factors ($F(x)$) should simplify the integral, while integrating the other ($G'(x)$) should not complicate things (too much).

Another example: $\sin x = \frac{\mathrm{d}}{\mathrm{d}x}(-\cos x)$ so

$$\int x\sin x\,\mathrm{d}x = x(-\cos x) - \int(-\cos x)\cdot 1\,\mathrm{d}x == -x\cos x + \sin x + C.$$

▶

◀ **7.3 Example – Repeated Integration by Parts.** Sometimes one integration by parts is not enough: since $e^{2x} = \frac{\mathrm{d}}{\mathrm{d}x}(\frac{1}{2}e^{2x})$ one has

$$\begin{aligned}\int \underbrace{x^2}_{F(x)}\underbrace{e^{2x}}_{G'(x)}\,\mathrm{d}x &= x^2\frac{e^{2x}}{2} - \int\frac{e^{2x}}{2}2x\,\mathrm{d}x \\ &= x^2\frac{e^{2x}}{2} - \left\{\frac{e^{2x}}{4}2x - \int\frac{e^{2x}}{4}2\,\mathrm{d}x\right\} \\ &= x^2\frac{e^{2x}}{2} - \left\{\frac{e^{2x}}{4}2x - \frac{e^{2x}}{8}2 + C\right\} \\ &= \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} - C\end{aligned}$$

(Be careful with all the minus signs that appear when you integrate by parts.)

The same procedure will work whenever you have to integrate

$$\int P(x)e^{ax}\,\mathrm{d}x$$

where $P(x)$ is a polynomial, and a is a constant. Each time you integrate by parts, you get this

$$\begin{aligned}\int P(x)e^{ax}\,\mathrm{d}x &= P(x)\frac{e^{ax}}{a} - \int \frac{e^{ax}}{a}P'(x)\,\mathrm{d}x\\ &= \frac{1}{a}P(x)e^{ax} - \frac{1}{a}\int P'(x)e^{ax}\,\mathrm{d}x.\end{aligned}$$

You have replaced the integral $\int P(x)e^{ax}\,\mathrm{d}x$ with the integral $\int P'(x)e^{ax}\,\mathrm{d}x$. This is the same kind of integral, but it is a little easier since the degree of the derivative $P'(x)$ is less than the degree of $P(x)$. ▶

◀ **7.4 Example – My cousin Bruce's computation.** Sometimes the factor $G'(x)$ is "invisible". Here is how you can get the antiderivative of $\ln x$ by integrating by parts:

$$\begin{aligned}\int \ln x\,\mathrm{d}x &= \int \underbrace{\ln x}_{F(x)}\cdot\underbrace{1}_{G'(x)}\,\mathrm{d}x\\ &= \ln x\cdot x - \int \frac{1}{x}\cdot x\,\mathrm{d}x\\ &= x\ln x - \int 1\,\mathrm{d}x\\ &= x\ln x - x + C.\end{aligned}$$

You can do $\int P(x)\ln x\,\mathrm{d}x$ in the same way if $P(x)$ is a polynomial. ▶

7.3. *Estimating definite integrals with cancellation.*

For *large* $R \gg 2$ let us consider the integral

$$I(R) := \int_1^R x^2\cos(x^4)dx.$$

Unfortunately it is not possible to compute this integral explicitly, but in many applications one encounters integral such as this one, and one would like to make statements about the size of the integrals. A very basic and natural question is: How big is $I(R)$ for large R?

Here is a *valid but very ineffective* estimate. We know that the values of the cos functions are between -1 and 1. Thus $-x^2 \le x^2\cos x^4 \le x^2$ and therefore the integral $\int_1^R x^2\cos(x^4)dx$ is at most $\int_1^R x^2dx = (R^3-1)/3$ and not smaller than $-\int_1^R x^2dx = -(R^3-1)/3$. That is not a very useful information . Because of the high oscillation of $\cos(x^4)$ there may be a lot of cancellation in this integral. Question: Does $I(R)$ actually grow as R becomes large? Or is it true that $I(R)$ stays below a fixed number, say, is it true that $|I(R)| \le 100$ for all R?

Integration by parts can be used to understand the behavior for large R much better: First notice that $\sin(x^4)$ is an antiderivative of $4x^3\cos(x^4)$. We like to create this term in the integral and write

$$I(R) = \int_1^R \frac{1}{4x} 4x^3 \cos(x^4)\,\mathrm{d}x.$$

Now integrate by parts to see that

$$\begin{aligned} I(R) &= \frac{\sin(x^4)}{4x}\Big|_1^R - \int_1^R \frac{\mathrm{d}}{\mathrm{d}x}\Big(\frac{1}{4x}\Big)\sin(x^4)\,\mathrm{d}x \\ &= \frac{\sin(R^4)}{4R} - \frac{\sin 1}{4} + \int_1^R \frac{1}{4x^2}\sin(x^4)\,\mathrm{d}x \end{aligned}$$

Now the integral $\int_1^R \frac{1}{4x^2}\sin(x^4)\,\mathrm{d}x$ is a number between $\int_1^R \frac{1}{4x^2}\,\mathrm{d}x$ and $-\int_1^R \frac{1}{4x^2}\,\mathrm{d}x$. Computing these two easy integrals we find that the integral $\int_1^R \frac{1}{4x^2}\sin(x^4)\,\mathrm{d}x$ is between $\frac{1}{4}(1-\frac{1}{R})$ and $-\frac{1}{4}(1-\frac{1}{R})$, that is the absolute value of this integral certainly is strictly less than $1/4$. The absolute value of each of the two boundary terms above (i.e. $|\frac{\sin(R^4)}{4R}|$ and $|\frac{\sin 1}{4}|$) also does not exceed $1/4$. Thus we have found that the integral $I(R)$ is a sum of three terms, the absolute value of each of them is $\le 1/4$ (and two of them are $< 1/4$). This means

$$|I(R)| < \frac{3}{4} \qquad \text{for all } R \ge 1.$$

For *large R* this is *much better* than the bound $(R^3-1)/3$ above. [3]

Remark: This type of application of integration by parts is used frequently in some more advanced problems in mathematics and some applications.

8. Reduction Formulas

Consider the integral

$$I_n = \int x^n e^{ax}\,\mathrm{d}x.$$

Integration by parts gives you

$$\begin{aligned} I_n &= x^n \frac{1}{a} e^{ax} - \int n x^{n-1} \frac{1}{a} e^{ax}\,\mathrm{d}x \\ &= \frac{1}{a} x^n e^{ax} - \frac{n}{a}\int x^{n-1} e^{ax}\,\mathrm{d}x. \end{aligned}$$

We haven't computed the integral, and in fact the integral that we still have to do is of the same kind as the one we started with (integral of $x^{n-1}e^{ax}$ instead of $x^n e^{ax}$). What we have derived is the following ***reduction formula***

$$I_n = \frac{1}{a} x^n e^{ax} - \frac{n}{a} I_{n-1} \qquad (R)$$

which holds for all n.

[3] If we change the question and ask about the behavior of $I(R)$ as $R \to 1$ then the bound $(R^3-1)/3$ is actually a good answer to that changed question.

For $n = 0$ the reduction formula says

$$I_0 = \frac{1}{a}e^{ax}, \text{ i.e. } \int e^{ax}\,dx = \frac{1}{a}e^{ax} + C.$$

When $n \neq 0$ the reduction formula tells us that we have to compute I_{n-1} if we want to find I_n. The point of a reduction formula is that the same formula also applies to I_{n-1}, and I_{n-2}, etc., so that after repeated application of the formula we end up with I_0, i.e., an integral we know.

◀ **8.1 Example.** To compute $\int x^3 e^{ax}\,dx$ we use the reduction formula three times:

$$\begin{aligned} I_3 &= \frac{1}{a}x^3e^{ax} - \frac{3}{a}I_2 \\ &= \frac{1}{a}x^3e^{ax} - \frac{3}{a}\left\{\frac{1}{a}x^2e^{ax} - \frac{2}{a}I_1\right\} \\ &= \frac{1}{a}x^3e^{ax} - \frac{3}{a}\left\{\frac{1}{a}x^2e^{ax} - \frac{2}{a}\left(\frac{1}{a}xe^{ax} - \frac{1}{a}I_0\right)\right\} \end{aligned}$$

Insert the known integral $I_0 = \frac{1}{a}e^{ax} + C$ and simplify the other terms and you get

$$\int x^3e^{ax}\,dx = \frac{1}{a}x^3e^{ax} - \frac{3}{a^2}x^2e^{ax} + \frac{6}{a^3}xe^{ax} - \frac{6}{a^4}e^{ax} + C.$$

▶

◀ **8.2 Reduction formula requiring two partial integrations.** Consider

$$S_n = \int x^n \sin x\,dx.$$

Then for $n \geq 2$ one has

$$\begin{aligned} S_n &= -x^n\cos x + n\int x^{n-1}\cos x\,dx \\ &= -x^n\cos x + nx^{n-1}\sin x - n(n-1)\int x^{n-2}\sin x\,dx. \end{aligned}$$

Thus we find the reduction formula

$$S_n = -x^n\cos x + nx^{n-1}\sin x - n(n-1)S_{n-2}.$$

Each time you use this reduction, the exponent n drops by 2, so in the end you get either S_1 or S_0, depending on whether you started with an odd or even n. ▶

◀ **8.3 A reduction formula where you have to solve for I_n.** We try to compute

$$I_n = \int (\sin x)^n\,dx$$

by a reduction formula. Integrating by parts twice we get

$$\begin{aligned} I_n &= \int (\sin x)^{n-1}\sin x\,dx \\ &= -(\sin x)^{n-1}\cos x - \int(-\cos x)(n-1)(\sin x)^{n-2}\cos x\,dx \\ &= -(\sin x)^{n-1}\cos x + (n-1)\int(\sin x)^{n-2}\cos^2 x\,dx. \end{aligned}$$

We now use $\cos^2 x = 1 - \sin^2 x$, which gives

$$\begin{aligned} I_n &= -(\sin x)^{n-1}\cos x + (n-1)\int \left\{\sin^{n-2} x - \sin^n x\right\} \mathrm{d}x \\ &= -(\sin x)^{n-1}\cos x + (n-1)I_{n-2} - (n-1)I_n. \end{aligned}$$

You can think of this as an equation for I_n, which, when you solve it tells you

$$nI_n = -(\sin x)^{n-1}\cos x + (n-1)I_{n-2}$$

and thus implies

$$I_n = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n} I_{n-2}. \qquad (\mathcal{S})$$

Since we know the integrals

$$I_0 = \int (\sin x)^0 \mathrm{d}x = \int \mathrm{d}x = x + C \text{ and } I_1 = \int \sin x \, \mathrm{d}x = -\cos x + C$$

the reduction formula $(\mathcal{S})$ allows us to calculate I_n for any $n \geq 0$. ▶

◀ **8.4 A reduction formula which will be handy later.** In the next section you will see how the integral of any "rational function" can be transformed into integrals of easier functions, the hardest of which turns out to be

$$I_n = \int \frac{\mathrm{d}x}{(1+x^2)^n}.$$

When $n = 1$ this is a standard integral, namely

$$I_1 = \int \frac{\mathrm{d}x}{1+x^2} = \arctan x + C.$$

When $n > 1$ integration by parts gives you a reduction formula. Here's the computation:

$$\begin{aligned} I_n &= \int (1+x^2)^{-n}\, \mathrm{d}x \\ &= \frac{x}{(1+x^2)^n} - \int x\,(-n)\big(1+x^2\big)^{-n-1} 2x \, \mathrm{d}x \\ &= \frac{x}{(1+x^2)^n} + 2n \int \frac{x^2}{(1+x^2)^{n+1}} \mathrm{d}x \end{aligned}$$

Apply

$$\frac{x^2}{(1+x^2)^{n+1}} = \frac{(1+x^2)-1}{(1+x^2)^{n+1}} = \frac{1}{(1+x^2)^n} - \frac{1}{(1+x^2)^{n+1}}$$

to get

$$\int \frac{x^2}{(1+x^2)^{n+1}} \mathrm{d}x = \int \left\{ \frac{1}{(1+x^2)^n} - \frac{1}{(1+x^2)^{n+1}} \right\} \mathrm{d}x = I_n - I_{n+1}.$$

Our integration by parts therefore told us that

$$I_n = \frac{x}{(1+x^2)^n} + 2n\big(I_n - I_{n+1}\big),$$

which you can solve for I_{n+1}. You find the reduction formula

$$I_{n+1} = \frac{1}{2n}\frac{x}{(1+x^2)^n} + \frac{2n-1}{2n} I_n.$$

As an example of how you can use it, we start with $I_1 = \arctan x + C$, and conclude that

$$\begin{aligned}\int \frac{dx}{(1+x^2)^2} &= I_2 = I_{1+1} \\ &= \frac{1}{2\cdot 1}\frac{x}{(1+x^2)^1} + \frac{2\cdot 1 - 1}{2\cdot 1} I_1 \\ &= \tfrac{1}{2}\frac{x}{1+x^2} + \tfrac{1}{2}\arctan x + C.\end{aligned}$$

Apply the reduction formula again, now with $n = 2$, and you get

$$\begin{aligned}\int \frac{dx}{(1+x^2)^3} &= I_3 = I_{2+1} \\ &= \frac{1}{2\cdot 2}\frac{x}{(1+x^2)^2} + \frac{2\cdot 2 - 1}{2\cdot 2} I_2 \\ &= \tfrac{1}{4}\frac{x}{(1+x^2)^2} + \tfrac{3}{4}\left\{\tfrac{1}{2}\frac{x}{1+x^2} + \tfrac{1}{2}\arctan x\right\} \\ &= \tfrac{1}{4}\frac{x}{(1+x^2)^2} + \tfrac{3}{8}\frac{x}{1+x^2} + \tfrac{3}{8}\arctan x + C.\end{aligned}$$

▶

9. Partial Fraction Expansion

A ***rational function*** is one which is a ratio of polynomials,

$$f(x) = \frac{P(x)}{Q(x)} = \frac{p_n x^n + p_{n-1}x^{n-1} + \cdots + p_1 x + p_0}{q_d x^d + q_{d-1}x^{d-1} + \cdots + q_1 x + q_0}.$$

Such rational functions can always be integrated, and the trick which allows you to do this is called a ***partial fraction expansion.*** The whole procedure consists of several steps which are explained in this section. The procedure itself has nothing to do with integration: it's just a way of rewriting rational functions. It is in fact useful in other situations, such as finding Taylor series (see Part 2 of these notes) and computing "inverse Laplace transforms" (see MATH 319.)

9.1. *Reduce to a proper rational function*

A ***proper rational function*** is a rational function $P(x)/Q(x)$ where the degree of $P(x)$ is strictly less than the degree of $Q(x)$. the method of partial fractions only applies to proper rational functions. Fortunately there's an additional trick for dealing with rational functions that are not proper.

If P/Q isn't proper, i.e. if degree$(P) \geq$ degree(Q), then you divide P by Q, with result

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where $S(x)$ is the quotient, and $R(x)$ is the remainder after division. In practice you would do a long division to find $S(x)$ and $R(x)$.

◀ **9.1 Example.** Consider the rational function

$$f(x) = \frac{x^3 - 2x + 2}{x^2 - 1}.$$

Here the numerator has degree 3 which is more than the degree of the denominator (which is 2). To apply the method of partial fractions we must first do a division with remainder. One has

$$\begin{array}{r|llllll}
 & x & +1 & = S(x) & & & \\ \hline
x^2 - x & x^3 & & -2x & +2 & & \\
 & x^3 & -x^2 & & & & \\
 & & x^2 & -2x & & & \\
 & & x^2 & -x & & & \\
 & & & -x & +2 & = R(x) &
\end{array}$$

so that

$$f(x) = \frac{x^3 - 2x + 2}{x^2 - 1} = x + 1 + \frac{-x + 2}{x^2 - 1}$$

When we integrate we get

$$\begin{aligned}
\int \frac{x^3 - 2x + 2}{x^2 - 1}\,\mathrm{d}x &= \int \left\{ x + 1 + \frac{-x + 2}{x^2 - 1} \right\} \mathrm{d}x \\
&= \frac{x^2}{2} + x + \int \frac{-x + 2}{x^2 - 1}\,\mathrm{d}x.
\end{aligned}$$

The rational function which still have to integrate, namely $\frac{-x+2}{x^2-1}$, is proper, i.e. its numerator has lower degree than its denominator. ▶

9.2. *Partial Fraction Expansion: The Easy Case*

To compute the partial fraction expansion of a proper rational function $P(x)/Q(x)$ you must factor the denominator $Q(x)$. Factoring the denominator is a problem as difficult as finding all of its roots; in Math 222 we shall only do problems where the denominator is already factored into linear and quadratic factors, or where this factorization is easy to find.

In the easiest partial fractions problems, all the roots of $Q(x)$ are real and distinct, so the denominator is factored into distinct linear factors, say

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(x - a_1)(x - a_2)\cdots(x - a_n)}.$$

To integrate this function we find constants $A_1, A_2, \ldots, A_n$ so that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} + \cdots + \frac{A_n}{x - a_n}. \qquad (\#)$$

Then the integral is

$$\int \frac{P(x)}{Q(x)}\,dx = A_1 \ln|x - a_1| + A_2 \ln|x - a_2| + \cdots + A_n \ln|x - a_n| + C.$$

One way to find the coefficients A_i in (#) is called the ***method of equating coefficients***. In this method we multiply both sides of (#) with $Q(x) = (x - a_1) \cdots (x - a_n)$. The result is a polynomial of degree n on both sides. Equating the coefficients of these polynomial gives a system of n linear equations for $A_1, \ldots, A_n$. You get the A_i by solving that system of equations.

Another much faster way to find the coefficients A_i is the ***Heaviside trick***[4]. Multiply equation (#) by $x - a_i$ and then plug in[5] $x = a_i$. On the right you are left with A_i so

$$A_i = \frac{P(x)(x-a_i)}{Q(x)}\bigg|_{x=a_i} = \frac{P(a_i)}{(a_i - a_1)\cdots(a_i - a_{i-1})(a_i - a_{i+1})\cdots(a_i - a_n)}.$$

◀ **9.2 Example continued.** To integrate $\dfrac{-x+2}{x^2-1}$ we factor the denominator,

$$x^2 - 1 = (x-1)(x+1).$$

The partial fraction expansion of $\dfrac{-x+2}{x^2-1}$ then is

$$\frac{-x+2}{x^2-1} = \frac{-x+2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}. \qquad (\dagger)$$

Multiply with $(x-1)(x+1)$ to get

$$-x + 2 = A(x+1) + B(x-1) = (A+B)x + (A-B).$$

The functions of x on the left and right are equal only if the coefficient of x and the constant term are equal. In other words we must have

$$A + B = -1 \text{ and } A - B = 2.$$

These are two linear equations for two unknowns A and B, which we now proceed to solve. Adding both equations gives $2A = 1$, so that $A = \frac{1}{2}$; from the first equation one then finds $B = -1 - A = -\frac{3}{2}$. So

$$\frac{-x+2}{x^2-1} = \frac{1/2}{x-1} - \frac{3/2}{x+1}.$$

Instead, we could also use the Heaviside trick: multiply (†) with $x - 1$ to get

$$\frac{-x+2}{x+1} = A + B\frac{x-1}{x+1}$$

Take the limit $x \to 1$ and you find

$$\frac{-1+2}{1+1} = A, \text{ i.e. } A = \frac{1}{2}.$$

Similarly, after multiplying (†) with $x + 1$ one gets

$$\frac{-x+2}{x-1} = A\frac{x+1}{x-1} + B,$$

---

[4] Named after Oliver Heaviside, a physicist and electrical engineer in the late 19th and early 20ieth century.

[5] More properly, you should take the limit $x \to a_i$. The problem here is that equation (#) has $x - a_i$ in the denominator, so that it does not hold for $x = a_i$. Therefore you cannot set x equal to a_i in any equation derived from (#), but you can take the limit $x \to a_i$, which in practice is just as good.

and letting $x \to -1$ you find

$$B = \frac{-(-1)+2}{(-1)-1} = -\frac{3}{2},$$

as before.

Either way, the integral is now easily found, namely,

$$\begin{aligned}\int \frac{x^3-2x+1}{x^2-1}\,dx &= \frac{x^2}{2} + x + \int \frac{-x+2}{x^2-1}\,dx \\ &= \frac{x^2}{2} + x + \int \left\{ \frac{1/2}{x-1} - \frac{3/2}{x+1} \right\} dx \\ &= \frac{x^2}{2} + x + \frac{1}{2}\ln|x-1| - \frac{3}{2}\ln|x+1| + C.\end{aligned}$$

▶

9.3. *Partial Fraction Expansion: The General Case*

Buckle up.

When the denominator $Q(x)$ contains repeated factors or quadratic factors (or both) the partial fraction decomposition is more complicated. In the most general case the denominator $Q(x)$ can be factored in the form

(5) $$Q(x) = (x-a_1)^{k_1} \cdots (x-a_n)^{k_n} (x^2+b_1x+c_1)^{\ell_1} \cdots (x^2+b_mx+c_m)^{\ell_m}$$

Here we assume that the factors $x-a_1, \ldots, x-a_n$ are all different, and we also assume that the factors $x^2+b_1x+c_1, \ldots, x^2+b_mx+c_m$ are all different.

It is a theorem from advanced algebra that you can always write the rational function $P(x)/Q(x)$ as a sum of terms like this

(6) $$\frac{P(x)}{Q(x)} = \cdots + \frac{A}{(x-a_i)^k} + \cdots + \frac{Bx+C}{(x^2+b_jx+c_j)^\ell} + \cdots$$

How did this sum come about?

For each linear factor $(x-a)^k$ in the denominator (5) you get terms

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_k}{(x-a)^k}$$

in the decomposition. There are as many terms as the exponent of the linear factor that generated them.

For each quadratic factor $(x^2+bx+c)^\ell$ you get terms

$$\frac{B_1x+C_1}{x^2+bx+c} + \frac{B_2x+C_2}{(x^2+bx+c)^2} + \cdots + \frac{B_mx+C_m}{(x^2+bx+c)^\ell}.$$

Again, there are as many terms as the exponent ℓ with which the quadratic factor appears in the denominator (5).

In general, you find the constants $A_{\dots}$, $B_{\dots}$ and $C_{\dots}$ by the method of equating coefficients.

◀ **9.3 Example.** To do the integral

$$\int \frac{x^2+3}{x^2(x+1)(x^2+1)^2}\,dx$$

apply the method of equating coefficients to the form

$$\frac{x^2+3}{x^2(x+1)(x^2+1)^2} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x+1} + \frac{B_1x+C_1}{x^2+1} + \frac{B_2x+C_2}{(x^2+1)^2}. \tag{$\mathcal{EX}$}$$

Solving this last problem will require solving a system of seven linear equations in the seven unknowns $A_1, A_2, A_3, B_1, C_1, B_2, C_2$. A computer program like `Maple` can do this easily, but it is a lot of work to do it by hand. In general, the method of equating coefficients requires solving n linear equations in n unknowns where n is the degree of the denominator $Q(x)$.

See Problem 99 for a worked example where the coefficients are found. ▶

!! Unfortunately, in the presence of quadratic factors or repeated linear factors the Heaviside trick does not give the whole answer; you must use the method of equating coefficients. **!!**

Once you have found the partial fraction decomposition $(\mathcal{EX})$ you still have to integrate the terms which appeared. The first three terms are of the form $\int A(x-a)^{-p}\,dx$ and they are easy to integrate:

$$\int \frac{A\,dx}{x-a} = A\ln|x-a| + C$$

and

$$\int \frac{A\,dx}{(x-a)^p} = \frac{A}{(1-p)(x-a)^{p-1}} + C$$

if $p > 1$. The next, fourth term in $(\mathcal{EX})$ can be written as

$$\begin{aligned}\int \frac{B_1x+C_1}{x^2+1}\,\mathrm{d}x &= B_1\int\frac{x}{x^2+1}\,\mathrm{d}x + C_1\int\frac{\mathrm{d}x}{x^2+1}\\ &= \frac{B_1}{2}\ln(x^2+1) + C_1\arctan x + C_{\text{integration const.}}\end{aligned}$$

While these integrals are already not very simple, the integrals

$$\int \frac{Bx+C}{(x^2+bx+c)^p}\,\mathrm{d}x \qquad \text{with } p>1$$

which can appear are particularly unpleasant. If you really must compute one of these, then complete the square in the denominator so that the integral takes the form

$$\int \frac{Ax+B}{((x+b)^2+a^2)^p}\,dx.$$

After the change of variables $u = x+b$ and factoring out constants you have to do the integrals

$$\int \frac{du}{(u^2+a^2)^p} \qquad \text{and} \qquad \int \frac{u\,du}{(u^2+a^2)^p}.$$

Use the reduction formula we found in example 8.4 to compute this integral.

An alternative approach is to use complex numbers (which are on the menu for this semester.) If you allow complex numbers then the quadratic factors $x^2 + bx + c$ can be factored, and your partial fraction expansion only contains terms of the form $A/(x-a)^p$, although A and a can now be complex numbers. The integrals are then easy, but the answer has complex numbers in it, and rewriting the answer in terms of real numbers again can be quite involved.

10. PROBLEMS

Basic Integrals

The following integrals are straightforward provided you know the list of standard antiderivatives. They can be done without using substitution or any other tricks, and you learned them in first semester calculus.

1. $\int \{6x^5 - 2x^{-4} - 7x + 3/x - 5 + 4e^x + 7^x\}\,dx$

2. $\int (x/a + a/x + x^a + a^x + ax)\,dx$

3. $\int \{\sqrt{x} - \sqrt[3]{x^4} + \frac{7}{\sqrt[3]{x^2}} - 6e^x + 1\}\,dx$

4. $\int \left\{2^x + \left(\tfrac{1}{2}\right)^x\right\}\,dx$

5. $\int_1^4 x^{-2}\,dx$ (hm...)

6. $\int_1^4 t^{-2}\,dt$ **(!)**

7. $\int_1^4 x^{-2}\,dt$ **(!!!)**

8. $\int_{-3}^0 (5y^4 - 6y^2 + 14)\,dy$

9. $\int_1^3 \left(\frac{1}{t^2} - \frac{1}{t^4}\right)\,dt$

10. $\int_1^2 \frac{t^6 - t^2}{t^4}\,dt$

11. $\int_1^2 \frac{x^2+1}{\sqrt{x}}\,dx$

12. $\int_0^2 (x^3 - 1)^2\,dx$

13. $\int_1^2 (x + 1/x)^2\,dx$

14. $\int_3^3 \sqrt{x^5 + 2}\,dx$

15. $\int_1^{-1} (x-1)(3x+2)\,dx$

16. $\int_1^4 (\sqrt{t} - 2/\sqrt{t})\,dt$

17. $\int_1^8 \left(\sqrt[3]{r} + \frac{1}{\sqrt[3]{r}}\right)\,dr$

18. $\int_{-1}^0 (x+1)^3\,dx$

19. $\int_1^e \frac{x^2+x+1}{x}\,dx$

20. $\int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2\,dx$

21. $\int_0^1 \left(\sqrt[4]{x^5} + \sqrt[5]{x^4}\right)\,dx$

22. $\int_1^8 \frac{x-1}{\sqrt[3]{x^2}}\,dx$

23. $\int_{\pi/4}^{\pi/3} \sin t\,dt$

24. $\int_0^{\pi/2} (\cos\theta + 2\sin\theta)\,d\theta$

25. $\int_0^{\pi/2} (\cos\theta + \sin 2\theta)\,d\theta$

26. $\int_{2\pi/3}^{\pi} \frac{\tan x}{\cos x}\,dx$

27. $\int_{\pi/3}^{\pi/2} \frac{\cot x}{\sin x}\,dx$

28. $\int_1^{\sqrt{3}} \frac{6}{1+x^2}\,dx$

29. $\int_0^{0.5} \frac{dx}{\sqrt{1-x^2}}$

30. $\int_4^8 (1/x)\,dx$

31. $\int_{\ln 3}^{\ln 6} 8e^x\, dx$

32. $\int_{8}^{9} 2^t\, dt$

33. $\int_{-e^2}^{-e} \frac{3}{x}\, dx$

34. $\int_{-2}^{3} |x^2 - 1|\, dx$

35. $\int_{-1}^{2} |x - x^2|\, dx$

36. $\int_{-1}^{2} (x - 2|x|)\, dx$

37. $\int_{0}^{2} (x^2 - |x - 1|)\, dx$

38. $\int_{0}^{2} f(x)\, dx$ where

$$f(x) = \begin{cases} x^4 & \text{if } 0 \le x < 1, \\ x^5, & \text{if } 1 \le x \le 2. \end{cases}$$

39. $\int_{-\pi}^{\pi} f(x)\, dx$ where

$$f(x) = \begin{cases} x, & \text{if } -\pi \le x \le 0, \\ \sin x, & \text{if } 0 < x \le \pi. \end{cases}$$

40. Compute

$$I = \int_0^2 2x\big(1 + x^2\big)^3\, dx$$

in two different ways:
(***i***) Expand $(1 + x^2)^3$, multiply with $2x$, and integrate each term.
(***ii***) Use the substitution $u = 1 + x^2$.

41. Compute

$$I_n = \int 2x\big(1 + x^2\big)^n\, dx.$$

42. If $f'(x) = x - 1/x^2$ and $f(1) = 1/2$ find $f(x)$.

Basic Substitutions

Use a substitution to evaluate the following integrals.

43. $\int_1^2 \frac{u\,\mathrm{d}u}{1 + u^2}$

44. $\int_1^2 \frac{x\,\mathrm{d}x}{1 + x^2}$

45. $\int_{\pi/4}^{\pi/3} \sin^2\theta \cos\theta\, \mathrm{d}\theta$

46. $\int_2^3 \frac{1}{r \ln r} \mathrm{d}r$

47. $\int \frac{\sin 2x}{1 + \cos^2 x}\, dx$

48. $\int \frac{\sin 2x}{1 + \sin x}\, dx$

49. $\int_0^1 z\sqrt{1 - z^2}\, \mathrm{d}z$

50. $\int_1^2 \frac{\ln 2x}{x}\, \mathrm{d}x$

51. $\int_{\xi=0}^{\sqrt{2}} \xi(1 + 2\xi^2)^{10}\, \mathrm{d}\xi$

52. $\int_2^3 \sin 2\rho \big(\cos 2\rho\big)^4\, \mathrm{d}\rho$

53. $\int \alpha e^{-\alpha^2}\, \mathrm{d}\alpha$

54. $\int \frac{e^{\frac{1}{t}}}{t^2}\, \mathrm{d}t$

Review of the Inverse Trigonometric Functions

55. The ***inverse sine function*** is the inverse function to the (restricted) sine function, i.e. when $\pi/2 \le \theta \le \pi/2$ we have

$$\theta = \arcsin(y) \iff y = \sin\theta.$$

The inverse sine function is sometimes called ***Arc Sine function*** and denoted $\theta = \arcsin(y)$. We avoid the notation $\sin^{-1}(x)$ which is used by some as it is ambiguous (it could stand for either $\arcsin x$ or for $(\sin x)^{-1} = 1/(\sin x)$).

(***i***) If $y = \sin\theta$, express $\sin\theta$, $\cos\theta$, and $\tan\theta$ in terms of y when $0 \le \theta < \pi/2$.

(***ii***) If $y = \sin\theta$, express $\sin\theta$, $\cos\theta$, and $\tan\theta$ in terms of y when $\pi/2 < \theta \le \pi$.

(***iii***) If $y = \sin\theta$, express $\sin\theta$, $\cos\theta$, and $\tan\theta$ in terms of y when $-\pi/2 < \theta < 0$.

(***iv***) Evaluate $\int \frac{dy}{\sqrt{1-y^2}}$ using the substitution $y = \sin\theta$, but give the final answer in terms of y.

56. Express in simplest form:

$$(\boldsymbol{i})\ \cos(\sin^{-1}(x)); \qquad (\boldsymbol{ii})\ \tan\left\{\arcsin\frac{\ln\frac{1}{4}}{\ln 16}\right\}; \qquad (\boldsymbol{iii})\ \sin\bigl(2\arctan a\bigr)$$

57. Draw the graph of $y = f(x) = \arcsin\bigl(\sin(x)\bigr)$, for $-2\pi \le x \le +2\pi$. Make sure you get the same answer as your graphing calculator.

58. Use the change of variables formula to evaluate $\int_{1/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}}$ first using the substitution $x = \sin u$ and then using the substitution $x = \cos u$.

59. The ***inverse tangent function*** is the inverse function to the (restricted) tangent function, i.e. for $\pi/2 < \theta < \pi/2$ we have

$$\theta = \arctan(w) \iff w = \tan\theta.$$

The inverse tangent function is sometimes called ***Arc Tangent function*** and denoted $\theta = \arctan(y)$. We avoid the notation $\tan^{-1}(x)$ which is used by some as it is ambiguous (it could stand for either $\arctan x$ or for $(\tan x)^{-1} = 1/(\tan x)$).

(***i***) If $w = \tan\theta$, express $\sin\theta$ and $\cos\theta$ in terms of w when

$$(\boldsymbol{ii})\ 0 \le \theta < \pi/2; \qquad (\boldsymbol{iii})\ \pi/2 < \theta \le \pi; \qquad (\boldsymbol{iv})\ -\pi/2 < \theta < 0.$$

(***v***) Evaluate $\int \frac{dw}{1+w^2}$ using the substitution $w = \tan\theta$, but give the final answer in terms of w.

Evaluate these integrals:

60. $\int \frac{dx}{\sqrt{1-x^2}}$

61. $\int \frac{dx}{\sqrt{4-x^2}}$

62. $\int \frac{x\,dx}{\sqrt{1-4x^4}}$

63. $\int_{-1/2}^{1/2} \frac{dx}{\sqrt{4-x^2}}$

64. $\int_{-1}^{1} \frac{dx}{\sqrt{4-x^2}}$

65. $\int_{0}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}}$

66. $\int \frac{dx}{x^2+1}$,

67. $\int \frac{dx}{x^2+a^2}$,

68. $\int \frac{dx}{7+3x^2}$,

69. $\int_{1}^{\sqrt{3}} \frac{dx}{x^2+1}$,

70. $\int_{a}^{a\sqrt{3}} \frac{dx}{x^2+a^2}$.

71. Evaluate $\int x^n \ln x \, dx$ where $n \neq -1$.

72. Evaluate $\int e^{ax} \sin bx \, dx$ where $a^2 + b^2 \neq 0$. [Hint: Integrate by parts twice.]

73. Evaluate $\int e^{ax} \cos bx \, dx$ where $a^2 + b^2 \neq 0$.

74. Prove the formula

$$\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx$$

and use it to evaluate $\int x^2 e^x \, dx$.

75. Prove the formula

$$\int \sin^n x \, dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x \, dx, \qquad n \neq 0$$

76. Evaluate $\int \sin^2 x \, dx$. Show that the answer is the same as the answer you get using the half angle formula.

77. Evaluate $\int_0^{\pi} \sin^{14} x \, dx$.

78. Prove the formula

$$\int \cos^n x \, dx = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} \int \cos^{n-2} x \, dx, \qquad n \neq 0$$

and use it to evaluate $\int_0^{\pi/4} \cos^4 x \, dx$.

79. Prove the formula

$$\int x^m (\ln x)^n \, dx = \frac{x^{m+1} (\ln x)^n}{m+1} - \frac{n}{m+1} \int x^m (\ln x)^{n-1} \, dx, \qquad m \neq -1,$$

and use it to evaluate the following integrals:

80. $\int \ln x \, dx$

81. $\int (\ln x)^2 \, dx$

82. $\int x^3 (\ln x)^2 \, dx$

83. Evaluate $\int x^{-1} \ln x \, dx$ by another method. [Hint: the solution is short!]

84. For an integer $n > 1$ derive the formula

$$\int \tan^n x \, dx = \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x \, dx$$

Using this, find $\int_0^{\pi/4} \tan^5 x \, dx$ by doing just one explicit integration.

Use the reduction formula from example 8.4 to compute these integrals:

85. $\int \frac{dx}{(1+x^2)^3}$

86. $\displaystyle\int \frac{dx}{(1+x^2)^4}$

87. $\displaystyle\int \frac{x dx}{(1+x^2)^4}$ [Hint: $\int x/(1+x^2)^n dx$ is easy.]

88. $\displaystyle\int \frac{1+x}{(1+x^2)^2} dx$

89. The reduction formula from example 8.4 is valid for all $n \neq 0$. In particular, n does not have to be an integer, and it does not have to be positive.

Find a relation between $\displaystyle\int \sqrt{1+x^2}\, dx$ and $\displaystyle\int \frac{dx}{\sqrt{1+x^2}}$ by setting $n = -\frac{1}{2}$.

Integration of Rational Functions

Express each of the following rational functions as a polynomial plus a proper rational function. (See §9.1 for definitions.)

90. $\displaystyle\frac{x^3}{x^3-4}$,

91. $\displaystyle\frac{x^3+2x}{x^3-4}$,

92. $\displaystyle\frac{x^3-x^2-x-5}{x^3-4}$.

93. $\displaystyle\frac{x^3-1}{x^2-1}$.

Completing the square

Write ax^2+bx+c in the form $a(x+p)^2+q$, i.e. find p and q in terms of a, b, and c (this procedure, which you might remember from high school algebra, is called "completing the square."). Then evaluate the integrals

94. $\displaystyle\int \frac{dx}{x^2+6x+8}$,

95. $\displaystyle\int \frac{dx}{x^2+6x+10}$,

96. $\displaystyle\int \frac{dx}{5x^2+20x+25}$.

97. Use the method of equating coefficients to find numbers A, B, C such that

$$\frac{x^2+3}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$$

and then evaluate the integral $\displaystyle\int \frac{x^2+3}{x(x+1)(x-1)} dx$.

98. Do the previous problem using the Heaviside trick.

99. Find the integral $\displaystyle\int \frac{x^2+3}{x^2(x-1)} dx$.

Evaluate the following integrals:

100. $\displaystyle\int_{-5}^{-2} \frac{x^4-1}{x^2+1} dx$

101. $\displaystyle\int \frac{x^3\, dx}{x^4+1}$

102. $\displaystyle\int \frac{x^5\, dx}{x^2-1}$

103. $\displaystyle\int \frac{x^5\, dx}{x^4-1}$

104. $\displaystyle\int \frac{e^{3x}\, dx}{e^{4x}-1}$

105. $\displaystyle\int \frac{e^x\, dx}{\sqrt{1+e^{2x}}}$

106. $\displaystyle\int \frac{e^x\, dx}{e^{2x}+2e^x+2}$

107. $\displaystyle\int \frac{dx}{x(x^2+1)}$

108. $\displaystyle\int \frac{\mathrm{d}x}{x(x^2+1)^2}$

109. $\displaystyle\int \frac{\mathrm{d}x}{x^2(x-1)}$

110. $\displaystyle\int \frac{1}{(x-1)(x-2)(x-3)}\,\mathrm{d}x$

111. $\displaystyle\int \frac{x^2+1}{(x-1)(x-2)(x-3)}\,\mathrm{d}x$

112. $\displaystyle\int \frac{x^3+1}{(x-1)(x-2)(x-3)}\,\mathrm{d}x$

113. (a) Compute $\displaystyle\int_1^2 \frac{\mathrm{d}x}{x(x-h)}$ where h is a small positive number.

(b) What happens to your answer to (*i*) when $h \to 0^+$?

(c) Compute $\displaystyle\int_1^2 \frac{\mathrm{d}x}{x^2}$.

Miscellaneous and Mixed Integrals

114. Find the area of the region bounded by the curves

$$x = 1, \qquad x = 2, \qquad y = \frac{2}{x^2-4x+5}, \qquad y = \frac{x^2-8x+7}{x^2-8x+16}.$$

115. Let $\mathcal{P}$ be the piece of the parabola $y = x^2$ on which $0 \le x \le 1$.

(*i*) Find the area between $\mathcal{P}$, the x-axis and the line $x = 1$.

(*ii*) Find the length of $\mathcal{P}$.

116. Let a be a positive constant and

$$F(x) = \int_0^x \sin(a\theta)\cos(\theta)\,\mathrm{d}\theta.$$

[Hint: use a trig identity for $\sin A \cos B$, or wait until we have covered complex exponentials and then come back to do this problem.]

(*i*) Find $F(x)$ if $a \neq 1$.

(*ii*) Find $F(x)$ if $a = 1$. (Don't divide by zero.)

Evaluate the following integrals:

117. $\displaystyle\int_0^a x \sin x\,\mathrm{d}x$

118. $\displaystyle\int_0^a x^2 \cos x\,\mathrm{d}x$

119. $\displaystyle\int_3^4 \frac{x\,\mathrm{d}x}{\sqrt{x^2-1}}$

120. $\displaystyle\int_{1/4}^{1/3} \frac{x\,\mathrm{d}x}{\sqrt{1-x^2}}$

121. $\displaystyle\int_3^4 \frac{\mathrm{d}x}{x\sqrt{x^2-1}}$

122. $\displaystyle\int \frac{x\,\mathrm{d}x}{x^2+2x+17}$

123. $\displaystyle\int \frac{x^4}{(x^2-36)^{1/2}}\,\mathrm{d}x$

124. $\displaystyle\int \frac{x^4}{x^2-36}\,\mathrm{d}x$

125. $\displaystyle\int \frac{x^4}{36-x^2}\,\mathrm{d}x$

126. $\displaystyle\int \frac{(x^2+1)\,\mathrm{d}x}{x^4-x^2}$

127. $\displaystyle\int \frac{(x^2+3)\,\mathrm{d}x}{x^4-2x^2}$

128. $\displaystyle\int \frac{\mathrm{d}x}{(x^2-3)^{1/2}}$

129. $\displaystyle\int e^x(x+\cos(x))\,\mathrm{d}x$

130. $\displaystyle\int (e^x+\ln(x))\,\mathrm{d}x$

131. $\displaystyle\int \frac{3x^2+2x-2}{x^3-1}\,\mathrm{d}x$

132. $\displaystyle\int \frac{x^4}{x^4-16}\,\mathrm{d}x$

133. $\displaystyle\int \frac{x}{(x-1)^3}\,dx$

134. $\displaystyle\int \frac{4}{(x-1)^3(x+1)}\,dx$

135. $\displaystyle\int \frac{1}{\sqrt{1-2x-x^2}}\,dx$

136. $\displaystyle\int \frac{dx}{\sqrt{x^2+2x+3}}$

137. $\displaystyle\int_1^e x\ln x\,dx$

138. $\displaystyle\int_{e^2}^{e^3} x^2\ln x\,dx$

139. $\displaystyle\int_1^e x(\ln x)^3\,dx$

140. $\displaystyle\int \arctan(\sqrt{x})\,dx$

141. $\displaystyle\int x(\cos x)^2\,dx$

142. $\displaystyle\int_0^\pi \sqrt{1+\cos(6w)}\,dw$

143. Find

$$\int \frac{dx}{x(x-1)(x-2)(x-3)}$$

and

$$\int \frac{(x^3+1)\,dx}{x(x-1)(x-2)(x-3)}$$

144. You don't always have to find the antiderivative to find a definite integral. This problem gives you two examples of how you can avoid finding the antiderivative.

(***i***) To find

$$I = \int_0^{\pi/2} \frac{\sin x\,dx}{\sin x+\cos x}$$

you use the substitution $u = \pi/2 - x$. The new integral you get must of course be equal to the integral I you started with, so if you *add the old and new integrals* you get $2I$. If you actually do this you will see that the sum of the old and new integrals is *very* easy to compute.

(***ii***) Use the same trick to find $\displaystyle\int_0^{\pi/2} \sin^2 x\,dx$

145. Graph the equation $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. Compute the area bounded by this curve.

146. THE BOW-TIE GRAPH. Graph the equation $y^2 = x^4 - x^6$. Compute the area bounded by this curve.

147. THE FAN-TAILED FISH. Graph the equation

$$y^2 = x^2\left(\frac{1-x}{1+x}\right).$$

Find the area enclosed by the loop. (HINT: Rationalize the denominator of the integrand.)

148. Find the area of the region bounded by the curves

$$x = 2, \qquad y = 0, \qquad y = x\ln\frac{x}{2}$$

149. Find the volume of the solid of revolution obtained by rotating around the $x-$axis the region bounded by the lines $x = 5$, $x = 10$, $y = 0$, and the curve

$$y = \frac{x}{\sqrt{x^2+25}}.$$

150. How to find the integral of $f(x) = \dfrac{1}{\cos x}$

(***i***) Verify the answer given in the table in the lecture notes.

(***ii***) Note that

$$\frac{1}{\cos x} = \frac{\cos x}{\cos^2 x} = \frac{\cos x}{1 - \sin^2 x},$$

and apply the substitution $s = \sin x$ followed by a partial fraction decomposition to compute $\int \frac{dx}{\cos x}$.

Taylor's Formula and Infinite Series

All continuous functions which vanish at $x = a$
are approximately equal at $x = a$,
but some are more approximately equal than others.

11. Taylor Polynomials

Suppose you need to do some computation with a complicated function $y = f(x)$, and suppose that the only values of x you care about are close to some constant $x = a$. Since polynomials are simpler than most other functions, you could then look for a polynomial $y = P(x)$ which somehow "matches" your function $y = f(x)$ for values of x close to a. And you could then replace your function f with the polynomial P, hoping that the error you make isn't too big. *Which* polynomial you will choose depends on when you think a polynomial "matches" a function. In this chapter we will say that a polynomial P of degree n matches a function f at $x = a$ ***if P has the same value and the same derivatives of order*** **$1, 2, \ldots, n$ *at* $x = a$ *as the function* f.** The polynomial which matches a given function at some point $x = a$ is the Taylor polynomial of f. It is given by the following formula.

Definition 11.1. *The Taylor polynomial of a function $y = f(x)$ of degree n at a point a is the polynomial*

(7) $$T_n^a f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Recall that $n! = 1 \cdot 2 \cdot 3 \cdots n$, and by definition $0! = 1$

Theorem 11.2. *The Taylor polynomial has the following property: it is the only polynomial $P(x)$ of degree n whose value and whose derivatives of orders $1, 2, \ldots,$ and n are the same as those of f, i.e. it's the only polynomial of degree n for which*

$$P(a) = f(a), \quad P'(a) = f'(a), \quad P''(a) = f''(a), \quad \ldots, \quad P^{(n)}(a) = f^{(n)}(a)$$

holds.

Proof. We do the case $a = 0$, for simplicity. Let n be given, consider a polynomial $P(x)$ of degree n, say,

$$P(x) = a_0 + a_1 x + a_2 x^2 + a^3 x^3 + \cdots + a_n x^n,$$

and let's see what its derivatives look like. They are:

$$\begin{array}{rcccccccccc}
P(x) & = & a_0 & + & a_1 x & + & a_2 x^2 & + & a_3 x^3 & + & a_4 x^4 & + \cdots \\
P'(x) & = & & & a_1 & + & 2a_2 x & + & 3a_3 x^2 & + & 4a_4 x^3 & + \cdots \\
P^{(2)}(x) & = & & & & & 1 \cdot 2a_2 & + & 2 \cdot 3a_3 x & + & 3 \cdot 4a_4 x^2 & + \cdots \\
P^{(3)}(x) & = & & & & & & & 1 \cdot 2 \cdot 3a_3 & + & 2 \cdot 3 \cdot 4a_4 x & + \cdots \\
P^{(4)}(x) & = & & & & & & & & & 1 \cdot 2 \cdot 3 \cdot 4a_4 & + \cdots
\end{array}$$

When you set $x = 0$ all the terms which have a positive power of x vanish, and you are left with the first entry on each line, i.e.

$$P(0) = a_0, \quad P'(0) = a_1, \quad P^{(2)}(0) = 2a_2, \quad P^{(3)}(0) = 2 \cdot 3a_3, \text{ etc.}$$

and in general

$$P^{(k)}(0) = k! a_k \text{ for } 0 \le k \le n.$$

For $k \geq n+1$ the derivatives $p^{(k)}(x)$ all vanish of course, since $P(x)$ is a polynomial of degree n.

Therefore, if we want P to have the same values and derivatives at $x = 0$ of orders $1,, \ldots,$ n as the function f, then we must have $k!a_k = P^{(k)}(0) = f^{(k)}(0)$ for all $k \leq n$. Thus

$$a_k = \frac{f^{(k)}(0)}{k!} \qquad \text{for } 0 \leq k \leq n.$$

□

12. **Examples**

Note that the zeroth order Taylor polynomial is just a constant,

$$T_0^a f(x) = f(a),$$

while the first order Taylor polynomial is

$$T_1^a f(x) = f(a) + f'(a)(x-a).$$

This is exactly the *linear approximation of* $f(x)$ for x close to a which was derived in 1st semester calculus.

The Taylor polynomial generalizes this first order approximation by providing "higher order approximations" to f.

Most of the time we will take $a = 0$ in which case we write $T_n f(x)$ instead of $T_n^a f(x)$, and we get a slightly simpler formula

$$T_n f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n. \tag{8}$$

You will see below that for many functions $f(x)$ the Taylor polynomials $T_n f(x)$ give better and better approximations as you add more terms (i.e. as you increase n). For this reason the limit when $n \to \infty$ is often considered, which leads to the *infinite sum*

$$T_\infty f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f''(0)}{3!}x^3 + \cdots$$

At this point we will not try to make sense of the "sum of infinitely many numbers".

◀ **12.1 Example: Compute the Taylor polynomials of degree 0, 1 and 2 of $f(x) = e^x$ at $a = 0$, and plot them.** One has

$$f(x) = e^x \implies f'(x) = e^x \implies f''(x) = e^x,$$

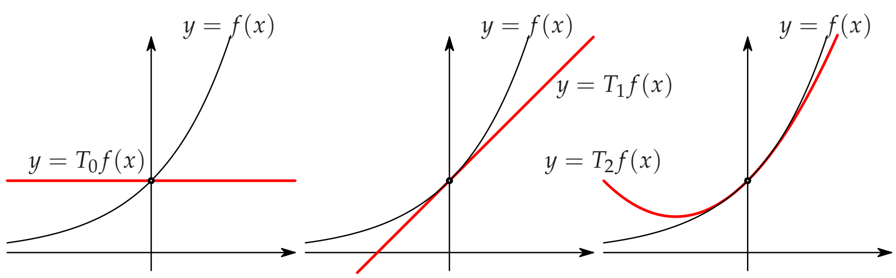


Figure 1: The Taylor polynomials of degree 0, 1 and 2 of $f(x) = e^x$ at $a = 0$. The zeroth order Taylor polynomial has the right value at $x = 0$ but it doesn't know whether or not the function f is increasing at $x = 0$. The first order Taylor polynomial has the right slope at $x = 0$, but it doesn't see if the graph of f is curved up or down at $x = 0$. The second order Taylor polynomial also has the right curvature at $x = 0$.

so that

$$f(0) = 1, \quad f'(0) = 1, \quad f''(0) = 1.$$

Therefore the first three Taylor polynomials of e^x at $a = 0$ are

$$T_0 f(x) = 1$$
$$T_1 f(x) = 1 + x$$
$$T_2 f(x) = 1 + x + \frac{1}{2}x^2.$$

The graphs are found in Figure 2. As you can see from the graphs, the Taylor polynomial $T_0 f(x)$ of degree 0 is close to e^x for small x, by virtue of the continuity of e^x

The Taylor polynomial of degree 0, i.e. $T_0 f(x) = 1$ captures the fact that e^x by virtue of its continuity does not change very much if x stays close to $x = 0$.

The Taylor polynomial of degree 1, i.e. $T_1 f(x) = 1 + x$ corresponds to the tangent line to the graph of $f(x) = e^x$, and so it also captures the fact that the function $f(x)$ is increasing near $x = 0$.

Clearly $T_1 f(x)$ is a better approximation to e^x than $T_0 f(x)$.

The graphs of both $y = T_0 f(x)$ and $y = T_1 f(x)$ are straight lines, while the graph of $y = e^x$ is curved (in fact, convex). The second order Taylor polynomial captures this convexity.

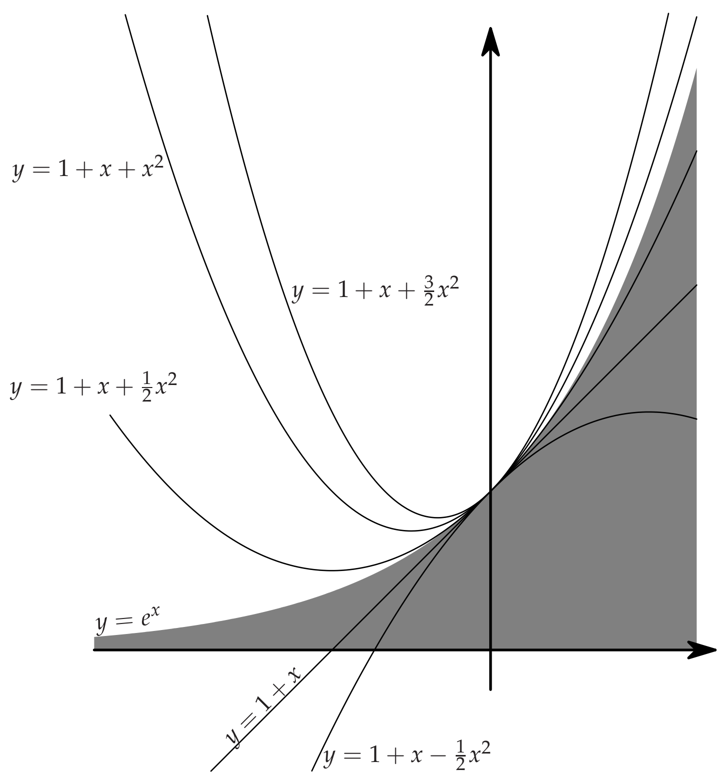


Figure 2: The top edge of the shaded region is the graph of $y = e^x$. The graphs are of the functions $y = 1 + x + Cx^2$ for various values of C. These graphs all are tangent at $x = 0$, but one of the parabolas matches the graph of $y = e^x$ better than any of the others.

In fact, the graph of $y = T_2 f(x)$ is a parabola, and since it has the same first and second derivative at $x = 0$, its curvature is the same as the curvature of the graph of $y = e^x$ at $x = 0$.

So it seems that $y = T_2 f(x) = 1 + x + x^2/2$ is an approximation to $y = e^x$ which beats both $T_0 f(x)$ and $T_1 f(x)$. ▶

◀ **12.2 Example: Find the Taylor polynomials of $f(x) = \sin x$.**

When you start computing the derivatives of $\sin x$ you find

$$f(x) = \sin x, \quad f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f^{(3)}(x) = -\cos x,$$

and thus

$$f^{(4)}(x) = \sin x.$$

So after four derivatives you're back to where you started, and the sequence of derivatives of $\sin x$ cycles through the pattern

$$\sin x,\ \cos x,\ -\sin x,\ -\cos x,\ \sin x,\ \cos x,\ -\sin x,\ -\cos x,\ \sin x,\ \ldots$$

on and on. At $x = 0$ you then get the following values for the derivatives $f^{(j)}(0)$,

j	0	1	2	3	4	5	6	7	$\cdots$
$f^{(j)}(0)$	0	1	0	-1	0	1	0	-1	$\cdots$

This gives the following Taylor polynomials

$$\begin{aligned} T_0 f(x) &= 0 \\ T_1 f(x) &= x \\ T_2 f(x) &= x \\ T_3 f(x) &= x - \frac{x^3}{3!} \\ T_4 f(x) &= x - \frac{x^3}{3!} \\ T_5 f(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} \end{aligned}$$

Note that since $f^{(2)}(0) = 0$ the Taylor polynomials $T_1 f(x)$ and $T_2 f(x)$ are the same! The second order Taylor polynomial in this example is really only a polynomial of degree 1. In general the Taylor polynomial $T_n f(x)$ of any function is a polynomial of degree at most n, and this example shows that the degree can sometimes be strictly less.

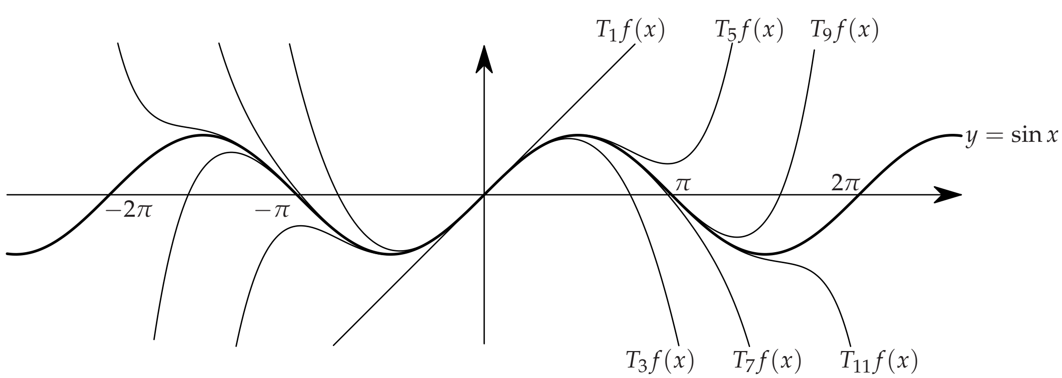


Figure 3: Taylor polynomials of $f(x) = \sin x$

▶

◀ **12.3 Example. Compute the Taylor polynomials of degree two and three of $f(x) = 1 + x + x^2 + x^3$ at $a = 3$.**

Solution: Remember that our notation for the n^{th} degree Taylor polynomial of a function f at a is $T_n^a f(x)$, and that it is defined by (7).

We have

$$f'(x) = 1 + 2x + 3x^2, \quad f''(x) = 2 + 6x, \quad f'''(x) = 6$$

Therefore $f(3) = 40$, $f'(3) = 34$, $f''(3) = 20$, $f''(3) = 6$, and thus

$$T_2^s f(x) = 40 + 34(x-3) + \frac{20}{2!}(x-3)^2 = 40 + 34(x-3) + 10(x-3)^2. \tag{9}$$

Why don't we expand the answer? You could do this (i.e. replace $(x-3)^2$ by $x^2 - 6x + 9$ throughout and sort the powers of x), but as we will see in this chapter, the Taylor polynomial $T_n^a f(x)$ is used as an approximation for $f(x)$ when x is close to a. In this example $T_2^3 f(x)$ is to be used when x is close to 3. If $x - 3$ is a small number then the successive powers $x - 3$, $(x-3)^2$, $(x-3)^3$, ... decrease rapidly, and so the terms in (9) are arranged in decreasing order.

We can also compute the third degree Taylor polynomial. It is

$$\begin{aligned} T_3^3 f(x) &= 40 + 34(x-3) + \frac{20}{2!}(x-3)^2 + \frac{6}{3!}(x-3)^3 \\ &= 40 + 34(x-3) + 10(x-3)^2 + (x-3)^3. \end{aligned}$$

If you expand this (this takes a little work) you find that

$$40 + 34(x-3) + 10(x-3)^2 + (x-3)^3 = 1 + x + x^2 + x^3.$$

So the third degree Taylor polynomial is the function f itself! Why is this so? Because of Theorem 11.2! Both sides in the above equation are third degree polynomials, and their derivatives of order 0, 1, 2 and 3 are the same at $x = 3$, so they must be the same polynomial. ▶

13. Some special Taylor polynomials

Here is a list of functions whose Taylor polynomials are sufficiently regular that you can write a formula for the nth term.

$$\begin{aligned} T_n e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \\ T_{2n+1}\{\sin x\} &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\ T_{2n}\{\cos x\} &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} \\ T_n\left\{\frac{1}{1-x}\right\} &= 1 + x + x^2 + x^3 + x^4 + \cdots + x^n \qquad \text{(Geometric Series)} \\ T_n\{\ln(1+x)\} &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} \end{aligned}$$

All of these Taylor polynomials can be computed directly from the definition, by repeatedly differentiating $f(x)$.

Another function whose Taylor polynomial you should know is $f(x) = (1+x)^a$, where a is a constant. You can compute $T_n f(x)$ directly from the definition, and when you do this you find

$$T_n\{(1+x)^a\} = 1 + ax + \frac{a(a-1)}{1\cdot 2}x^2 + \frac{a(a-1)(a-2)}{1\cdot 2\cdot 3}x^3 + \cdots + \frac{a(a-1)\cdots(a-n+1)}{1\cdot 2\cdots n}x^n. \tag{10}$$

This formula is called ***Newton's binomial formula***. The coefficient of x^n is called a ***binomial coefficient,*** and it is written

$$\binom{a}{n} = \frac{a(a-1)\cdots(a-n+1)}{n!}. \tag{11}$$

When a is an integer $\binom{a}{n}$ is also called "a choose n."

Note that you already knew special cases of the binomial formula: when a is a positive integer the binomial coefficients are just the numbers in ***Pascal's triangle***. When $a = -1$ the binomial formula is the Geometric series.

14. The Remainder Term

The Taylor polynomial $T_n f(x)$ is almost never exactly equal to $f(x)$, but often it is a good approximation, especially if x is small. To see how good the approximation is we define the "error term" or, "remainder term".

Definition 14.1. *If f is an n times differentiable function on some interval containing a, then*

$$R_n^a f(x) = f(x) - T_n^a f(x)$$

is called the n^{th} order remainder (or error) term in the Taylor polynomial of f.

If $a = 0$, as will be the case in most examples we do, then we omit the superscript a and write

$$R_n f(x) = f(x) - T_n f(x).$$

◀ **14.2 Example.** If $f(x) = \sin x$ then we have found that $T_3 f(x) = x - \frac{1}{6}x^3$, so that

$$R_3\{\sin x\} = \sin x - x + \tfrac{1}{6}x^3.$$

This is a completely correct formula for the remainder term, but it's rather useless: there's nothing about this expression that suggests that $x - \frac{1}{6}x^3$ is a much better approximation to $\sin x$ than, say, $x + \frac{1}{6}x^3$. ▶

The usual situation is that there is no simple formula for the remainder term.

◀ **14.3 An unusual example, in which there *is* a simple formula for $R_n f(x)$.** Consider $f(x) = 1 - x + 3x^2 - 15x^3$.

Then you find

$$T_2 f(x) = 1 - x + 3x^2, \text{ so that } R_2 f(x) = f(x) - T_2 f(x) = -15x^3.$$

The moral of this example is this: *Given a polynomial $f(x)$ you find its n^{th} degree Taylor polynomial by taking all terms of degree $\le n$ in $f(x)$; the remainder $R_n f(x)$ then consists of the remaining terms.* ▶

◀ **14.4 Another unusual, but important example where you can compute $R_n f(x)$.** Consider the function

$$f(x) = \frac{1}{1-x}.$$

Then repeated differentiation gives

$$f'(x) = \frac{1}{(1-x)^2}, \quad f^{(2)}(x) = \frac{1\cdot 2}{(1-x)^3}, \quad f^{(3)}(x) = \frac{1\cdot 2\cdot 3}{(1-x)^4}, \quad \dots$$

and thus

$$f^{(n)}(x) = \frac{1\cdot 2\cdot 3\cdots n}{(1-x)^{n+1}}.$$

Consequently,

$$f^{(n)}(0) = n! \implies \frac{1}{n!}f^{(n)}(0) = 1,$$

and you see that the Taylor polynomials of this function are really simple, namely

$$T_n f(x) = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n.$$

But this sum should be really familiar: it is just the ***Geometric Sum*** (each term is x times the previous term). Its sum is given by[6]

$$T_n f(x) = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n = \frac{1-x^{n+1}}{1-x},$$

which we can rewrite as

$$T_n f(x) = \frac{1}{1-x} - \frac{x^{n+1}}{1-x} = f(x) - \frac{x^{n+1}}{1-x}.$$

The remainder term therefore is

$$R_n f(x) = f(x) - T_n f(x) = \frac{x^{n+1}}{1-x}.$$

▶

15. Lagrange's Formula for the Remainder Term

Theorem 15.1. *Let f be an $n+1$ times differentiable function on some interval I containing the point a. Then for every x in the interval I there is a ξ between a and x such that*

$$R_n^a f(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}.$$

(ξ between 0 and x means either $a < \xi < x$ or $x < \xi < a$, depending on whether a is smaller or larger than x).

This theorem is similar to the Mean Value Theorem. The proofs of it are a bit involved. In the next section we shall give a proof under the slightly more restrictive hypothesis that the last $(n+1)$st derivative is continuous. Another proof which is somewhat shorter but more tricky (and related to the mean value theorem for derivatives) is given in §20 below.

There are calculus textbooks which, after presenting this remainder formula, give a whole bunch of problems which ask you to find ξ for given f and x. Such problems completely miss the point of Lagrange's formula. The point is that *even though you usually can't compute the mystery point ξ precisely, Lagrange's formula for the remainder term allows you to **estimate** it.* Here is the most common way to estimate the remainder:

[6]Multiply both sides with $1-x$ to verify this, in case you had forgotten the formula!

Theorem 15.2 (Estimate of remainder term). *If f is an $n+1$ times differentiable function on an interval containing $x = a$, and if you have a constant M such that*

$$(\dagger) \qquad \left| f^{(n+1)}(t) \right| \leq M \text{ for all } t \text{ between } a \text{ and } x,$$

then

$$|R_n^a f(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!}.$$

Proof. We don't know what ξ is in Lagrange's formula, but it doesn't matter, for wherever it is, it must lie between a and x so that our assumption (†) implies $|f^{(n+1)}(\xi)| \leq M$. Put that in Lagrange's formula and you get the stated inequality. □

◀ **15.3 How to compute e in a few decimal places.** Consider $f(x) = e^x$. We computed the Taylor polynomials before, with respect to $a = 1$. If you set $x = 1$, then you get $e = f(1) = T_n f(1) + R_n f(1)$, and thus, taking $n = 8$,

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} + R_8(1).$$

By Lagrange's formula there is a ξ between 0 and 1 such that

$$R_8(1) = \frac{f^{(9)}(\xi)}{9!}\, 1^9 = \frac{e^\xi}{9!}.$$

(remember: $f(x) = e^x$, so all its derivatives are also e^x.) We don't really know where ξ is, but since it lies between 0 and 1 we know that $1 < e^\xi < e$. So the remainder term $R_8(1)$ is positive and no more than $e/9!$. Estimating $e < 3$, we find

$$\frac{1}{9!} < R_8(1) < \frac{3}{9!}.$$

Thus we see that

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{7!} + \frac{1}{8!} + \frac{1}{9!} < e < 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{7!} + \frac{1}{8!} + \frac{3}{9!}$$

or, in decimals,

$$2.718\,281\ldots < e < 2.718\,287\ldots$$

▶

◀ **15.4 Error in the approximation $\sin x \approx x$.** In many calculations involving $\sin x$ for small values of x one makes the simplifying approximation $\sin x \approx x$, justified by the known limit

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Question: How big is the error in this approximation?

To answer this question, we use Lagrange's formula for the remainder term again.

Let $f(x) = \sin x$. Then the first degree Taylor polynomial of f (with respect to $a = 0$) is

$$T_1 f(x) = x.$$

The approximation $\sin x \approx x$ is therefore exactly what you get if you approximate $f(x) = \sin x$ by its first degree Taylor polynomial. Lagrange tells us that

$$f(x) = T_1 f(x) + R_1 f(x), \text{ i.e. } \sin x = x + R_1 f(x),$$

where, since $f''(x) = -\sin x$,

$$R_1 f(x) = \frac{f''(\xi)}{2!} x^2 = -\tfrac{1}{2} \sin \xi \cdot x^2$$

for some ξ between 0 and x.

As always with Lagrange's remainder term, we don't know where ξ is precisely, so we have to estimate the remainder term. The easiest way to do this (but not the best: see below) is to say that no matter what ξ is, $\sin\xi$ will always be between -1 and 1. Hence the remainder term is bounded by

$$|R_1 f(x)| \le \tfrac{1}{2}x^2, \tag{¶}$$

and we find that

$$x - \tfrac{1}{2}x^2 \le \sin x \le x + \tfrac{1}{2}x^2.$$

Question: How small must we choose x to be sure that the approximation $\sin x \approx x$ isn't off by more than 1% ?

If we want the error to be less than 1% of the estimate, then we should require $\frac{1}{2}x^2$ to be less than 1% of $|x|$, i.e.

$$\tfrac{1}{2}x^2 < 0.01 \cdot |x| \Leftrightarrow |x| < 0.02$$

So we have shown that, if you choose $|x| < 0.02$, then the error you make in approximating $\sin x$ by just x is no more than 1%.

A final comment about this example: the estimate for the error we got here can be improved quite a bit in two different ways:

(1) You could notice that one has $|\sin x| \le x$ for all x, so if ξ is between 0 and x, then $|\sin\xi| \le |\xi| \le |x|$, which gives you the estimate

$$|R_1 f(x)| \le \tfrac{1}{2}|x|^3 \quad \text{instead of } \tfrac{1}{2}x^2 \text{ as in (¶).}$$

(2) For this particular function the two Taylor polynomials $T_1 f(x)$ and $T_2 f(x)$ are the same (because $f''(0) = 0$). So $T_2 f(x) = x$, and we can write

$$\sin x = f(x) = x + R_2 f(x),$$

In other words, the error in the approximation $\sin x \approx x$ is also given by the *second* order remainder term, which according to Lagrange is given by

$$R_2 f(x) = \frac{-\cos\xi}{3!}x^3 \overset{|\cos\xi|\le 1}{\Longrightarrow} |R_2 f(x)| \le \tfrac{1}{6}|x|^3,$$

which is the best estimate for the error in $\sin x \approx x$ we have so far.

▶

16. Taylor's theorem with integral remainder

We state some alternative formulas for the remainder terms which are useful for various theoretical reasons and can be used to prove Theorem 15.1.

Theorem 16.1. *Let f be an $n+1$ times differentiable function on some interval I containing the point a and assume that $f^{(n+1)}$ is continuous on I. Then, for every x in I we have the formula*

$$f(x) = T_n^a f(x) + R_n^a f(x)$$

where T_n^a is the Taylor polynomial as defined in (7) and the remainder can be expressed as

$$R_n^a f(x) = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,\mathrm{d}t. \tag{12}$$

16.1. *Proof of the first three cases of Taylor's theorem*

First we look at the case $n = 0$. Then the right hand side of (12) is $\int_a^x \frac{(x-t)^0}{0!} f^{(1)}(t)\,\mathrm{d}t$, that is $\int_a^x f'(t)\,\mathrm{d}t$. Thus the theorem states in this case that

$$(\#_0) \qquad f(x) = f(a) + \int_a^x f'(t)\,\mathrm{d}t.$$

But this fact we know well; it is just a restatement of the fundamental theorem of calculus.

Now let's examine the case $n = 1$ of the theorem. We need to verify that

$$(\#_1) \qquad f(x) = f(a) + f'(a)(x-a) + \int_a^x (x-t) f''(t)\,\mathrm{d}t.$$

To get this we apply integration by parts to the integral $\int_a^x f'(t)\,dt$.. Recall the integration by parts formula from Theorem 7.1, i.e.

$$(13) \qquad \int_a^x u(t)v'(t)\,\mathrm{d}t = u(x)v(x) - u(a)v(a) - \int_a^x u'(t)v(t)\,\mathrm{d}t\,.$$

We use it with the choices $u(t) = f'(t)$, $v(t) = t - x$. We then get

$$\begin{aligned}\int_a^x f'(t)\,\mathrm{d}t &= \Big[(t-x)f'(t)\Big]_a^x - \int_a^x (t-x) f''(t)\,\mathrm{d}t \\ &= (x-a)f'(a) + \int_a^x (x-t) f''(t)\,\mathrm{d}t.\end{aligned}$$

If we plug this in for $\int_a^x f'(t)\mathrm{d}t$ in formula ($\#_0$) we obtain formula ($\#_1$).

For the case $n = 2$ we repeat this integration by parts argument one more time. It is our objective to show that

$$(\#_2) \qquad f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{1}{2}\int_a^x (x-t)^2 f'''(t)\,\mathrm{d}t.$$

Comparing ($\#_1$) and ($\#_2$) we see that ($\#_2$) follows from

$$(*_1) \qquad \int_a^x (x-t) f''(t)\,dt = \frac{f''(a)}{2}(x-a)^2 + \frac{1}{2}\int_a^x (x-t)^2 f'''(t)\,\mathrm{d}t.$$

To verify ($*_1$) we use it the integration by parts formula with the choices of $u(t) = f^{''}(t)$ and $v(t) = -\frac{1}{2}(x-t)^2$ (i.e. $v'(t) = (x-t)$). Thus the left hand side of ($*_1$) is equal to

$$f''(x)(-\frac{1}{2}(x-x)^2) - f''(a)(-\frac{1}{2}(x-a)^2) - \int_a^x \frac{-(x-t)^2}{2} f'''(t)\,\mathrm{d}t$$

which is equal to the right hand side of ($*_1$).

16.2. *Successive integration by parts yield the general case.*

The general case follows by an iteration of the above integration by parts procedure. One shows the formula

$$(*_k) \qquad \frac{1}{k!}\int_a^x (x-t)^k f^{(k+1)}(t)\,\mathrm{d}t = \frac{f^{(k+1)}(a)}{(k+1)!}(x-a)^{k+1} \frac{1}{(k+1)!}\int_a^x (x-t)^{k+1} f^{(k+2)}(t)\,\mathrm{d}t$$

for $k = 0, 1, 2, \ldots, n-1$. That means if we have verified the Taylor formula with Taylor polynomial of degree k and integral remainder then this formula is used to add the last term for the Taylor polynomial of degree $k+1$ and to generate the new integral remainder. Note that above we have already verified ($*_k$) for $k = 0$ and $k = 1$. To check ($*_k$) in general we use the above integration by parts formula (13), with the choices of $u(t) = f^{(k+1)}(t)$ and $v(t) = \frac{-(x-t)^{k+1}}{k+1}$. We compute

$$\int_a^x (x-t)^k f^{(k+1)}(t)\,\mathrm{d}t = \Big[\frac{-(x-t)^{k+1}}{k+1} f^{(k+1)}(t)\Big]_a^x - \int_a^x \frac{-(x-t)^{k+1}}{k+1} f^{(k+2)}(t)\,\mathrm{d}t$$
$$= \frac{(x-a)^{k+1}}{k+1} f^{(k+1)}(a) + \int_a^x \frac{(x-t)^{k+1}}{k+1} f^{(k+2)}(t)\,\mathrm{d}t.$$

Devide by $k!$ and ($*_k$) follows.

Remark: Some students may have systematically learned about proof by induction. You will then recognize that we prove Taylor's theorem with integral remainder by induction and the formula ($*_k$) is the main step in the induction.

◀ **16.2 An alternative form of the remainder term.** Another integral formula which is often useful involves an integral over the interval $[0,1]$:

$$R_n^a f(x) = \frac{(x-a)^{n+1}}{(n+1)!} \int_0^1 (n+1)(1-s)^n f^{(n+1)}(a+s(x-a))\,\mathrm{d}s. \tag{14}$$

This follows from (12) by a simple substitution, namely, for fixed x we change variables

$$t = a + s(x-a) \qquad \text{with } \mathrm{d}t = (x-a)\,\mathrm{d}s.$$

Then $s = \frac{t-a}{x-a}$ so that the point $t = a$ corresponds to $s = 0$ and the point $t = x$ corresponds to $s = 1$. Thus

$$\int_{t=a}^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,\mathrm{d}t$$
$$= \int_{s=0}^1 \frac{(x-a-s(x-a))^n}{n!} f^{(n+1)}(a+s(x-a))(x-a)\,\mathrm{d}s$$
$$= \frac{(x-a)^{n+1}}{n!} \int_{s=0}^1 (1-s)^n f^{(n+1)}(a+s(x-a))\,\mathrm{d}s$$

which is equal to the right hand side of (14) (just use $\frac{n+1}{(n+1)!} = \frac{1}{n!}$). ▶

◀ **16.3 Application: An estimation related to Theorem 15.2.** Why does one care about the particular form (14) of the remainder term. The answer is that it yields an argument for the important estimation (15.2) without having to prove the Lagrange formula first (the point here is that while Lagrange's formula is easy to state, its proof is not easy).

We check that if $f^{(n+1)}$ is continuous and satisfies the bound $|f^{(n+1)}(t)| \le M$ for all t between a and x, then $|R_n^a f(x)| \le \frac{M|x-a|^{n+1}}{(n+1)!}$. Note that for s between 0 and 1 the point $a + s(x-a)$ lies between a and x. Therefore $-M \le f^{(n+1)}(a+s(x-a)) \le M$ for such s. We also know that $(1-s)^n$ is nonnegative for s between 0 and 1. Thus we get

$$\int_0^1 (1-s)^n(-M)\,\mathrm{d}s \le \int_0^1 (1-s)^n f^{(n+1)}(a+s(x-a))\,\mathrm{d}s \le \int_0^1 (1-s)^n M\,\mathrm{d}s$$

and this chain of inequalities does not change if we multipliy all three terms with $n+1$. But we can compute $\int_0^1 (n+1)(1-s)^n\,\mathrm{d}s = 1$ and thus

$$-M \le \int_0^1 (n+1)(1-s)^n f^{(n+1)}(a+s(x-a))\,\mathrm{d}s \le M.$$

This implies the estimate $|R_n^a f(x)| \le \frac{M|x-a|^{n+1}}{(n+1)!}$ that we wanted to prove. ▶

16.3. *On Lagrange's form of the remainder*

We wish to prove the formula in Theorem 15.1 directly from Theorem 16.1. However this requires an assumption on the derivative $f^{(n+1)}(t)$ to make sure that the integral defining the remainder makes even sense. The additional assumption that we use is that $f^{(n+1)}$ is continuous. This is good enough for what is needed for most applications. We now restate this special case of the theorem as [7]

Theorem 15.1* *Let f be an $n+1$ times differentiable function on some interval I containing the point a and assume that $f^{(n+1)}$ is continuous on I. Then for every x in the interval I there is a ξ between a and x such that*

$$f(x) = T_n^a f(x) + R_n^a f(x) \quad \text{with } R_n f(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}.$$

Proof. We will only write out the case where $x > a$ (and the case where $x < a$ requires notational changes).

Let b be the minimum of $f^{(n+1)}(t)$ in the interval $[a, x]$ (as above) and let B be the maximum of $f^{(n+1)}$ in this interval[8]. We choose two points t_1, t_2 in $[a, x]$ at which the minimum and the maximum occurs, say, $b = f^{(n+1)}(t_1)$ and $B = f^{(n+1)}(t_2)$. Since $b \le f^{(n+1)}(t) \le B$ for $a \le t \le x$ and since $(x-t)^n$ is positive on $[a, x]$ we also have

$$b\int_a^x \frac{(x-t)^n}{n!}\,\mathrm{d}t \le \int_a^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}\,\mathrm{d}t \le B\int_a^x \frac{(x-t)^n}{n!}\,\mathrm{d}t.$$

we can evaluate the integral

$$\int_a^x \frac{(x-t)^n}{n!}\,\mathrm{d}t = \frac{(x-a)^{n+1}}{(n+1)!}.$$

Thus

$$\frac{b}{(n+1)!} \le \frac{1}{(x-a)^{n+1}}\int_a^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}\,\mathrm{d}t \le \frac{B}{(n+1)!}.$$

So this means that the expression $\frac{1}{(x-a)^{n+1}}\int_a^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}$ lies between $\frac{f^{(n+1)}(t_1)}{(n+1)!}$ and $\frac{f^{(n+1)}(t_2)}{(n+1)!}$. An application of the intermediate value theorem to the continuous function $\frac{f^{(n+1)}}{(n+1)!}$ shows that the middle term in the last displayed formula is an (intermediate) value for this function, more precisely, that there exists a number ξ between a and x such that

$$\frac{f^{(n+1)}(\xi)}{(n+1)!} = \frac{1}{(x-a)^{n+1}}\int_a^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}\,\mathrm{d}t.$$

Thus, by formula (12) in Theorem 16.1, we get the Lagrange formula

$$R_n^a(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}.$$

□

[7] The proof of the original and slightly more general Theorem 15.1 is written up in §20. It does not use integrals.

[8] A theorem in advanced calculus states that every continuous function on a closed bounded interval (such as $[a, x]$) has a minimum and a maximum in this interval. We use this theorem here for the function $f^{(n+1)}$ which we have assumed to be continuous

17. The limit as $x \to 0$, keeping n fixed

17.1. *Little-oh*

Lagrange's formula for the remainder term lets us write a function $y = f(x)$, which is defined on some interval containing $x = 0$, in the following way

$$f(x) = f(0) + f'(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1} \tag{15}$$

The last term contains the ξ from Lagrange's theorem, which depends on x, and of which you only know that it lies between 0 and x. For many purposes it is not necessary to know the last term in this much detail – often it is enough to know that "in some sense" the last term is the smallest term, in particular, as $x \to 0$ it is much smaller than x, or x^2, or, …, or x^n:

Theorem 17.1. *If the $n+1$st derivative $f^{(n+1)}(x)$ is continuous at $x = 0$ then the remainder term $R_n f(x) = f^{(n+1)}(\xi)x^{n+1}/(n+1)!$ satisfies*

$$\lim_{x\to 0} \frac{R_n f(x)}{x^k} = 0$$

for any $k = 0, 1, 2, \ldots, n$.

Proof. Since ξ lies between 0 and x, one has $\lim_{x\to 0} f^{(n+1)}(\xi) = f^{(n+1)}(0)$, and therefore

$$\lim_{x\to 0} \frac{R_n f(x)}{x^k} = \lim_{x\to 0} f^{(n+1)}(\xi)\frac{x^{n+1}}{x^k} = \lim_{x\to 0} f^{(n+1)}(\xi) \cdot x^{n+1-k} = f^{(n+1)}(0) \cdot 0 = 0.$$

□

So we can rephrase (15) by saying

$$f(x) = f(0) + f'(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \text{ remainder}$$

where the remainder is much smaller than x^n, x^{n-1}, …, x^2, x or 1. In order to express the condition that some function is "much smaller than x^n," at least for very small x, Landau introduced the following notation which many people find useful.

Definition 17.2. *"$o(x^n)$" is an abbreviation for any function $h(x)$ which satisfies*

$$\lim_{x\to 0} \frac{h(x)}{x^n} = 0.$$

So you can rewrite (15) as

$$f(x) = f(0) + f'(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + o(x^n).$$

The nice thing about Landau's little-oh is that you can compute with it, as long as you obey the following (at first sight rather strange) rules which will be proved in class

$$\begin{aligned}
x^n \cdot o(x^m) &= o(x^{n+m}) \\
o(x^n) \cdot o(x^m) &= o(x^{n+m}) \\
x^m &= o(x^n) && \text{if } n < m \\
o(x^n) + o(x^m) &= o(x^n) && \text{if } n < m \\
o(Cx^n) &= o(x^n) && \text{for any constant } C
\end{aligned}$$

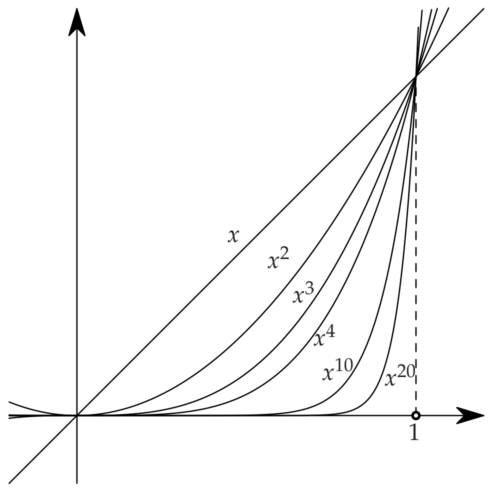


Figure 4: How the powers stack up. All graphs of $y = x^n$ $(n > 1)$ are tangent to the x-axis at the origin. But the larger the exponent n the "flatter" the graph of $y = x^n$ is.

◀ **17.3 Example: prove one of these little-oh rules.** Let's do the first one, i.e. let's show that $x^n \cdot o(x^m)$ is $o(x^{n+m})$ as $x \to 0$.

Remember, if someone writes $x^n \cdot o(x^m)$, then the $o(x^m)$ is an abbreviation for some function $h(x)$ which satisfies $\lim_{x\to 0} h(x)/x^m = 0$. So the $x^n \cdot o(x^m)$ we are given here really is an abbreviation for $x^n h(x)$. We then have

$$\lim_{x\to 0} \frac{x^n h(x)}{x^{n+m}} = \lim_{x\to 0} \frac{h(x)}{x^m} = 0, \text{ since } h(x) = o(x^m).$$

▶

◀ **17.4 Can you see that $x^3 = o(x^2)$ by looking at the graphs of these functions?** A picture is of course never a proof, but have a look at figure 4 which shows you the graphs of $y = x$, x^2, x^3, x^4, x^5 and x^{10}. As you see, when x approaches 0, the graphs of higher powers of x approach the x-axis (much?) faster than do the graphs of lower powers.

You should also have a look at figure 5 which exhibits the graphs of $y = x^2$, as well as several linear functions $y = Cx$ (with $C = 1, \frac{1}{2}, \frac{1}{5}$ and $\frac{1}{10}$.) For each of these linear functions one has $x^2 < Cx$ if x is small enough; *how* small is actually small enough depends on C. The smaller the constant C, the closer you have to keep x to 0 to be sure that x^2 is smaller than Cx. Nevertheless, no matter how small C is, the parabola will eventually always reach the region below the line $y = Cx$. ▶

◀ **17.5 Example: Little-oh arithmetic is a little funny.** Both x^2 and x^3 are functions which are $o(x)$, i.e.

$$x^2 = o(x) \quad \text{and} \quad x^3 = o(x)$$

Nevertheless $x^2 \neq x^3$. So in working with little-oh we are giving up on the principle that says that two things which both equal a third object must themselves be equal; in other words, $a = b$ and $b = c$ implies $a = c$, but not when you're using little-ohs! You can also put it like this: just because two quantities both are much smaller than x, they don't have to be equal. In particular,

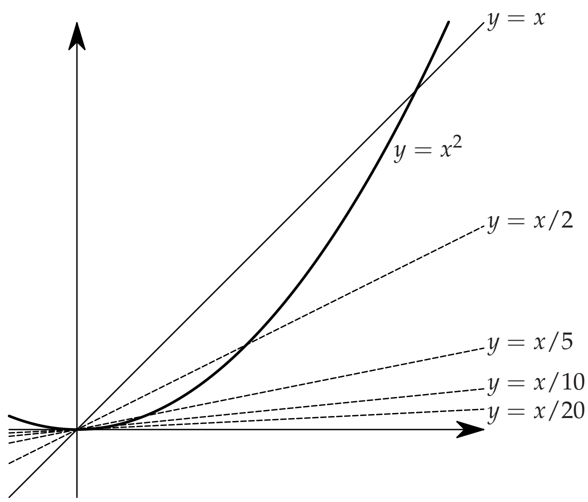


Figure 5: x^2 is smaller than any multiple of x, if x is small enough. Compare the quadratic function $y = x^2$ with a linear function $y = Cx$. Their graphs are a parabola and a straight line. Parts of the parabola may lie above the line, but as $x \searrow 0$ the parabola will always duck underneath the line.

you can never cancel little-ohs!!!

In other words, the following is pretty wrong

$$o(x^2) - o(x^2) = 0.$$

Why? The two $o(x^2)$'s both refer to functions $h(x)$ which satisfy $\lim_{x\to 0} h(x)/x^2 = 0$, but there are many such functions, and the two $o(x^2)$'s could be abbreviations for different functions $h(x)$.

Contrast this with the following computation, which at first sight looks wrong even though it is actually right:

$$o(x^2) - o(x^2) = o(x^2).$$

In words: if you subtract two quantities both of which are negligible compared to x^2 for small x then the result will also be negligible compared to x^2 for small x.

▶

17.2. *Computations with Taylor polynomials*

The following theorem is very useful because it lets you compute Taylor polynomials of a function without differentiating it.

Theorem 17.6. *If $f(x)$ and $g(x)$ are $n+1$ times differentiable functions then*

$$T_n f(x) = T_n g(x) \iff f(x) = g(x) + o(x^n). \tag{16}$$

In other words, if two functions have the same nth degree Taylor polynomial, then their difference is much smaller than x^n, at least, if x is small.

In principle the definition of $T_n f(x)$ lets you compute as many terms of the Taylor polynomial as you want, but in many (most) examples the computations quickly get out of hand. To see what can happen go though the following example:

◀ **17.7 How *NOT* to compute the Taylor polynomial of degree 12 of $f(x) = 1/(1+x^2)$.** Diligently computing derivatives one by one you find

$$\begin{aligned}
f(x) &= \frac{1}{1+x^2} && \text{so } f(0)=1\\
f'(x) &= \frac{-2x}{(1+x^2)^2} && \text{so } f'(0)=0\\
f''(x) &= \frac{6x^2-2}{(1+x^2)^3} && \text{so } f''(0)=-2\\
f^{(3)}(x) &= 24\frac{x-x^3}{(1+x^2)^4} && \text{so } f^{(3)}(0)=0\\
f^{(4)}(x) &= 24\frac{1-10x^2+5x^4}{(1+x^2)^5} && \text{so } f^{(4)}(0)=24=4!\\
f^{(5)}(x) &= 240\frac{-3x+10x^3-3x^5}{(1+x^2)^6} && \text{so } f^{(4)}(0)=0\\
f^{(6)}(x) &= -720\frac{-1+21x^2-35x^4+7x^6}{(1+x^2)^7} && \text{so } f^{(4)}(0)=720=6!\\
&\vdots
\end{aligned}$$

I'm getting tired of differentiating – can you find $f^{(12)}(x)$? After a lot of work we give up at the sixth derivative, and all we have found is

$$T_6\left\{\frac{1}{1+x^2}\right\} = 1-x^2+x^4-x^6.$$

By the way,

$$f^{(12)}(x) = 479001600\frac{1-78\,x^2+715\,x^4-1716\,x^6+1287\,x^8-286\,x^{10}+13\,x^{12}}{(1+x^2)^{13}}$$

and $479001600 = 12!$. ▶

◀ **17.8 The right approach to finding the Taylor polynomial of any degree of $f(x) = 1/(1+x^2)$.** Start with the Geometric Series: if $g(t) = 1/(1-t)$ then

$$g(t) = 1+t+t^2+t^3+t^4+\cdots+t^n+o(t^n).$$

Now substitute $t=-x^2$ in this limit,

$$g(-x^2) = 1-x^2+x^4-x^6+\cdots+(-1)^n x^{2n}+o\left(\left(-x^2\right)^n\right)$$

Since $o\left(\left(-x^2\right)^n\right) = o(x^{2n})$ and

$$g(-x^2) = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2},$$

we have found

$$\frac{1}{1+x^2} = 1-x^2+x^4-x^6+\cdots+(-1)^n x^{2n}+o(x^{2n})$$

By Theorem (17.6) this implies

$$T_{2n}\left\{\frac{1}{1+x^2}\right\} = 1-x^2+x^4-x^6+\cdots+(-1)^n x^{2n}.$$

▶

◀ **17.9 Example of multiplication of Taylor series.** Finding the Taylor series of $e^{2x}/(1+x)$ directly from the definition is another recipe for headaches. Instead, you should exploit your knowledge of the Taylor series of both factors e^{2x} and $1/(1+x)$:

$$\begin{aligned} e^{2x} &= 1 + 2x + \frac{2^2x^2}{2!} + \frac{2^3x^3}{3!} + \frac{2^4x^4}{4!} + o(x^4) \\ &= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + o(x^4) \\ \frac{1}{1+x} &= 1 - x + x^2 - x^3 + x^4 + o(x^4). \end{aligned}$$

Then multiply these two

$$\begin{aligned} e^{2x}\cdot\frac{1}{1+x} &= \Big(1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + o(x^4)\Big)\cdot\Big(1 - x + x^2 - x^3 + x^4 + o(x^4)\Big) \\ &= \begin{array}{ccccccccccc} 1 & - & x & + & x^2 & - & x^3 & + & x^4 & + & o(x^4) \\ & + & 2x & - & 2x^2 & + & 2x^3 & - & 2x^4 & + & o(x^4) \\ & & & + & 2x^2 & - & 2x^3 & + & 2x^4 & + & o(x^4) \\ & & & & & + & \frac{4}{3}x^3 & - & \frac{4}{3}x^4 & + & o(x^4) \\ & & & & & & & + & \frac{2}{3}x^4 & + & o(x^4) \end{array} \\ &= 1 + x + x^2 + \frac{1}{3}x^3 + \frac{1}{3}x^4 + o(x^4) \quad (x \to 0) \end{aligned}$$

▶

◀ **17.10 Taylor's formula and Fibonacci numbers.** The Fibonacci numbers are defined as follows: the first two are $f_0 = 1$ and $f_1 = 1$, and the others are defined by the equation

(Fib) $$\boxed{f_n = f_{n-1} + f_{n-2}}$$

So

$$\begin{aligned} f_2 &= f_1 + f_0 = 1 + 1 = 2, \\ f_3 &= f_2 + f_1 = 2 + 1 = 3, \\ f_4 &= f_3 + f_2 = 3 + 2 = 5, \\ &\text{etc.} \end{aligned}$$

The equation (Fib) lets you compute the whole sequence of numbers, one by one, when you are given only the first few numbers of the sequence (f_0 and f_1 in this case). Such an equation for the elements of a sequence is called a ***recursion relation***.

Now consider the function

$$f(x) = \frac{1}{1 - x - x^2}.$$

Let

$$T_\infty f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots$$

be its Taylor series.

Due to Lagrange's remainder theorem you have, for any n,

$$\frac{1}{1 - x - x^2} = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots + c_nx^n + o(x^n) \quad (x \to 0).$$

Multiply both sides with $1-x-x^2$ and you get

$$\begin{aligned}
1 &= (1-x-x^2)\cdot(c_0+c_1x+c_2x^2+\cdots+c_n+o(x^n)) \quad (x\to 0)\\
&= \begin{array}{ccccccccccc} c_0 & + & c_1x & + & c_2x^2 & + & \cdots & + & c_nx^n & + & o(x^n)\\ & - & c_0x & - & c_1x^2 & - & \cdots & - & c_{n-1}x^n & + & o(x^n)\\ & & & - & c_0x^2 & - & \cdots & - & c_{n-2}x^n & - & o(x^n)\end{array} \qquad (x\to 0)\\
&= c_0+(c_1-c_0)x+(c_2-c_1-c_0)x^2+(c_3-c_2-c_1)x^3+\cdots\\
&\qquad \cdots+(c_n-c_{n-1}-c_{n-2})x^n+o(x^n) \quad (x\to 0)
\end{aligned}$$

Compare the coefficients of powers x^k on both sides for $k=0,1,\ldots,n$ and you find

$$c_0=1,\quad c_1-c_0=0 \implies c_1=c_0=1,\quad c_2-c_1-c_0=0 \implies c_2=c_1+c_0=2$$

and in general

$$c_n-c_{n-1}-c_{n-2}=0 \implies c_n=c_{n-1}+c_{n-2}$$

Therefore the coefficients of the Taylor series $T_\infty f(x)$ are exactly the Fibonacci numbers:

$$c_n=f_n \text{ for } n=0,1,2,3,\ldots$$

Since it is much easier to compute the Fibonacci numbers one by one than it is to compute the derivatives of $f(x)=1/(1-x-x^2)$, this is a better way to compute the Taylor series of $f(x)$ than just directly from the definition. ▶

◀ 17.11 More about the Fibonacci numbers.

In this example you'll see a trick that lets you compute the Taylor series of ***any rational function***. You already know the trick: find the partial fraction decomposition of the given rational function. Ignoring the case that you have quadratic expressions in the denominator, this lets you represent your rational function as a sum of terms of the form

$$\frac{A}{(x-a)^p}.$$

These are easy to differentiate any number of times, and thus they allow you to write their Taylor series.

Let's apply this to the function $f(x)=1/(1-x-x^2)$ from the example 17.10. First we factor the denominator.

$$1-x-x^2=0 \iff x^2+x-1=0 \iff x=\frac{-1\pm\sqrt{5}}{2}.$$

The number

$$\phi=\frac{1+\sqrt{5}}{2}\approx 1.618\,033\,988\,749\,89\ldots$$

is called the ***Golden Ratio***. It satisfies[9]

$$\phi+\frac{1}{\phi}=\sqrt{5}.$$

The roots of our polynomial x^2+x-1 are therefore

$$x_-=\frac{-1-\sqrt{5}}{2}=-\phi,\qquad x_+=\frac{-1+\sqrt{5}}{2}=\frac{1}{\phi}.$$

and we can factor $1-x-x^2$ as follows

$$1-x-x^2=-(x^2+x-1)=-(x-x_-)(x-x_+)=-(x-\frac{1}{\phi})(x+\phi).$$

[9] To prove this, use $\dfrac{1}{\phi}=\dfrac{2}{1+\sqrt{5}}=\dfrac{2}{1+\sqrt{5}}\dfrac{1-\sqrt{5}}{1-\sqrt{5}}=\dfrac{-1+\sqrt{5}}{2}$.

So $f(x)$ can be written as

$$f(x) = \frac{1}{1-x-x^2} = \frac{-1}{(x-\frac{1}{\phi})(x+\phi)} = \frac{A}{x-\frac{1}{\phi}} + \frac{B}{x+\phi}$$

The Heaviside trick will tell you what A and B are, namely,

$$A = \frac{-1}{\frac{1}{\phi}+\phi} = \frac{-1}{\sqrt{5}}, \qquad B = \frac{1}{\frac{1}{\phi}+\phi} = \frac{1}{\sqrt{5}}$$

The nth derivative of $f(x)$ is

$$f^{(n)}(x) = \frac{A(-1)^n n!}{\left(x-\frac{1}{\phi}\right)^{n+1}} + \frac{B(-1)^n n!}{(x+\phi)^{n+1}}$$

Setting $x = 0$ and dividing by $n!$ finally gives you the coefficient of x^n in the Taylor series of $f(x)$. The result is the following formula for the nth Fibonacci number

$$c_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!}\frac{A(-1)^n n!}{\left(-\frac{1}{\phi}\right)^{n+1}} + \frac{1}{n!}\frac{B(-1)^n n!}{(\phi)^{n+1}} = -A\phi^{n+1} - B\left(\frac{1}{\phi}\right)^{n+1}$$

Using the values for A and B you find

$$\boxed{f_n = c_n = \frac{1}{\sqrt{5}}\left\{\phi^{n+1} - \frac{1}{\phi^{n+1}}\right\}}$$

▶

17.3. *Differentiating Taylor polynomials*

If

$$T_n f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

is the Taylor polynomial of a function $y = f(x)$, then what is the Taylor polynomial of its derivative $f'(x)$?

Theorem 17.12. *The Taylor polynomial of degree $n-1$ of $f'(x)$ is given by*

$$T_{n-1}\{f'(x)\} = a_1 + 2a_2 x + \cdots + na_n x^{n-1}.$$

In other words, "the Taylor polynomial of the derivative is the derivative of the Taylor polynomial."

Proof. Let $g(x) = f'(x)$. Then $g^{(k)}(0) = f^{(k+1)}(0)$, so that

$$T_{n-1}g(x) = g(0) + g'(0)x + g^{(2)}(0)\frac{x^2}{2!} + \cdots + g^{(n-1)}(0)\frac{x^{n-1}}{(n-1)!}$$

$$(\$) \qquad = f'(0) + f^{(2)}(0)x + f^{(3)}(0)\frac{x^2}{2!} + \cdots + f^{(n)}(0)\frac{x^{n-1}}{(n-1)!}$$

On the other hand, if $T_n f(x) = a_0 + a_1 x + \cdots + a_n x^n$, then $a_k = f^{(k)}(0)/k!$, so that

$$ka_k = \frac{k}{(k-1)!}f^{(k)}(0) = \frac{f^{(k)}(0)}{(k-1)!}.$$

In other words,

$$1 \cdot a_1 = f'(0),\ 2a_2 = f^{(2)}(0),\ 3a_3 = \frac{f^{(3)}(0)}{2!},\ \text{etc.}$$

So, continuing from ($) you find that

$$T_{n-1}\{f'(x)\} = T_{n-1}g(x) = a_1 + 2a_2 x + \cdots + na_n x^{n-1}$$

as claimed. □

◀ **17.13 Example.** We compute the Taylor polynomial of $f(x) = 1/(1-x)^2$ by noting that

$$f(x) = F'(x), \text{ where } F(x) = \frac{1}{1-x}.$$

Since

$$T_{n+1}F(x) = 1 + x + x^2 + x^3 + \cdots + x^{n+1},$$

theorem 17.12 implies that

$$T_n\left\{\frac{1}{(1-x)^2}\right\} = 1 + 2x + 3x^2 + 4x^3 + \cdots + (n+1)x^n$$

▶

◀ **17.14 Example: Taylor polynomials of** $\arctan x$**.** Let $f(x) = \arctan x$. Then know that

$$f'(x) = \frac{1}{1+x^2}.$$

By substitution of $t = -x^2$ in the Taylor polynomial of $1/(1-t)$ we had found

$$T_{2n}\{f'(x)\} = T_{2n}\left\{\frac{1}{1+x^2}\right\} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + o\left(x^{2n}\right).$$

This Taylor polynomial must be the derivative of $T_{2n+1}f(x)$, so we have

$$T_{2n+1}\{\arctan x\} = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1}.$$

▶

18. The limit $n \to \infty$, keeping x fixed

18.1. *Sequences and their limits*

We shall call a ***sequence*** any ordered sequence of numbers $a_1, a_2, a_3, \ldots$: for each positive integer n we have to specify a number a_n.

◀ **18.1 Examples of sequences.**

definition ↓	**first few number in the sequence** ↓
$a_n = n$	$1, 2, 3, 4, \ldots$
$b_n = 0$	$0, 0, 0, 0, \ldots$
$c_n = \dfrac{1}{n}$	$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$
$d_n = \left(-\frac{1}{3}\right)^n$	$-\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, \ldots$
$E_n = 1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \dfrac{1}{3!} + \cdots + \dfrac{1}{n!}$	$1, 2, 2\frac{1}{2}, 2\frac{2}{3}, 2\frac{17}{24}, 2\frac{43}{60}, \ldots$
$S_n = T_{2n+1}\{\sin x\} = x - \dfrac{x^3}{3!} + \cdots + (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$	$x, x - \dfrac{x^3}{3!}, x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!}, \ldots$

▶

The last two sequences are derived from the Taylor polynomials of e^x (at $x = 1$) and $\sin x$ (at any x). The last example S_n really is a sequence of functions, i.e. for every choice of x you get a different sequence.

Definition 18.2. *A sequence of numbers* $(a_n)_{n=1}^{\infty}$ *converges to a limit* L*, if for every* $\epsilon > 0$ *there is a number* N_ϵ *such that for all* $n > N_\epsilon$ *one has*

$$|a_n - L| < \epsilon.$$

One writes

$$\lim_{n\to\infty} a_n = L$$

◀ **18.3 Example:** $\lim_{n\to\infty} \frac{1}{n} = 0$**.**

The sequence $c_n = 1/n$ converges to 0. To prove this let $\epsilon > 0$ be given. We have to find an N_ϵ such that

$$|c_n| < \epsilon \text{ for all } n > N_\epsilon.$$

The c_n are all positive, so $|c_n| = c_n$, and hence

$$|c_n| < \epsilon \iff \frac{1}{n} < \epsilon \iff n > \frac{1}{\epsilon},$$

which prompts us to choose $N_\epsilon = 1/\epsilon$. The calculation we just did shows that if $n > \frac{1}{\epsilon} = N_\epsilon$, then $|c_n| < \epsilon$. That means that $\lim_{n\to\infty} c_n = 0$. ▶

◀ **18.4 Example:** $\lim_{n\to\infty} a^n = 0$ **if** $|a| < 1$**.**

As in the previous example one can show that $\lim_{n\to\infty} 2^{-n} = 0$, and more generally, that for any constant a with $-1 < a < 1$ one has

$$\lim_{n\to\infty} a^n = 0.$$

Indeed,

$$|a^n| = |a|^n = e^{n \ln |a|} < \epsilon$$

holds if and only if

$$n \ln |a| < \ln \epsilon.$$

Since $|a| < 1$ we have $\ln |a| < 0$ so that dividing by $\ln |a|$ reverses the inequality, with result

$$|a^n| < \epsilon \iff n > \frac{\ln \epsilon}{\ln |a|}$$

The choice $N_\epsilon = (\ln \epsilon)/(\ln |a|)$ therefore guarantees that $|a^n| < \epsilon$ whenever $n > N_\epsilon$. ▶

One can show that the operation of taking limits of sequences obeys the same rules as taking limits of functions.

Theorem 18.5. *If*

$$\lim_{n\to\infty} a_n = A \text{ and } \lim_{n\to\infty} b_n = B,$$

then one has

$$\begin{aligned} \lim_{n\to\infty} a_n \pm b_n &= A \pm B \\ \lim_{n\to\infty} a_n b_n &= AB \\ \lim_{n\to\infty} \frac{a_n}{b_n} &= \frac{A}{B} \quad \text{(assuming } B \neq 0\text{).} \end{aligned}$$

The so-called "sandwich theorem" for ordinary limits also applies to limits of sequences. Namely, one has

Theorem 18.6 ("Sandwich theorem"). *If* a_n *is a sequence which satisfies* $b_n < a_n < c_N$ *for all* n*, and if* $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 0$*, then* $\lim_{n\to\infty} a_n = 0$*.*

Finally, one can show this:

Theorem 18.7. *If $f(x)$ is a function which is continuous at $x = A$, and a_n is a sequence which converges to A, then*

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(A).$$

◀ **18.8 Example.** Since $\lim_{n\to\infty} 1/n = 0$ and since $f(x) = \cos x$ is continuous at $x = 0$ we have

$$\lim_{n\to\infty} \cos\frac{1}{n} = \cos 0 = 1.$$

▶

◀ **18.9 Example.** You can compute the limit of any rational function of n by dividing numerator and denominator by the highest occurring power of n. Here is an example:

$$\lim_{n\to\infty} \frac{2n^2-1}{n^2+3n} = \lim_{n\to\infty} \frac{2-\left(\frac{1}{n}\right)^2}{1+3\cdot\frac{1}{n}} = \frac{2-0^2}{1+3\cdot 0^2} = 2$$

▶

◀ **18.10 Application of the Sandwich theorem.** We show that $\lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} = 0$ in two different ways.

Method 1: Since $\sqrt{n^2+1} > \sqrt{n^2} = n$ we have

$$0 < \frac{1}{\sqrt{n^2+1}} < \frac{1}{n}.$$

The sequences "0" and $\frac{1}{n}$ both go to zero, so the Sandwich theorem implies that $1/\sqrt{n^2+1}$ also goes to zero.

Method 2: Divide numerator and denominator both by n to get

$$a_n = \frac{1/n}{\sqrt{1+(1/n)^2}} = f\left(\frac{1}{n}\right), \quad \text{where } f(x) = \frac{x}{\sqrt{1+x^2}}.$$

Since $f(x)$ is continuous at $x = 0$, and since $\frac{1}{n} \to 0$ as $n \to \infty$, we conclude that a_n converges to 0. ▶

◀ **18.11** $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ **for any real number** x**.**

If $|x| \le 1$ then this is easy, for we would have $|x^n| \le 1$ for all $n \ge 0$ and thus

$$\left|\frac{x^n}{n!}\right| \le \frac{1}{n!} = \frac{1}{1\cdot\underbrace{2\cdot 3\cdots(n-1)\cdot n}_{n-1\text{ factors}}} \le \frac{1}{1\cdot\underbrace{2\cdot 2\cdots 2\cdot 2}_{n-1\text{ factors}}} = \frac{1}{2^{n-1}}$$

which shows that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$, by the Sandwich Theorem.

For arbitrary x you first choose an integer $N \ge 2x$. Then for all $n \ge N$ one has

$$\begin{aligned} \frac{x^n}{n!} &\le \frac{|x|\cdot|x|\cdots|x|\cdot|x|}{1\cdot 2\cdot 3\cdots n} && \text{use } |x| \le \frac{N}{2} \\ &\le \frac{N\cdot N\cdot N\cdots N\cdot N}{1\cdot 2\cdot 3\cdots n}\left(\frac{1}{2}\right)^n \end{aligned}$$

Split fraction into two parts, one containing the first N factors from both numerator and denominator, the other the remaining factors:

$$\underbrace{\frac{N}{1}\cdot\frac{N}{2}\cdot\frac{N}{3}\cdots\frac{N}{N}}_{=N^N/N!}\cdot\frac{N}{N+1}\cdots\frac{N}{n} = \frac{N^N}{N!}\cdot\underbrace{\frac{N}{N+1}}_{<1}\cdot\underbrace{\frac{N}{N+2}}_{<1}\cdots\underbrace{\frac{N}{n}}_{<1} \le \frac{N^N}{N!}$$

Hence we have

$$\left|\frac{x^n}{n!}\right| \le \frac{N^N}{N!}\left(\frac{1}{2}\right)^n$$

if $2|x| \le N$ and $n \ge N$.

Here everything is independent of n, except for the last factor $(\frac{1}{2})^n$ which causes the whole thing to converge to zero as $n \to \infty$. ▶

18.2. *Convergence of Taylor Series*

Definition 18.12. *Let $y = f(x)$ be some function defined on an interval $a < x < b$ containing* 0. *We say the Taylor series $T_\infty f(x)$ converges to $f(x)$ for a given x if*

$$\lim_{n\to\infty} T_n f(x) = f(x).$$

The most common notations which express this condition are

$$f(x) = \sum_{k=0}^{\infty} f^{(k)}(0)\frac{x^k}{k!}$$

or

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f^{(3)}(0)\frac{x^3}{3!} + \cdots$$

In both cases convergence justifies the idea that you can add infinitely many terms, as suggested by both notations.

There is no easy and general criterion which you could apply to a given function $f(x)$ that would tell you if its Taylor series converges for any particular x (except $x = 0$). On the other hand, it turns out that for many functions the Taylor series does converge to $f(x)$ for all x in some interval $-\rho < x < \rho$. In this section we will check this for two examples: the "geometric series" and the exponential function.

What does the Taylor series look like when you set $x = 0$?

Before we do the examples I want to make this point about how we're going to prove that the Taylor series converges: Instead of taking the limit of the $T_n f(x)$ as $n \to \infty$, you are usually better off looking at the remainder term. Since $T_n f(x) = f(x) - R_n f(x)$ you have

$$\lim_{n\to\infty} T_n f(x) = f(x) \iff \lim_{n\to\infty} R_n f(x) = 0$$

So: to check that the Taylor series of $f(x)$ converges to $f(x)$ we must show that the remainder term $R_n f(x)$ goes to zero as $n \to \infty$.

◀ **18.13 Example: The GEOMETRIC SERIES converges for** $-1 < x < 1$**.** If $f(x) = 1/(1-x)$ then by the formula for the Geometric Sum you have

$$\begin{aligned} f(x) &= \frac{1}{1-x} \\ &= \frac{1 - x^{n+1} + x^{n+1}}{1-x} \\ &= 1 + x + x^2 + \cdots + x^n + \frac{x^{n+1}}{1-x} \\ &= T_n f(x) + \frac{x^{n+1}}{1-x}. \end{aligned}$$

We are not dividing by zero since $|x| < 1$ so that $1 - x \ne 0$. The remainder term is

$$R_n f(x) = \frac{x^{n+1}}{1-x}.$$

Since $|x| < 1$ we have

($|x| < 1$ implies $1 - x > 0$)

$$\lim_{n\to\infty} |R_n f(x)| = \lim_{n\to\infty} \frac{|x|^{n+1}}{1-x} = \frac{\lim_{n\to\infty} |x|^{n+1}}{1-x} = \frac{0}{1-x} = 0.$$

Thus we have shown that the series converges for all $-1 < x < 1$, i.e.

$$\boxed{\frac{1}{1-x} = \lim_{n\to\infty} \left\{1 + x + x^2 + \cdots + x^n\right\} = 1 + x + x^2 + x^3 + \cdots}$$

▶

◀ **18.14 Convergence of the exponential Taylor series.** Let $f(x) = e^x$. It turns out the Taylor series of e^x converges to e^x for every value of x. Here's why: we had found that

$$T_n e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!},$$

and by Lagrange's formula the remainder is given by

$$R_n e^x = e^{\xi} \frac{x^{n+1}}{(n+1)!},$$

where ξ is some number between 0 and x.

If $x > 0$ then $0 < \xi < x$ so that $e^{\xi} \le e^x$; if $x < 0$ then $x < \xi < 0$ implies that $e^{\xi} < e^0 = 1$. Either way one has $e^{\xi} \le e^{|x|}$, and thus

$$|R_n e^x| \le e^{|x|} \frac{|x|^{n+1}}{(n+1)!}.$$

We have shown before that $\lim_{n\to\infty} x^{n+1}/(n+1)! = 0$, so the Sandwich theorem again implies that $\lim_{n\to\infty} |R_n e^x| = 0$.

Conclusion:

$$\boxed{e^x = \lim_{n\to\infty} \left\{1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\right\} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots}$$

▶

Do Taylor series always converge? And if the series of some function $y = f(x)$ converges, must it then converge to $f(x)$? Although the Taylor series of most function we run into converge to the functions itself, the following example shows that it doesn't have to be so.

◀ **18.15 The day that all Chemistry stood still.** The rate at which a chemical reaction "A→B" proceeds depends among other things on the temperature at which the reaction is taking place. This dependence is described by the ***Arrhenius law*** which states that the rate at which a reaction takes place is proportional to

$$f(T) = e^{-\frac{\Delta E}{kT}}$$

where ΔE is the amount of energy involved in each reaction, k is Boltzmann's constant, and T is the temperature in degrees Kelvin. If you ignore the constants ΔE and k (i.e. if you set them equal to one by choosing the right units) then the reaction rate is proportional to

$$f(T) = e^{-1/T}.$$

If you have to deal with reactions at low temperatures you might be inclined to replace this function with its Taylor series at $T = 0$, or at least the first non-zero term in this series. If you were to do this you'd be in for a surprise. To see what happens, let's look at the following function,

$$f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \le 0 \end{cases}$$

This function goes to zero ***very*** quickly as $x \to 0$. In fact one has

$$\lim_{x \searrow 0} \frac{f(x)}{x^n} = \lim_{x \searrow 0} \frac{e^{-1/x}}{x^n} = \lim_{t \to \infty} t^n e^{-t} = 0. \qquad \text{(set } t = 1/x\text{)}$$

This implies

$$f(x) = o(x^n) \quad (x \to 0)$$

for any $n = 1, 2, 3 \ldots$. As $x \to 0$, this function vanishes faster than any power of x.

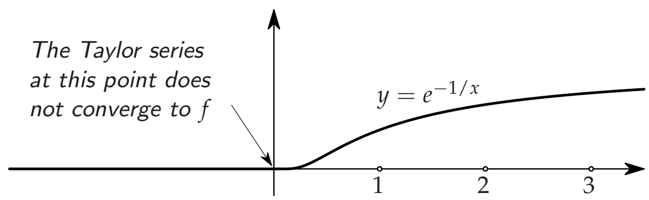


Figure 6: An innocent looking function with an unexpected Taylor series. See example 18.15 which shows that even when a Taylor series of some function f converges you can't be sure that it converges to f – it could converge to a different function.

If you try to compute the Taylor series of f you need its derivatives at $x = 0$ of all orders. These can be computed (not easily), and the result turns out to be that ***all derivatives of*** f ***vanish at*** $x = 0$***,***

$$f(0) = f'(0) = f''(0) = f^{(3)}(0) = \cdots = 0.$$

The Taylor series of f is therefore

$$T_\infty f(x) = 0 + 0 \cdot x + 0 \cdot \frac{x^2}{2!} + 0 \cdot \frac{x^3}{3!} + \cdots = 0.$$

Clearly this series converges (all terms are zero, after all), but instead of converging to the function $f(x)$ we started with, it converges to the function $g(x) = 0$.

What does this mean for the chemical reaction rates and Arrhenius' law? We wanted to "simplify" the Arrhenius law by computing the Taylor series of $f(T)$ at $T = 0$, but we have just seen that all terms in this series are zero. Therefore replacing the Arrhenius reaction rate by its Taylor series at $T = 0$ has the effect of setting all reaction rates equal to zero.

▶

19. Leibniz' formulas for $\ln 2$ and $\pi/4$

Leibniz showed that

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \ln 2$$

and

$$\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \frac{\pi}{4}$$

Both formulas arise by setting $x = 1$ in the Taylor series for

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} - \cdots$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} - \cdots$$

This is only justified if you show that the series actually converge, which we'll do here, at least for the first of these two formulas. The proof of the second is similar. The following is not Leibniz' original proof.

You begin with the geometric sum

$$1 - x + x^2 - x^3 + \cdots + (-1)^n x^n = \frac{1}{1+x} + \frac{(-1)^{n+1}x^{n+1}}{1+x}$$

Then you integrate both sides from $x = 0$ to $x = 1$ and get

$$\begin{aligned}\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^n \frac{1}{n+1} &= \int_0^1 \frac{\mathrm{d}x}{1+x} + (-1)^{n+1} \int_0^1 \frac{x^{n+1}\mathrm{d}x}{1+x} \\ &= \ln 2 + (-1)^{n+1} \int_0^1 \frac{x^{n+1}\mathrm{d}x}{1+x}\end{aligned}$$

(Use $\int_0^1 x^k \mathrm{d}x = \frac{1}{k+1}$.) Instead of computing the last integral you estimate it by saying

$$0 \le \frac{x^{n+1}}{1+x} \le x^{n+1} \implies 0 \le \int_0^1 \frac{x^{n+1}\mathrm{d}x}{1+x} \le \int_0^1 x^{n+1}\mathrm{d}x = \frac{1}{n+2}$$

Hence

$$\lim_{n\to\infty} (-1)^{n+1} \int_0^1 \frac{x^{n+1}\mathrm{d}x}{1+x} = 0,$$

and we get

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^n \frac{1}{n+1} &= \ln 2 + \lim_{n\to\infty} (-1)^{n+1} \int_0^1 \frac{x^{n+1}\mathrm{d}x}{1+x} \\ &= \ln 2.\end{aligned}$$

20. **More proofs**

20.1. *Proof of Lagrange's formula*

We give the proof of Theorem 15.1 based on an idea related to the mean value theorem. We prove the formula for the case where $a = 0$. Once this is done you can apply the special case for the function $f^*(x) = f(a + x)$ to deduce the general case.

We have to distinguish the cases $x > 0$ and $x < 0$. We assume $x > 0$ (and leave the case $x < 0$ to you). Consider the function

$$g(t) = f(0) + f'(0)t + \frac{f''(0)}{2}t^2 + \cdots + \frac{f^{(n)}(0)}{n!}t^n + Kt^{n+1} - f(t),$$

where

$$K \stackrel{\text{def}}{=} -\frac{f(0) + f'(0)x + \cdots + \frac{f^{(n)}(0)}{n!}x^n - f(x)}{x^{n+1}} \tag{17}$$

We have chosen this particular K to be sure that

$$g(x) = 0.$$

Just by computing the derivatives you also find that

$$g(0) = g'(0) = g''(0) = \cdots = g^{(n)}(0) = 0,$$

while

$$g^{(n+1)}(t) = (n+1)!K - f^{(n+1)}(t). \tag{18}$$

We now apply *Rolle's Theorem* n times:

- since $g(t)$ vanishes at $t = 0$ and at $t = x$ there exists an x_1 with $0 < x_1 < x$ such that $g'(x_1) = 0$

- since $g'(t)$ vanishes at $t = 0$ and at $t = x_1$ there exists an x_2 with $0 < x_2 < x_1$ such that $g'(x_2) = 0$
- since $g''(t)$ vanishes at $t = 0$ and at $t = x_2$ there exists an x_3 with $0 < x_3 < x_2$ such that $g''(x_3) = 0$

  $\vdots$
- since $g^{(n)}(t)$ vanishes at $t = 0$ and at $t = x_n$ there exists an x_{n+1} with $0 < x_{n+1} < x_n$ such that $g^{(n)}(x_{n+1}) = 0$.

We now set $\xi = x_{n+1}$, and observe that we have shown that $g^{(n+1)}(\xi) = 0$, so by (18) we get

$$K = \frac{f^{(n+1)}(\xi)}{(n+1)!}.$$

Apply that to (17) and you finally get

$$f(x) = f(0) + f'(0)x + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}.$$

20.2. *Proof of Theorem 17.6*

Lemma 20.1. *If $h(x)$ is a k times differentiable function on some interval containing 0, and if for some integer $k < n$ one has $h(0) = h'(0) = \cdots = h^{(k-1)}(0) = 0$, then*

$$\lim_{x\to 0} \frac{h(x)}{x^k} = \frac{h^{(k)}(0)}{k!}. \tag{19}$$

Proof. Just apply l'Hopital's rule k times. You get

$$\lim_{x\to 0} \frac{h(x)}{x^k} \overset{=\frac{0}{0}}{=} \lim_{x\to 0} \frac{h'(x)}{kx^{k-1}} \overset{=\frac{0}{0}}{=} \lim_{x\to 0} \frac{h^{(2)}(x)}{k(k-1)x^{k-2}} \overset{=\frac{0}{0}}{=} \cdots$$

$$\cdots = \lim_{x\to 0} \frac{h^{(k-1)}(x)}{k(k-1)\cdots 2x^1} \overset{=\frac{0}{0}}{=} \frac{h^{(k)}(0)}{k(k-1)\cdots 2\cdot 1}$$

□

First define the function $h(x) = f(x) - g(x)$. If $f(x)$ and $g(x)$ are n times differentiable, then so is $h(x)$.

The condition $T_n f(x) = T_n g(x)$ means that

$$f(0) = g(0), \quad f'(0) = g'(0), \quad \ldots, \quad f^{(n)}(0) = g^{(n)}(0),$$

which says, in terms of $h(x)$,

$$h(0) = h'(0) = h''(0) = \cdots = h^{(n)}(0) = 0, \tag{†}$$

i.e.

$$T_n h(x) = 0.$$

We now prove the first pat of the theorem: suppose $f(x)$ and $g(x)$ have the same nth degree Taylor polynomial. Then we have just argued that $T_n h(x) = 0$, and Lemma 20.1 (with $k = n$) says that $\lim_{x\to 0} h(x)/x^n = 0$, as claimed.

To conclude we show the converse also holds. So suppose that $\lim_{x\to 0} h(x)/x^n = 0$. We'll show that (†) follows. If (†) were not true then there would be a smallest integer $k \le n$ such that

$$h(0) = h'(0) = h''(0) = \cdots = h^{(k-1)}(0) = 0, \text{ but } h^{(k)}(0) \neq 0.$$

This runs into the following contradiction with Lemma 20.1

$$0 \neq \frac{h^{(k)}(0)}{k!} = \lim_{x\to 0}\frac{h(x)}{x^k} = \lim_{x\to 0}\frac{h(x)}{x^n}\cdot\frac{x^n}{x^k} = 0\cdot\underbrace{\lim_{x\to 0} x^{n-k}}_{(*)} = 0.$$

Here the limit $(*)$ exists because $n \geq k$.

21. PROBLEMS

Taylor's formula

151. Find a second order polynomial (i.e. a quadratic function) $Q(x)$ such that $Q(7) = 43, Q'(7) = 19, Q''(7) = 11$.

152. A fourth order polynomial $P(x)$ satisfies $P(0) = 1, P'(0) = -3, P''(0) = -8, P'''(0) = 24$. Find $P(x)$.

153. Let $f(x) = \sqrt{x+25}$. Find the polynomial $P(x)$ of degree three such that $P^{(k)}(0) = f^{(k)}(0)$ for $k = 0, 1, 2, 3$.

154. Let $f(x) = 1 + x - x^2 - x^3$. Compute and graph $T_0f(x)$, $T_1f(x)$, $T_2f(x)$, $T_3f(x)$, and $T_4f(x)$, as well as $f(x)$ itself (so, for each of these functions find where they are positive or negative, where they are increasing/decreasing, and find the inflection points on their graph.)

155. Find $T_3 \sin x$ and $T_5 \sin x$.

Graph $T_3 \sin x$ and $T_5 \sin x$ as well as $y = \sin x$ in one picture. (As before, find where these functions are positive or negative, where they are increasing/decreasing, and find the inflection points on their graph. This problem can&should be done without a graphing calculator.)

Compute $T_0^a f(x)$, $T_1^a f(x)$ and $T_2^a f(x)$ for the following functions.

156. $f(x) = x^3$, $a = 0$; then for $a = 1$ and $a = 2$.

157. $f(x) = \dfrac{1}{x}$, $a = 1$. Also do $a = 2$.

158. $f(x) = \sqrt{x}$, $a = 1$.

159. $f(x) = \ln x$, $a = 1$. Also $a = e^2$.

160. $f(x) = \ln\sqrt{x}$, $a = 1$.

161. $f(x) = \sin(2x)$, $a = 0$, also $a = \pi/4$.

162. $f(x) = \cos(x)$, $a = \pi$.

163. $f(x) = (x-1)^2$, $a = 0$, and also $a = 1$.

164. $f(x) = \dfrac{1}{e^x}$, $a = 0$.

165. Find the nth degree Taylor polynomial $T_n^a f(x)$ of the following functions $f(x)$

n	a	$f(x)$
2	0	$1 + x - x^3$
3	0	$1 + x - x^3$
25	0	$1 + x - x^3$
25	2	$1 + x - x^3$
2	1	$1 + x - x^3$
1	1	x^2
2	1	x^2
5	1	$1/x$
5	0	$1/(1+x)$
3	0	$1/(1-3x+2x^2)$

For which of these combinations $(n, a, f(x))$ is $T_n^a f(x)$ the same as $f(x)$?

* * *

Compute the Taylor series $T_\infty f(t)$ for the following functions (α is a constant). Give a formula for the coefficient of x^n in $T_\infty f(t)$. (*Be smart. Remember properties of the logarithm, definitions of the hyperbolic functions, partial fraction decomposition.*)

166. e^t

167. $e^{\alpha t}$

168. $\sin(3t)$

169. $\sinh t$

170. $\cosh t$

171. $\dfrac{1}{1+2t}$

172. $\dfrac{3}{(2-t)^2}$

173. $\ln(1+t)$

174. $\ln(2+2t)$

175. $\ln\sqrt{1+t}$

176. $\ln(1+2t)$

177. $\ln\sqrt{\dfrac{1+t}{1-t}}$

178. $\dfrac{1}{1-t^2}$ [hint:PFD!]

179. $\dfrac{t}{1-t^2}$

180. $\sin t + \cos t$

181. $2\sin t \cos t$

182. $\tan t$ (3 terms only)

183. $1 + t^2 - \dfrac{2}{3}t^4$

184. $(1+t)^5$

185. $\sqrt[3]{1+t}$

186. Compute the Taylor series of the following two functions

$$f(x) = \sin a \cos x + \cos a \sin x$$

and

$$g(x) = \sin(a+x)$$

where a is a constant.

187. Compute the Taylor series of the following two functions

$$h(x) = \cos a \cos x - \sin a \sin x$$

and

$$k(x) = \cos(a+x)$$

where a is a constant.

188. The following questions ask you to rediscover ***Newton's Binomial Formula***, which is just the Taylor series for $(1+x)^n$. Newton's formula generalizes the formulas for $(a+b)^2$, $(a+b)^3$, etc that you get using Pascal's triangle. It allows non integer exponents which are allowed to be either positive and negative. Reread section 13 before doing this problem.

(a) Find the Taylor series of $f(x) = \sqrt{1+x}$ $(= (1+x)^{1/2})$

(b) Find the coefficient of x^4 in the Taylor series of $f(x) = (1+x)^\pi$ (don't do the arithmetic!)

(c) Let p be any real number. Compute the terms of degree 0, 1, 2 and 3 of the Taylor series of

$$f(x) = (1+x)^p$$

(d) Compute the Taylor polynomial of degree n of $f(x) = (1+x)^p$.

(e) Write the result of (d) for the exponents $p = 2, 3$ and also, for $p = -1, -2, -3$ and finally for $p = \frac{1}{2}$. The ***Binomial Theorem*** states that this series converges when $|x| < 1$.

Lagrange's formula for the remainder

189. Find the fourth degree Taylor polynomial $T_4\{\cos x\}$ for the function $f(x) = \cos x$ and estimate the error $|\cos x - P_4(x)|$ for $|x| < 1$.

190. Find the 4th degree Taylor polynomial $T_4\{\sin x\}$ for the function $f(x) = \sin x$. Estimate the error $|\sin x - T_4\{\sin x\}|$ for $|x| < 1$.

191. (*Computing the cube root of* 9) The cube root of $8 = 2 \times 2 \times 2$ is easy, and 9 is only one more than 8. So you could try to compute $\sqrt[3]{9}$ by viewing it as $\sqrt[3]{8+1}$.

(a) Let $f(x) = \sqrt[3]{8+x}$. Find $T_2 f(x)$, and estimate the error $|\sqrt[3]{9} - T_2 f(1)|$.

(b) Repeat part (***i***) for "$n = 3$", i.e. compute $T_3 f(x)$ and estimate $|\sqrt[3]{9} - T_3 f(1)|$.

(c) Follow the method of problem 191 to compute $\sqrt{10}$:

(d) Use Taylor's formula with $f(x) = \sqrt{9+x}$, $n = 1$, to calculate $\sqrt{10}$ approximately. Show that the error is less than $1/216$.

(e) Repeat with $n = 2$. Show that the error is less than 0.0003.

192. Find the eighth degree Taylor polynomial $T_8 f(x)$ about the point 0 for the function $f(x) = \cos x$ and estimate the error $|\cos x - T_8 f(x)|$ for $|x| < 1$.

Now find the ninth degree Taylor polynomial, and estimate $|\cos x - T_9 f(x)|$ for $|x| \le 1$.

Little-oh and manipulating Taylor polynomials

Are the following statements *True or False?* In mathematics this means that you should either *show that the statement always holds* or else *give at least one counterexample*, thereby showing that the statement is not always true.

193. $(1+x^2)^2 - 1 = o(x)$?

194. $(1+x^2)^2 - 1 = o(x^2)$?

195. $\sqrt{1+x} - \sqrt{1-x} = o(x)$?

196. $o(x) + o(x) = o(x)$?

197. $o(x) - o(x) = o(x)$?

198. $o(x) \cdot o(x) = o(x)$?

199. $o(x^2) + o(x) = o(x^2)$?

200. $o(x^2) - o(x^2) = o(x^3)$?

201. $o(2x) = o(x)$?

202. $o(x) + o(x^2) = o(x)$?

203. $o(x) + o(x^2) = o(x^2)$?

204. $1 - \cos x = o(x)$?

205. For which value(s) of k is $\sqrt{1+x^2} = 1 + o(x^k)$ (as $x \to 0$)?

For which value(s) of k is $\sqrt[3]{1+x^2} = 1 + o(x^k)$ (as $x \to 0$)?

For which value(s) of k is $1 - \cos x^2 = o(x^k)$ (as $x \to 0$)?

206. Let g_n be the coefficient of x^n in the Taylor series of the function

$$g(x) = \frac{1}{2 - 3x + x^2}$$

(a) Compute g_0 and g_1 directly from the definition of the Taylor series.

(b) Show that the recursion relation $g_n = 3g_{n-1} - 2g_{n-2}$ holds for all $n \ge 2$.

(c) Compute g_2, g_3, g_4, g_5.

(d) Using a partial fraction decomposition of $g(x)$ find a formula for $g^{(n)}(0)$, and hence for g_n.

207. Answer the same questions as in the previous problem, for the functions

$$h(x) = \frac{x}{2 - 3x + x^2}$$

and

$$k(x) = \frac{2 - x}{2 - 3x + x^2}.$$

208. Let h_n be the coefficient of x^n in the Taylor series of

$$h(x) = \frac{1+x}{2 - 5x + 2x^2}.$$

(a) Find a recursion relation for the h_n.

(b) Compute $h_0, h_1, \ldots, h_8$.

(c) Derive a formula for h_n valid for all n, by using a partial fraction expansion.

(d) Is h_{2009} more or less than a million? A billion?

Find the Taylor series for the following functions, by substituting, adding, multiplying, applying long division and/or differentiating known series for $\frac{1}{1+x}$, e^x, $\sin x$, $\cos x$ *and* $\ln x$.

209. e^{at}

210. e^{1+t}

211. e^{-t^2}

212. $\dfrac{1+t}{1-t}$

213. $\dfrac{1}{1+2t}$

214. $\dfrac{\ln(1+x)}{x}$

215. $\dfrac{e^t}{1-t}$

216. $\dfrac{1}{\sqrt{1-t}}$

217. $\dfrac{1}{\sqrt{1-t^2}}$ (recommendation: use the answer to problem 216)

218. $\arcsin t$
(use problem 216 again)

219. Compute $T_4[e^{-t}\cos t]$ (See example 17.9.)

220. $T_4[e^{-t}\sin 2t]$

221. $\dfrac{1}{2-t-t^2}$

222. $\sqrt[3]{1+2t+t^2}$

223. $\ln(1-t^2)$

224. $\sin t \cos t$

Limits of Sequences

Compute the following limits:

225. $\lim_{n\to\infty} \dfrac{n}{2n-3}$

226. $\lim_{n\to\infty} \dfrac{n^2}{2n-3}$

227. $\lim_{n\to\infty} \dfrac{n^2}{2n^2+n-3}$

228. $\lim_{n\to\infty} \dfrac{2^n+1}{1-2^n}$

229. $\lim_{n\to\infty} \dfrac{2^n+1}{1-3^n}$

230. $\lim_{n\to\infty} \dfrac{e^n+1}{1-2^n}$

231. $\lim_{n\to\infty} \dfrac{n^2}{(1.01)^n}$

232. $\lim_{n\to\infty} \dfrac{1000^n}{n!}$

233. $\lim_{n\to\infty} \dfrac{n!+1}{(n+1)!}$

234. Compute $\lim_{n\to\infty} \dfrac{(n!)^2}{(2n)!}$ [Hint: write out all the factors in numerator and denominator.]

235. Let f_n be the nth Fibonacci number. Compute

$$\lim_{n\to\infty} \frac{f_n}{f_{n-1}}$$

Convergence of Taylor Series

236. Prove that the Taylor series for $f(x) = \cos x$ converges to $f(x)$ for all real numbers x (by showing that the remainder term goes to zero as $n \to \infty$).

237. Prove that the Taylor series for $g(x) = \sin(2x)$ converges to $g(x)$ for all real numbers x .

238. Prove that the Taylor series for $h(x) = \cosh(x)$ converges to $h(x)$ for all real numbers x .

239. Prove that the Taylor series for $k(x) = e^{2x+3}$ converges to $k(x)$ for all real numbers x .

240. Prove that the Taylor series for $\ell(x) = \cos\left(x - \frac{\pi}{7}\right)$ converges to $\ell(x)$ for all real numbers x.

241. If the Taylor series of a function $y = f(x)$ converges for all x, does it have to converge to $f(x)$, or could it converge to some other function?

242. For which real numbers x does the Taylor series of $f(x) = \dfrac{1}{1-x}$ converge to $f(x)$?

243. For which real numbers x does the Taylor series of $f(x) = \dfrac{1}{1-x^2}$ converge to $f(x)$? (hint: a substitution may help.)

244. For which real numbers x does the Taylor series of $f(x) = \dfrac{1}{1+x^2}$ converge to $f(x)$?

245. For which real numbers x does the Taylor series of $f(x) = \dfrac{1}{3+2x}$ converge to $f(x)$?

246. For which real numbers x does the Taylor series of $f(x) = \dfrac{1}{2-x-x^2}$ converge to $f(x)$? (hint: use PFD and the geoemtric series to find the remainder term.)

247. Show that the Taylor series for $f(x) = \ln(1+x)$ converges when $-1 < x < 1$ by integrating the Geometric Series

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots + (-1)^n t^n + (-1)^{n+1}\frac{t^{n+1}}{1+t}$$

from $t = 0$ to $t = x$. (See §19.)

248. Show that the Taylor series for $f(x) = e^{-x^2}$ converges for all real numbers x. (Set $t = -x^2$ in the Taylor series with remainder for e^t.)

249. Show that the Taylor series for $f(x) = \sin(x^4)$ converges for all real numbers x. (Set $t = x^4$ in the Taylor series with remainder for $\sin t$.)

250. Show that the Taylor series for $f(x) = 1/(1+x^3)$ converges whenever $-1 < x < 1$ (Use the GEOMETRIC SERIES.)

251. For which x does the Taylor series of $f(x) = 2/(1+4x^2)$ converge? (Again, use the GEOMETRIC SERIES.)

252. The error function from statistics is defined by

$$\operatorname{erf}(x) = \frac{1}{\sqrt{\pi}}\int_0^x e^{-t^2/2}\,\mathrm{d}t$$

(a) Find the Taylor series of the error function from the Taylor series of $f(r) = e^r$ (set $r = -t^2/2$ and integrate).

(b) Estimate the error term and show that the Taylor series of the error function converges for all real x.

253. Prove Leibniz' formula for $\frac{\pi}{4}$ by mimicking the proof in section 19. Specifically, find a formula for the remainder in :

$$\frac{1}{1+t^2} = 1 - t^2 + \cdots + (-1)^n t^{2n} + R_{2n}(t)$$

and integrate this from $t = 0$ to $t = 1$.

Approximating integrals

254. (a) Compute $T_2\{\sin t\}$ and give an upper bound for $R_2\{\sin t\}$ for $0 \le t \le 0.5$

(b) Use part (a) to approximate $\int_0^{0.5} \sin(x^2)\,dx$, and give an upper bound for the error in your approximation.

255. Approximate $\int_0^{0.1} \arctan x\,dx$ and estimate the error in your approximation by analyzing $T_2 f(t)$ and $R_2 f(t)$ where $f(t) = \arctan t$.

256. Approximate $\int_0^{0.1} x^2 e^{-x^2}\,dx$ and estimate the error in your approximation by analyzing $T_3 f(t)$ and $R_3 f(t)$ where $f(t) = te^{-t}$.

257. Estimate $\int_0^{0.5} \sqrt{1+x^4}\,dx$ with an error of less than 10^{-4}.

258. Estimate $\int_0^{0.1} \arctan x\,dx$ with an error of less than 0.001.

Complex Numbers and the Complex Exponential

22. Complex numbers

The equation $x^2 + 1 = 0$ has no solutions, because for any real number x the square x^2 is nonnegative, and so $x^2 + 1$ can never be less than 1. In spite of this it turns out to be very useful to ***assume*** that there is a number i for which one has

(20) $$i^2 = -1.$$

Any ***complex number*** is then an expression of the form $a + bi$, where a and b are old-fashioned real numbers. The number a is called the ***real part*** of $a + bi$, and b is called its ***imaginary part.***

Traditionally the letters z and w are used to stand for complex numbers.

Since any complex number is specified by two real numbers one can visualize them by plotting a point with coordinates (a, b) in the plane for a complex number $a + bi$. The plane in which one plot these complex numbers is called the Complex plane, or Argand plane.

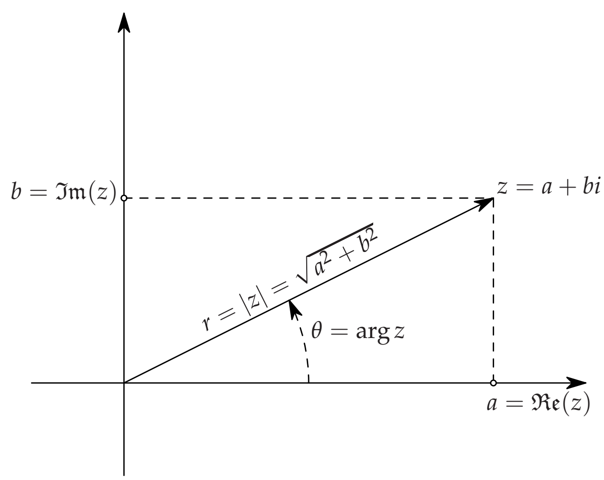


Figure 7: A complex number.

You can add, multiply and divide complex numbers. Here's how:

To add (subtract) $z = a + bi$ and $w = c + di$

$$z + w = (a + bi) + (c + di) = (a + c) + (b + d)i,$$
$$z - w = (a + bi) - (c + di) = (a - c) + (b - d)i.$$

To multiply z and w proceed as follows:

$$\begin{aligned} zw &= (a + bi)(c + di) \\ &= a(c + di) + bi(c + di) \\ &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

where we have use the defining property $i^2 = -1$ to get rid of i^2.

To divide two complex numbers one always uses the following trick.

$$\begin{aligned}\frac{a+bi}{c+di} &= \frac{a+bi}{c+di}\cdot\frac{c-di}{c-di} \\ &= \frac{(a+bi)(c-di)}{(c+di)(c-di)}\end{aligned}$$

Now

$$(c+di)(c-di) = c^2 - (di)^2 = c^2 - d^2i^2 = c^2 + d^2,$$

so

$$\begin{aligned}\frac{a+bi}{c+di} &= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2} \\ &= \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}\, i\end{aligned}$$

Obviously you do not want to memorize this formula: instead you remember the trick, i.e. to divide $c+di$ into $a+bi$ you multiply numerator and denominator with $c-di$.

For any complex number $w = c+di$ the number $c-di$ is called its ***complex conjugate.*** Notation:

$$w = c+di, \quad \bar{w} = c-di.$$

A frequently used property of the complex conjugate is the following formula

(21) $$w\bar{w} = (c+di)(c-di) = c^2 - (di)^2 = c^2 + d^2.$$

The following notation is used for the ***real and imaginary parts*** of a complex number z. If $z = a+bi$ then

$$a = \text{the Real Part of } z = \mathfrak{Re}(z), \qquad b = \text{the Imaginary Part of } z = \mathfrak{Im}(z).$$

Note that both $\mathfrak{Re}z$ and $\mathfrak{Im}z$ are real numbers. A common mistake is to say that $\mathfrak{Im}z = bi$. The "i" should **not** be there.

23. Argument and Absolute Value

For any given complex number $z = a+bi$ one defines the ***absolute value*** or ***modulus*** to be

$$|z| = \sqrt{a^2+b^2},$$

so $|z|$ is the distance from the origin to the point z in the complex plane (see figure 7).

The angle θ is called the ***argument*** of the complex number z. Notation:

$$\arg z = \theta.$$

The argument is defined in an ambiguous way: it is only defined up to a multiple of 2π. E.g. the argument of -1 could be π, or $-\pi$, or 3π, or, etc. In general one says $\arg(-1) = \pi + 2k\pi$, where k may be any integer.

From trigonometry one sees that for any complex number $z = a+bi$ one has

$$a = |z|\cos\theta, \text{ and } b = |z|\sin\theta,$$

so that

$$|z| = |z|\cos\theta + i|z|\sin\theta = |z|\big(\cos\theta + i\sin\theta\big).$$

and

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{b}{a}.$$

◀ **23.1 Example.** Find argument and absolute value of $z = 2+i$.

Solution: $|z| = \sqrt{2^2 + 1^2} = \sqrt{5}$. z lies in the first quadrant so its argument θ is an angle between 0 and $\pi/2$. From $\tan\theta = \frac{1}{2}$ we then conclude $\arg(2+i) = \theta = \arctan\dfrac{1}{2}$. ▶

24. **Geometry of Arithmetic**

Since we can picture complex numbers as points in the complex plane, we can also try to visualize the arithmetic operations "addition" and "multiplication." To add z and w one

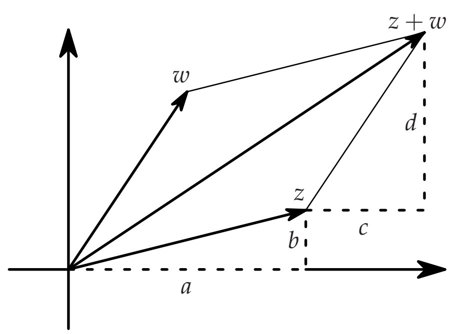


Figure 8: Addition of $z = a + bi$ and $w = c + di$

forms the parallelogram with the origin, z and w as vertices. The fourth vertex then is $z + w$. See figure 8.

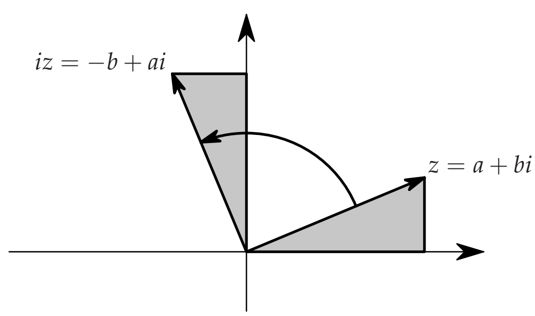


Figure 9: Multiplication of $a + bi$ by i.

To understand multiplication we first look at multiplication with i. If $z = a + bi$ then

$$iz = i(a+bi) = ia + bi^2 = ai - b = -b + ai.$$

Thus, to form iz from the complex number z one rotates z counterclockwise by 90 degrees. See figure 9.

If a is any real number, then multiplication of $w = c + di$ by a gives

$$aw = ac + adi,$$

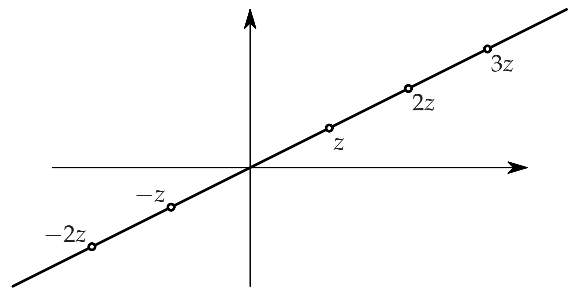


Figure 10: Multiplication of a real and a complex number

so aw points in the same direction, but is a times as far away from the origin. If $a < 0$ then aw points in the opposite direction. See figure 10.

Next, to multiply $z = a + bi$ and $w = c + di$ we write the product as

$$zw = (a + bi)w = aw + biw.$$

Figure 11 shows $a + bi$ on the right. On the left, the complex number w was first drawn,

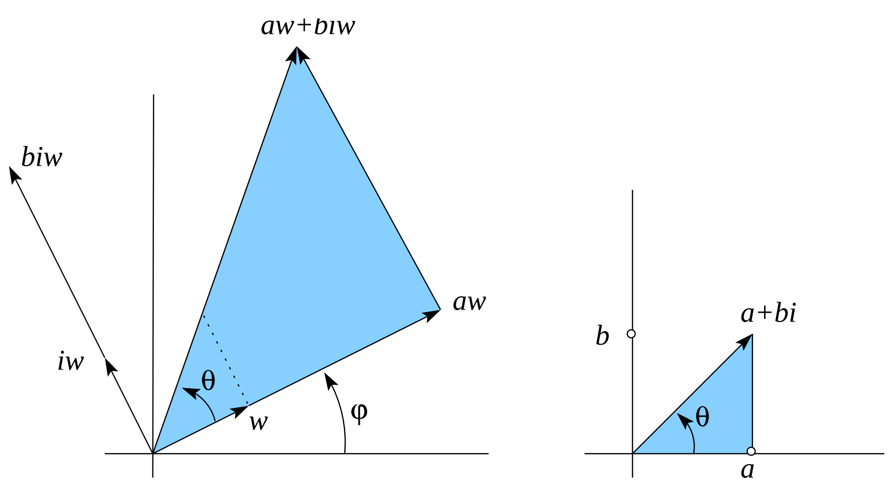


Figure 11: Multiplication of two complex numbers

then aw was drawn. Subsequently iw and biw were constructed, and finally $zw = aw + biw$ was drawn by adding aw and biw.

One sees from figure 11 that since iw is perpendicular to w, the line segment from 0 to biw is perpendicular to the segment from 0 to aw. Therefore the larger shaded triangle on the left is a right triangle. The length of the adjacent side is $a|w|$, and the length of the opposite side is $b|w|$. The ratio of these two lengths is $a : b$, which is the same as for the shaded right triangle on the right, so we conclude that these two triangles are similar.

The triangle on the left is $|w|$ times as large as the triangle on the right. The two angles marked θ are equal.

Since $|zw|$ is the length of the hypothenuse of the shaded triangle on the left, it is $|w|$ times the hypothenuse of the triangle on the right, i.e. $|zw| = |w| \cdot |z|$.

The argument of zw is the angle $\theta + \varphi$; since $\theta = \arg z$ and $\varphi = \arg w$ we get the following two formulas

$$|zw| = |z| \cdot |w| \tag{22}$$

$$\arg(zw) = \arg z + \arg w, \tag{23}$$

in other words,

when you multiply complex numbers, their lengths get multiplied and their arguments get added.

25. **Applications in Trigonometry**

25.1. *Unit length complex numbers*

For any θ the number $z = \cos\theta + i\sin\theta$ has length 1: it lies on the unit circle. Its argument is $\arg z = \theta$. Conversely, any complex number on the unit circle is of the form $\cos\phi + i\sin\phi$, where ϕ is its argument.

25.2. *The Addition Formulas for Sine & Cosine*

For any two angles θ and ϕ one can multiply $z = \cos\theta + i\sin\theta$ and $w = \cos\phi + i\sin\phi$. The product zw is a complex number of absolute value $|zw| = |z| \cdot |w| = 1 \cdot 1$, and with argument $\arg(zw) = \arg z + \arg w = \theta + \phi$. So zw lies on the unit circle and must be $\cos(\theta + \phi) + i\sin(\theta + \phi)$. Thus we have

$$(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) = \cos(\theta + \phi) + i\sin(\theta + \phi). \tag{24}$$

By multiplying out the Left Hand Side we get

$$\begin{aligned}(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) &= \cos\theta\cos\phi - \sin\theta\sin\phi \\ &\quad + i(\sin\theta\cos\phi + \cos\theta\sin\phi).\end{aligned} \tag{25}$$

Compare the Right Hand Sides of (24) and (25), and you get the addition formulas for Sine and Cosine:

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$
$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

25.3. *De Moivre's formula*

For any complex number z the argument of its square z^2 is $\arg(z^2) = \arg(z \cdot z) = \arg z + \arg z = 2\arg z$. The argument of its cube is $\arg z^3 = \arg(z \cdot z^2) = \arg(z) + \arg z^2 = \arg z + 2\arg z = 3\arg z$. Continuing like this one finds that

$$\arg z^n = n\ \arg z \tag{26}$$

for any integer n.

Applying this to $z = \cos\theta + i\sin\theta$ you find that z^n is a number with absolute value $|z^n| = |z|^n = 1^n = 1$, and argument $n\arg z = n\theta$. Hence $z^n = \cos n\theta + i\sin n\theta$. So we have found

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta. \tag{27}$$

This is ***de Moivre's formula.***

For instance, for $n = 2$ this tells us that

$$\cos 2\theta + i\sin 2\theta = (\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta.$$

Comparing real and imaginary parts on left and right hand sides this gives you the double angle formulas $\cos\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$.

For $n = 3$ you get, using the *Binomial Theorem*, or Pascal's triangle,

$$\begin{aligned}(\cos\theta + i\sin\theta)^3 &= \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta \\ &= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)\end{aligned}$$

so that

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

and

$$\sin 3\theta = \cos^2\theta\sin\theta - \sin^3\theta.$$

In this way it is fairly easy to write down similar formulas for $\sin 4\theta$, $\sin 5\theta$, etc....

26. Calculus of complex valued functions

A ***complex valued function*** on some interval $I = (a, b) \subseteq \mathbb{R}$ is a function $f : I \to \mathbb{C}$. Such a function can be written as in terms of its real and imaginary parts,

$$f(x) = u(x) + iv(x), \tag{28}$$

in which $u, v : I \to \mathbb{R}$ are two real valued functions.

One defines limits of complex valued functions in terms of limits of their real and imaginary parts. Thus we say that

$$\lim_{x\to x_0} f(x) = L$$

if $f(x) = u(x) + iv(x)$, $L = A + iB$, and both

$$\lim_{x\to x_0} u(x) = A \text{ and } \lim_{x\to x_1} v(x) = B$$

hold. From this definition one can prove that the usual limit theorems also apply to complex valued functions.

Theorem 26.1. *If $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = M$, then one has*

$$\begin{aligned}\lim_{x\to x_0} f(x) \pm g(x) &= L \pm M, \\ \lim_{x\to x_0} f(x)g(x) &= LM, \\ \lim_{x\to x_0} \frac{f(x)}{g(x)} &= \frac{L}{M}, \text{ provided } M \neq 0.\end{aligned}$$

The ***derivative*** of a complex valued function $f(x) = u(x) + iv(x)$ is defined by simply differentiating its real and imaginary parts:

$$f'(x) = u'(x) + iv'(x). \tag{29}$$

Again, one finds that the sum,product and quotient rules also hold for complex valued functions.

Theorem 26.2. *If $f, g : I \to \mathbb{C}$ are complex valued functions which are differentiable at some $x_0 \in I$, then the functions $f \pm g$, fg and f/g are differentiable (assuming $g(x_0) \neq 0$ in the case of the quotient.) One has*

$$\begin{aligned}(f \pm g)'(x_0) &= f'(x_0) \pm g'(x_0) \\ (fg)'(x_0) &= f'(x_0)g(x_0) + f(x_0)g'(x_0) \\ \left(\frac{f}{g}\right)'(x_0) &= \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}\end{aligned}$$

Note that the chain rule does not appear in this list! See problem 287 for more about the chain rule.

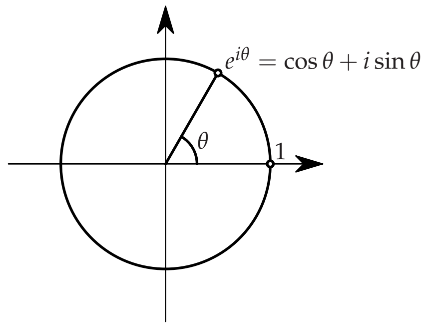


Figure 12: Euler's definition of $e^{i\theta}$

27. The Complex Exponential Function

We finally give a definition of e^{a+bi}. First we consider the case $a = 0$:

Definition 27.1. *For any real number t we set*

$$e^{it} = \cos t + i \sin t.$$

See Figure 12.

◀ **27.2 Example.** $e^{\pi i} = \cos\pi + i\sin\pi = -1$. This leads to Euler's famous formula

$$e^{\pi i} + 1 = 0,$$

which combines the five most basic quantities in mathematics: e, π, i, 1, and 0. ▶

Reasons why the definition 27.1 seems a good definition.

Reason 1. We haven't defined e^{it} before and we can do anything we like.

Reason 2. Substitute it in the Taylor series for e^x:

$$\begin{aligned} e^{it} &= 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \cdots \\ &= 1 + it - \frac{t^2}{2!} - i\frac{t^3}{3!} + \frac{t^4}{4!} + i\frac{t^5}{5!} - \cdots \\ &= 1 - t^2/2! + t^4/4! - \cdots \\ &\qquad + i\big(t - t^3/3! + t^5/5! - \cdots\big) \\ &= \cos t + i \sin t. \end{aligned}$$

This is not a proof, because before we had only proved the convergence of the Taylor series for e^x if x was a real number, and here we have pretended that the series is also good if you substitute $x = it$.

Reason 3. As a function of t the definition 27.1 gives us the correct derivative. Namely, using the chain rule (i.e. pretending it still applies for complex functions) we would get

$$\frac{de^{it}}{dt} = ie^{it}.$$

Indeed, this is correct. To see this proceed from our definition 27.1:

$$\begin{aligned}\frac{de^{it}}{dt} &= \frac{d\cos t + i\sin t}{dt} \\ &= \frac{d\cos t}{dt} + i\frac{d\sin t}{dt} \\ &= -\sin t + i\cos t \\ &= i(\cos t + i\sin t)\end{aligned}$$

Reason 4. The formula $e^x \cdot e^y = e^{x+y}$ still holds. Rather, we have $e^{it+is} = e^{it}e^{is}$. To check this replace the exponentials by their definition:

$$e^{it}e^{is} = (\cos t + i\sin t)(\cos s + i\sin s) = \cos(t+s) + i\sin(t+s) = e^{i(t+s)}.$$

Requiring $e^x \cdot e^y = e^{x+y}$ to be true for all complex numbers helps us decide what e^{a+bi} shoud be for arbitrary complex numbers $a + bi$.

Definition 27.3. *For any complex number $a + bi$ we set*

$$e^{a+bi} = e^a \cdot e^{ib} = e^a(\cos b + i\sin b).$$

One verifies as above in "reason 3" that this gives us the right behaviour under differentiation. Thus, for any complex number $r = a + bi$ the function

$$y(t) = e^{rt} = e^{at}(\cos bt + i\sin bt)$$

satisfies

$$y'(t) = \frac{de^{rt}}{dt} = re^{rt}.$$

28. Complex solutions of polynomial equations

28.1. *Quadratic equations*

The well-known quadratic formula tells you that the equation

$$ax^2 + bx + c = 0 \tag{30}$$

has two solutions, given by

$$x_\pm = \frac{-b \pm \sqrt{D}}{2a}, \qquad D = b^2 - 4ac. \tag{31}$$

If the coefficients a, b, c are real numbers and if the ***discriminant*** D is positive, then this formula does indeed give two real solutions x_+ and x_-. However, if $D < 0$, then there are no real solutions, but there are two complex solutions, namely

$$x_\pm = \frac{-b}{2a} \pm i\frac{\sqrt{-D}}{2a}$$

◀ **28.1 Solve $x^2 + 2x + 5 = 0$.**

Solution: Use the quadratic formula, or complete the square:

$$\begin{aligned} & x^2+2x+5=0 \\ \iff & x^2+2x+1=-4 \\ \iff & (x+1)^2=-4 \\ \iff & x+1=\pm 2i \\ \iff & x=-1\pm 2i. \end{aligned}$$

▶

So, if you allow complex solutions then every quadratic equation has two solutions, unless the two solutions coincide (the case $D=0$, in which there is only one solution.)

28.2. *Complex roots of a number*

For any given complex number w there is a method of finding all complex solutions of the equation

(32) $$z^n = w$$

if $n = 2, 3, 4, \cdots$ is a given integer.

To find these solutions you write w in polar form, i.e. you find $r > 0$ and θ such that $w = re^{i\theta}$. Then

$$z = r^{1/n}e^{i\theta/n}$$

is a solution to (32). But it isn't the only solution, because the angle θ for which $w = r^{i\theta}$ isn't unique – it is only determined up to a multiple of 2π. Thus if we have found one angle θ for which $w = r^{i\theta}$, then we can also write

$$w = re^{i(\theta+2k\pi)}, \qquad k = 0, \pm1, \pm2, \cdots$$

The n^{th} roots of w are then

$$z_k = r^{1/n}e^{i\left(\frac{\theta}{n}+2\frac{k}{n}\pi\right)}$$

Here k can be any integer, so it looks as if there are infinitely many solutions. However, if you increase k by n, then the exponent above increases by $2\pi i$, and hence z_k does not change. In a formula:

$$z_n = z_0, \quad z_{n+1} = z_1, \quad z_{n+2} = z_2, \quad \ldots \quad z_{k+n} = z_k$$

So if you take $k = 0, 1, 2, \cdots, n-1$ then you have had all the solutions.

The solutions z_k always form a regular polygon with n sides.

◀ **28.2 Example: find all sixth roots of $w = 1$.** We are to solve $z^6 = 1$. First write 1 in polar form,

$$1 = 1\cdot e^{0i} = 1\cdot e^{2k\pi i}, \qquad (k = 0, \pm1, \pm2, \ldots).$$

Then we take the 6^{th} root and find

$$z_k = 1^{1/6}e^{2k\pi i/6} = e^{k\pi i/3}, \qquad (k = 0, \pm1, \pm2, \ldots).$$

The six roots are

$$\begin{array}{lll} z_0 = 1 & z_1 = e^{\pi i/3} = \frac{1}{2} + \frac{i}{2}\sqrt{3} & z_2 = e^{2\pi i/3} = -\frac{1}{2} + \frac{i}{2}\sqrt{3} \\ z_3 = -1 & z_4 = e^{\pi i/3} = -\frac{1}{2} - \frac{i}{2}\sqrt{3} & z_5 = e^{\pi i/3} = \frac{1}{2} - \frac{i}{2}\sqrt{3} \end{array}$$

▶

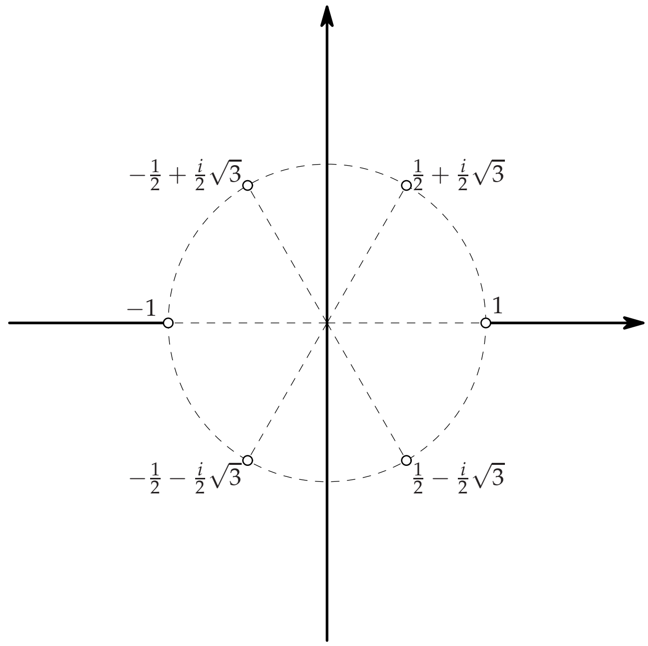


Figure 13: The sixth roots of 1. There are six of them, and they re arranged in a regular hexagon.

29. Other handy things you can do with complex numbers

29.1. *Partial fractions*

Consider the partial fraction decomposition

$$\frac{x^2+3x-4}{(x-2)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4}$$

The coefficient A is easy to find: multiply with $x-2$ and set $x=2$ (or rather, take the limit $x \to 2$) to get

$$A = \frac{2^2+3\cdot 2-4}{2^2+4} = \cdots.$$

Before we had no similar way of finding B and C quickly, but now we can apply the same trick: multiply with x^2+4,

$$\frac{x^2+3x-4}{(x-2)} = Bx + C + (x^2+4)\frac{A}{x-2},$$

and substitute $x = 2i$. This make $x^2+4=0$, with result

$$\frac{(2i)^2+3\cdot 2i-4}{(2i-2)} = 2iB + C.$$

Simplify the complex number on the left:

$$\begin{aligned}\frac{(2i)^2+3\cdot 2i-4}{(2i-2)} &= \frac{-4+6i-4}{-2+2i}\\ &= \frac{-8+6i}{-2+2i}\\ &= \frac{(-8+6i)(-2-2i)}{(-2)^2+2^2}\\ &= \frac{28+4i}{8}\\ &= \frac{7}{2}+\frac{i}{2}\end{aligned}$$

So we get $2iB+C=\frac{7}{2}+\frac{i}{2}$; since B and C are real numbers this implies

$$B=\frac{1}{4}, \qquad C=\frac{7}{2}.$$

29.2. *Certain trigonometric and exponential integrals*

You can compute

$$I=\int e^{3x}\cos 2x\,\mathrm{d}x$$

by integrating by parts twice. You can also use that $\cos 2x$ is the real part of e^{2ix}. Instead of computing the real integral I, we look at the following related complex integral

$$J=\int e^{3x}e^{2ix}\mathrm{d}x$$

which we get from I by replacing $\cos 2x$ with e^{2ix}. Since $e^{2ix}=\cos 2x+i\sin 2x$ we have

$$J=\int e^{3x}(\cos 2x+i\sin 2x)\mathrm{d}x=\int e^{3x}\cos 2x\mathrm{d}x+i\int e^{3x}\sin 2x\mathrm{d}x$$

i.e.,

$$J=I+\text{something imaginary.}$$

The point of all this is that J is easier to compute than I:

$$J=\int e^{3x}e^{2ix}\mathrm{d}x=\int e^{3x+2ix}\mathrm{d}x=\int e^{(3+2i)x}\mathrm{d}x=\frac{e^{(3+2i)x}}{3+2i}+C$$

where we have used that

$$\int e^{ax}\mathrm{d}x=\frac{1}{a}e^{ax}+C$$

holds even if a is complex is a complex number such as $a=3+2i$.

To find I you have to compute the real part of J, which you do as follows:

$$\begin{aligned}\frac{e^{(3+2i)x}}{3+2i} &= e^{3x}\frac{\cos 2x+i\sin 2x}{3+2i}\\ &= e^{3x}\frac{(\cos 2x+i\sin 2x)(3-2i)}{(3+2i)(3-2i)}\\ &= e^{3x}\frac{3\cos 2x+2\sin 2x+i(\cdots)}{13}\end{aligned}$$

so

$$\int e^{3x}\cos 2x\mathrm{d}x=e^{3x}\big(\tfrac{3}{13}\cos 2x+\tfrac{2}{13}\sin 2x\big)+C.$$

29.3. *Complex amplitudes*

A harmonic oscillation is given by

$$y(t) = A\cos(\omega t - \phi),$$

where A is the ***amplitude***, ω is the ***frequency***, and ϕ is the ***phase*** of the oscillation. If you add two harmonic oscillations with the same frequency ω, then you get another harmonic oscillation with frequency ω. You can prove this using the addition formulas for cosines, but there's another way using complex exponentials. It goes like this.

Let $y(t) = A\cos(\omega t - \phi)$ and $z(t) = B\cos(\omega t - \theta)$ be the two harmonic oscillations we wish to add. They are the real parts of

$$Y(t) = A\{\cos(\omega t - \phi) + i\sin(\omega t - \phi)\} = Ae^{i\omega t - i\phi} = Ae^{-i\phi}e^{i\omega t}$$
$$Z(t) = B\{\cos(\omega t - \theta) + i\sin(\omega t - \theta)\} = Be^{i\omega t - i\theta} = Be^{-i\theta}e^{i\omega t}$$

Therefore $y(t) + z(t)$ is the real part of $Y(t) + Z(t)$, i.e.

$$y(t) + z(t) = \mathfrak{Re}\big(Y(t)\big) + \mathfrak{Re}\big(Z(t)\big) = \mathfrak{Re}\big(Y(t) + Z(t)\big).$$

The quantity $Y(t) + Z(t)$ is easy to compute:

$$Y(t) + Z(t) = Ae^{-i\phi}e^{i\omega t} + Be^{-i\theta}e^{i\omega t} = \left(Ae^{-i\phi} + Be^{-i\theta}\right)e^{i\omega t}.$$

If you now do the complex addition

$$Ae^{-i\phi} + Be^{-i\theta} = Ce^{-i\psi},$$

i.e. you add the numbers on the right, and compute the absolute value C and argument $-\psi$ of the sum, then we see that $Y(t) + Z(t) = Ce^{i(\omega t - \psi)}$. Since we were looking for the real part of $Y(t) + Z(t)$, we get

$$y(t) + z(t) = A\cos(\omega t - \phi) + B\cos(\omega t - \theta) = C\cos(\omega t - \psi).$$

The complex numbers $Ae^{-i\phi}$, $Be^{-i\theta}$ and $Ce^{-i\psi}$ are called the complex amplitudes for the harmonic oscillations $y(t)$, $z(t)$ and $y(t) + z(t)$.

The recipe for adding harmonic oscillations can therefore be summarized as follows: ***Add the complex amplitudes.***

30. PROBLEMS

Computing and Drawing Complex Numbers

259. Compute the following complex numbers by hand.

Draw ***all*** numbers in the complex (or "Argand") plane (use graph paper or quad paper if necessary).

Compute absolute value and argument of all numbers involved.

i^2; i^3; i^4; $1/i$;

$(1+2i)(2-i)$;

$(1+i)(1+2i)(1+3i)$;

$(\frac{1}{2}\sqrt{2} + \frac{i}{2}\sqrt{2})^2$; $(\frac{1}{2} + \frac{i}{2}\sqrt{3})^3$;

$\dfrac{1}{1+i}$; $5/(2-i)$;

260. [Deriving the addition formula for $\tan(\theta + \phi)$] Let $\theta, \phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be two angles.

(a) What are the arguments of

$z = 1 + i\tan\theta$ and $w = 1 + i\tan\phi$?

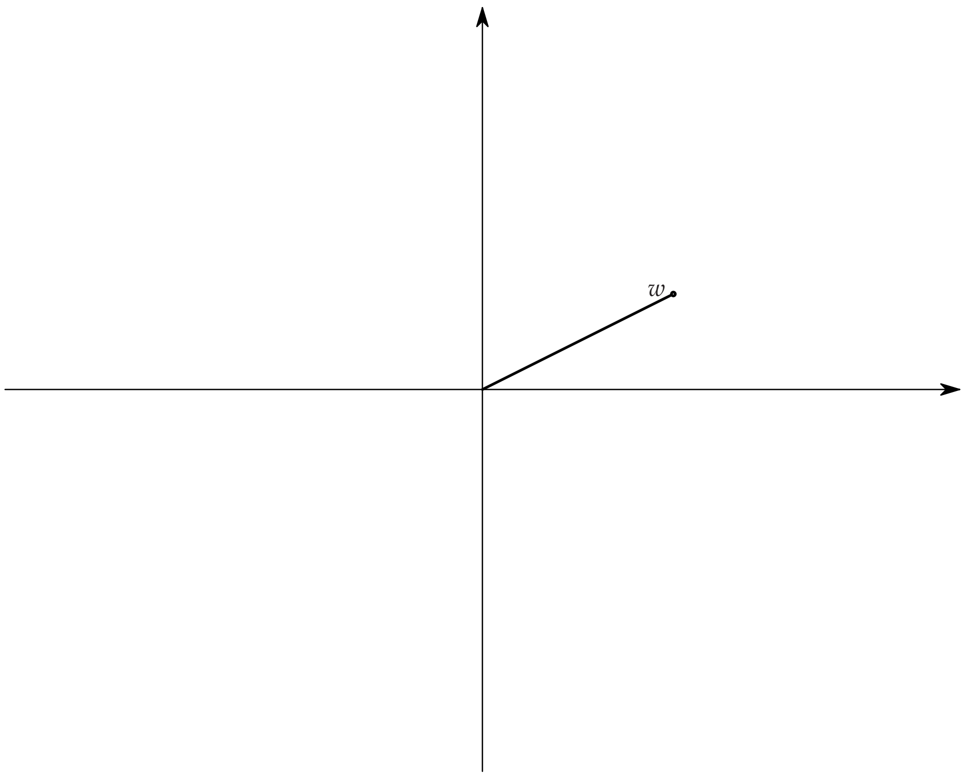
(Draw both z and w.)

(b) Compute zw.

(c) What is the argument of zw?

(d) Compute $\tan(\arg zw)$.

261. Find formulas for $\cos 4\theta$, $\sin 4\theta$, $\cos 5\theta$ and $\sin 6\theta$ in terms of $\cos\theta$ and $\sin\theta$, by using *de Moivre*'s formula.



262. In the following picture draw $2w$, $\frac{3}{4}w$, iw, $-2iw$, $(2+i)w$ and $(2-i)w$. (Try to make a nice drawing, use a ruler.)

Make a new copy of the picture, and draw $\bar{w}$, $-\bar{w}$ and $-w$.

Make yet another copy of the drawing. Draw $1/w$, $1/\bar{w}$, and $-1/w$. For this drawing you need to know where the unit circle is in your drawing: Draw a circle centered at the origin with radius of your choice, and let this be the unit circle. [Depending on which circle you draw you will get a different answer!]

263. Verify directly from the definition of addition and multiplication of complex numbers that

(a) $z+w=w+z$

(b) $zw=wz$

(c) $z(v+w)=zv+zw$

holds for *all* complex numbers v, w, and z.

264. True or False? (In mathematics this means that you should either give a proof that the statement is always true, or else give a counterexample, thereby showing that the statement is not always true.)

For any complex numbers z and w one has

(a) $\mathfrak{Re}(z)+\mathfrak{Re}(w)=\mathfrak{Re}(z+w)$

(b) $\overline{z+w}=\bar{z}+\bar{w}$

(c) $\mathfrak{Im}(z)+\mathfrak{Im}(w)=\mathfrak{Im}(z+w)$

(d) $\overline{zw}=(\bar{z})(\bar{w})$

(e) $\mathfrak{Re}(z)\mathfrak{Re}(w)=\mathfrak{Re}(zw)$

(f) $\overline{z/w}=(\bar{z})/(\bar{w})$

(g) $\mathfrak{Re}(iz)=\mathfrak{Im}(z)$

(h) $\mathfrak{Re}(iz)=i\mathfrak{Re}(z)$

(i) $\mathfrak{Re}(iz)=\mathfrak{Im}(z)$

(j) $\mathfrak{Re}(iz)=i\mathfrak{Im}(z)$

(k) $\mathfrak{Im}(iz)=\mathfrak{Re}(z)$

(l) $\mathfrak{Re}(\bar{z})=\mathfrak{Re}(z)$

265. The imaginary part of a complex number is known to be twice its real part. The absolute value of this number is 4. Which number is this?

266. The real part of a complex number is known to be half the absolute value of that number. The imaginary part of the number is 1. Which number is it?

The Complex Exponential

267. Compute and **draw** the following numbers in the complex plane

$e^{\pi i/3}$; $e^{\pi i/2}$; $\sqrt{2}e^{3\pi i/4}$; $e^{17\pi i/4}$.

$e^{\pi i}+1$; $e^{i\ln 2}$.

$\dfrac{1}{e^{\pi i/4}}$; $\dfrac{e^{-\pi i}}{e^{\pi i/4}}$; $\dfrac{e^{2-\pi i/2}}{e^{\pi i/4}}$

$e^{2009\pi i}$; $e^{2009\pi i/2}$.

$-8e^{4\pi i/3}$; $12e^{\pi i}+3e^{-\pi i}$.

268. Compute the absolute value and argument of $e^{(\ln 2)(1+i)}$.

269. Suppose z can be any complex number.

(a) Is it true that e^z is always a positive number?

(b) Is it true that $e^z \neq 0$?

270. Verify directly from the definition that

$$e^{-it}=\frac{1}{e^{it}}$$

holds for *all real* values of t.

271. Show that

$$\cos t=\frac{e^{it}+e^{-it}}{2},\qquad \sin t=\frac{e^{it}-e^{-it}}{2i}$$

272. Show that

$$\cosh x=\cos ix,\quad \sinh x=\frac{1}{i}\sin ix.$$

273. The general solution of a second order linear differential equation contains expressions of the form $Ae^{i\beta t}+Be^{-i\beta t}$. These can be rewritten as $C_1\cos\beta t+C_2\sin\beta t$.

If $Ae^{i\beta t}+Be^{-i\beta t}=2\cos\beta t+3\sin\beta t$, then what are A and B?

274. (a) Show that you can write a "cosine-wave" with amplitude A and phase ϕ as follows

$$A\cos(t-\phi)=\mathfrak{Re}\left(ze^{it}\right),$$

where the "complex amplitude" is given by $z=Ae^{-i\phi}$. (See §29.3).

(b) Show that a "sine-wave" with amplitude A and phase ϕ as follows

$$A\sin(t-\phi)=\mathfrak{Re}\left(ze^{it}\right),$$

where the "complex amplitude" is given by $z=-iAe^{-i\phi}$.

275. Find A and ϕ where $A\cos(t-\phi)=2\cos(t)+2\cos(t-\frac{2}{3}\pi)$.

276. Find A and ϕ where $A\cos(t-\phi)=12\cos(t-\frac{1}{6}\pi)+12\sin(t-\frac{1}{3}\pi)$.

277. Find A and ϕ where $A\cos(t-\phi)=12\cos(t-\pi/6)+12\cos(t-\pi/3)$.

278. Find A and ϕ such that $A\cos(t-\phi)=\cos\left(t-\frac{1}{6}\pi\right)+\sqrt{3}\cos\left(t-\frac{2}{3}\pi\right)$.

Real and Complex Solutions of Algebraic Equations

279. ***Find and draw*** all real and complex solutions of

(a) $z^2+6z+10=0$

(b) $z^3+8=0$

(c) $z^3-125=0$

(d) $2z^2+4z+4=0$

(e) $z^4+2z^2-3=0$

(f) $3z^6=z^3+2$

(g) $z^5-32=0$

(h) $z^5-16z=0$

Calculus of Complex Valued Functions

280. Compute the derivatives of the following functions

$$f(x) = \frac{1}{x+i} \qquad g(x) = \log x + i \arctan x$$

$$h(x) = e^{ix^2} \qquad k(x) = \log \frac{i+x}{i-x}$$

Try to simplify your answers.

281. (a) Compute

$$\int (\cos 2x)^4 \, dx$$

by using $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and expanding the fourth power.

(b) Assuming $a \in \mathbb{R}$, compute

$$\int e^{-2x} (\sin ax)^2 \, dx.$$

(same trick: write $\sin ax$ in terms of complex exponentials; make sure your final answer has no complex numbers.)

282. Use $\cos\alpha = (e^{i\alpha} + e^{-i\alpha})/2$, etc. to evaluate these indefinite integrals:

(a) $\int \cos^2 x \, \mathrm{d}x$

(b) $\int \cos^4 x \, \mathrm{d}x$,

(c) $\int \cos^2 x \sin x \, \mathrm{d}x$,

(d) $\int \sin^3 x \, \mathrm{d}x$,

(e) $\int \cos^2 x \sin^2 x \, \mathrm{d}x$,

(f) $\int \sin^6 x \, \mathrm{d}x$

(g) $\int \sin(3x)\cos(5x) \, \mathrm{d}x$

(h) $\int \sin^2(2x)\cos(3x) \, \mathrm{d}x$

(i) $\int_0^{\pi/4} \sin(3x)\cos(x) \, \mathrm{d}x$

(j) $\int_0^{\pi/3} \sin^3(x)\cos^2(x) \, \mathrm{d}x$

(k) $\int_0^{\pi/2} \sin^2(x)\cos^2(x) \, \mathrm{d}x$

(l) $\int_0^{\pi/3} \sin(x)\cos^2(x) \, \mathrm{d}x$

283. Compute the following integrals when $m \neq n$ are distinct integers.

(a) $\int_0^{2\pi} \sin(mx)\cos(nx) \, \mathrm{d}x$

(b) $\int_0^{2\pi} \sin(nx)\cos(nx) \, \mathrm{d}x$

(c) $\int_0^{2\pi} \cos(mx)\cos(nx) \, \mathrm{d}x$

(d) $\int_0^{\pi} \cos(mx)\cos(nx) \, \mathrm{d}x$

(e) $\int_0^{2\pi} \sin(mx)\sin(nx) \, \mathrm{d}x$

(f) $\int_0^{\pi} \sin(mx)\sin(nx) \, \mathrm{d}x$

These integrals are basic to the theory of *Fourier series*, which occurs in many applications, especially in the study of wave motion (light, sound, economic cycles, clocks, oceans, etc.). They say that different frequency waves are "independent".

284. Show that $\cos x + \sin x = C\cos(x+\beta)$ for suitable constants C and β and use this to evaluate the following integrals.

(a) $\int \frac{\mathrm{d}x}{\cos x + \sin x}$

(b) $\int \frac{\mathrm{d}x}{(\cos x + \sin x)^2}$

(c) $\int \frac{\mathrm{d}x}{A\cos x + B\sin x}$

where A and B are any constants.

285. Compute the integrals

$$\int_0^{\pi/2} \sin^2 kx \sin^2 lx \, \mathrm{d}x,$$

where k and l are positive integers.

286. Show that for any integers k, l, m

$$\int_0^{\pi} \sin kx \sin lx \sin mx \, \mathrm{d}x = 0$$

if and only if $k + l + m$ is even.

287. (***i***) Prove the following version of the CHAIN RULE: If $f : I \to \mathbb{C}$ is a differentiable complex valued function, and $g : J \to I$ is a differentiable real valued function, then $h = f \circ g : J \to \mathbb{C}$ is a differentiable function, and one has

$$h'(x) = f'(g(x))g'(x).$$

(***ii***) Let $n \geq 0$ be a nonnegative integer. Prove that if $f : I \to \mathbb{C}$ is a differentiable function, then $g(x) = f(x)^n$ is also differentiable, and one has

$$g'(x) = nf(x)^{n-1}f'(x).$$

Note that the chain rule from part (a) does ***not*** apply! *Why?*

Differential Equations

31. What is a DiffEq?

A ***differential equation*** is an equation involving an unknown function and its derivatives. The ***order*** of the differential equation is the order of the highest derivative which appears. A ***linear differential equation*** is one of form

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = k(x)$$

where the coefficients $a_1(x), \ldots, a_n(x)$ and the right hand side $k(x)$ are given functions of x and y is the unknown function. Here

$$y^{(k)} = \frac{\mathrm{d}^k y}{\mathrm{d}x^k}$$

denotes the kth derivative of y so this equation has order n. We shall mainly study the case $n = 1$ where the equation has form

$$y' + a(x)y = k(x)$$

and the case $n = 2$ with constant coefficients where the equation has form

$$y'' + ay' + by = k(x).$$

When the right hand side $k(x)$ is zero the equation is called ***homogeneous linear*** and otherwise it is called ***inhomogeneous linear*** (or ***nonhomogeneous linear*** by some people). For a homogeneous linear equation the sum of two solutions is a solution and a constant multiple of a solution is a solution. This property of linear equations is called the ***principle of superposition***.

32. First Order Separable Equations

A ***separable differential equation*** is a diffeq of the form

$$y'(x) = F(x)G(y(x)), \quad \text{or} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = F(x)G(y). \tag{33}$$

To solve this equation divide by $G(y(x))$ to get

$$\frac{1}{G(y(x))}\frac{\mathrm{d}y}{\mathrm{d}x} = F(x). \tag{34}$$

Next find a function $H(y)$ whose derivative with respect to y is

$$H'(y) = \frac{1}{G(y)} \qquad \left(\text{solution: } H(y) = \int \frac{dy}{G(y)}.\right) \tag{35}$$

Then the chain rule implies that (34) can be written as

$$\frac{\mathrm{d}H(y(x))}{\mathrm{d}x} = F(x).$$

In words: $H(y(x))$ is an antiderivative of $F(x)$, which means we can find $H(y(x))$ by integrating $F(x)$:

$$H(y(x)) = \int F(x)dx + C. \tag{36}$$

Once you've found the integral of $F(x)$ this gives you $y(x)$ in implicit form: the equation (36) gives you $y(x)$ as an *implicit function* of x. To get $y(x)$ itself you must solve the equation (36) for $y(x)$.

A quick way of organizing the calculation goes like this:

To solve $\frac{\mathrm{d}y}{\mathrm{d}x} = F(x)G(y)$ you first *separate the variables*,

$$\frac{\mathrm{d}y}{G(y)} = F(x)\,\mathrm{d}x,$$

and then integrate,

$$\int \frac{\mathrm{d}y}{G(y)} = \int F(x)\,\mathrm{d}x.$$

The result is an implicit equation for the solution y with one undetermined integration constant.

Determining the constant. The solution you get from the above procedure contains an arbitrary constant C. If the value of the solution is specified at some given x_0, i.e. if $y(x_0)$ is known then you can express C in terms of $y(x_0)$ by using (36).

A snag: You have to divide by $G(y)$ which is problematic when $G(y) = 0$. This has as consequence that in addition to the solutions you found with the above procedure, there are at least a few more solutions: the zeroes of $G(y)$ (see Example 32.2 below). In addition to the zeroes of $G(y)$ there sometimes can be more solutions, as we will see in Example 34.2 on "Leaky Bucket Dating."

◀ **32.1 Example.** We solve

$$\frac{\mathrm{d}z}{\mathrm{d}t} = (1 + z^2)\cos t.$$

Separate variables and integrate

$$\int \frac{\mathrm{d}z}{1+z^2} = \int \cos t\,\mathrm{d}t,$$

to get

$$\arctan z = \sin t + C.$$

Finally solve for z and you find the general solution

$$z(t) = \tan\big(\sin(t) + C\big).$$

▶

◀ **32.2 Example: The snag in action** If you apply the method to $y'(x) = Ky$ with K a constant, you get $y(x) = e^{K(x+C)}$. No matter how you choose C you never get the function $y(x) = 0$, even though $y(x) = 0$ satisfies the equation. This is because here $G(y) = Ky$, and $G(y)$ vanishes for $y = 0$. ▶

33. First Order Linear Equations

There are two systematic methods which solve a first order linear inhomogeneous equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + a(x)y = k(x). \tag{‡}$$

You can multiply the equation with an "integrating factor", or you do a substitution $y(x) = c(x)y_0(x)$, where y_0 is a solution of the *homogeneous equation* (that's the equation you get by setting $k(x) \equiv 0$).

33.1. *The Integrating Factor*

Let
$$A(x) = \int a(x)\,\mathrm{d}x, \qquad m(x) = e^{A(x)}.$$
Multiply the equation (‡) by the "integrating factor" $m(x)$ to get
$$m(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a(x)m(x)y = m(x)k(x).$$
By the chain rule the integrating factor satisfies
$$\frac{\mathrm{d}m(x)}{\mathrm{d}x} = A'(x)m(x) = a(x)m(x).$$
Therefore one has
$$\frac{\mathrm{d}m(x)y}{\mathrm{d}x} = m(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a(x)m(x)y = m(x)\left\{\frac{\mathrm{d}y}{\mathrm{d}x} + a(x)y\right\} = m(x)k(x).$$
Integrating and then dividing by the integrating factor gives the solution
$$y = \frac{1}{m(x)}\left(\int m(x)k(x)\,\mathrm{d}x + C\right).$$
In this derivation we have to divide by $m(x)$, but since $m(x) = e^{A(x)}$ and since exponentials never vanish we know that $m(x) \neq 0$, no matter which problem we're doing, so it's OK, we can always divide by $m(x)$.

33.2. *Variation of constants for 1st order equations*

Here is the second method of solving the inhomogeneous equation (‡). Recall again that the ***homogeneous equation*** associated with (‡) is
$$\frac{\mathrm{d}y}{\mathrm{d}x} + a(x)y = 0. \tag{†}$$
The general solution of this equation is
$$y(x) = Ce^{-A(x)}.$$
where the coefficient C is an arbitrary constant. To solve the inhomogeneous equation (‡) we replace the constant C by an unknown function $C(x)$, i.e. we look for a solution in the form
$$y = C(x)y_0(x) \text{ where } y_0(x) \stackrel{\text{def}}{=} e^{-A(x)}.$$
(This is how the method gets its name: we are allowing the constant C to vary.)

Then $y_0'(x) + a(x)y_0(x) = 0$ (because $y_0(x)$ solves (†)) and
$$y'(x) + a(x)y(x) = C'(x)y_0(x) + C(x)y_0'(x) + a(x)C(x)y_0(x) = C'(x)y_0(x)$$
so $y(x) = C(x)y_0(x)$ is a solution if $C'(x)y_0(x) = k(x)$, i.e.
$$C(x) = \int \frac{k(x)}{y_0(x)}\,\mathrm{d}x.$$
Once you notice that $y_0(x) = \dfrac{1}{m(x)}$, you realize that the resulting solution
$$y(x) = C(x)y_0(x) = y_0(x)\int \frac{k(x)}{y_0(x)}\,\mathrm{d}x$$
is the same solution we found before, using the integrating factor.

Either method implies the following:

Theorem 33.1. *The initial value problem*

$$\frac{\mathrm{d}y}{\mathrm{d}x} + a(x)y = 0, \qquad y(0) = y_0,$$

*has **exactly one** solution. It is given by*

$$y = y_0 e^{-A(x)}, \text{ where } A(x) = \int_0^x a(t)\,\mathrm{d}t.$$

The theorem says three things: (1) there is a solution, (2) there is a formula for the solution, (3) there aren't any other solutions (if you insist on the initial value $y(0) = y_0$.) The last assertion is just as important as the other two, so I'll spend a whole section trying to explain why.

34. Dynamical Systems and Determinism

A differential equation which describes how something (e.g. the position of a particle) evolves in time is called a ***dynamical system***. In this situation the independent variable is *time*, so it is customary to call it t rather than x; the dependent variable, which depends on time is often denoted by x. In other words, one has a differential equation for a function $x = x(t)$. The simplest examples have form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,t). \tag{37}$$

In applications such a differential equation expresses a *law* according to which the quantity $x(t)$ evolves with time (synonyms: "evolutionary law", "dynamical law", "evolution equation for x").

A good law is ***deterministic***, which means that any solution of (37) is completely determined by its value at one particular time t_0: if you know x at time $t = t_0$, then the "evolution law" (37) should predict the values of $x(t)$ at all other times, both in the past ($t < t_0$) and in the future ($t > t_0$).

Our experience with solving differential equations so far (§32 and §33) tells us that the general solution to a differential equation like (37) contains an unknown integration constant C. Let's call the general solution $x(t;C)$ to emphasize the presence of this constant. If the value of x at some time t_0 is known to be, say, x_0, then you get an equation

$$x(t_0;C) = x_0 \tag{38}$$

which you can try to solve for C. If this equation always has exactly one solution C then the evolutionary law (37) is deterministic (the value of $x(t_0)$ always determines $x(t)$ at all other times t); if for some prescribed value x_0 at some time t_0 the equation (38) has several solutions, then the evolutionary law (37) is not deterministic (because knowing $x(t)$ at time t_0 still does not determine the whole solution $x(t)$ at times other than t_0).

◀ **34.1 Example: Carbon Dating.** Suppose we have a fossil, and we want to know how old it is.

All living things contain carbon, which naturally occurs in two isotopes, C_{14} (unstable) and C_{12} (stable). A long as the living thing is alive it eats & breaths, and its ratio of C_{12} to C_{14} is kept constant. Once the thing dies the isotope C_{14} decays into C_{12} at a steady rate.

Let $x(t)$ be the ratio of C_{14} to C_{12} at time t. The laws of radioactive decay says that there is a constant $k > 0$ such that

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -kx(t).$$

Solve this differential equation (it is both separable and first order linear: you choose your method) to find the general solution

$$x(t;C) = Ce^{-kt}.$$

After some lab work it is found that the current C_{14}/C_{12} ratio of our fossil is x_{now}. Thus we have

$$x_{\text{now}} = Ce^{-kt_{\text{now}}} \implies C = x_{\text{now}}e^{t_{\text{now}}}.$$

Therefore our fossil's C_{14}/C_{12} ratio at any other time t is/was

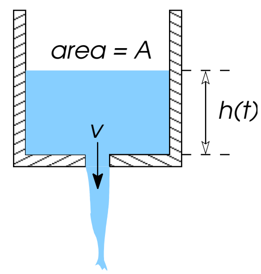
$$x(t) = x_{\text{now}}e^{k(t_{\text{now}}-t)}.$$

This allows you to compute the time at which the fossil died. At this time the C_{14}/C_{12} ratio must have been the common value in all living things, which can be measured, let's call it x_{life}. So at the time t_{demise} when our fossil became a fossil you would have had $x(t_{\text{demise}}) = x_{\text{life}}$. Hence the age of the fossil would be given by

$$x_{\text{life}} = x(t_{\text{demise}}) = x_{\text{now}}e^{k(t_{\text{now}}-t_{\text{demise}})} \implies \boxed{t_{\text{now}} - t_{\text{demise}} = \frac{1}{k}\ln\frac{x_{\text{life}}}{x_{\text{now}}}}$$ ▶

◀ **34.2 Example: On Dating a Leaky Bucket.** A bucket is filled with water. There's a hole in the bottom of the bucket so the water streams out at a certain rate.

$h(t)$	the height of water in the bucket
A	area of cross section of bucket
a	area of hole in the bucket
v	velocity with which water goes through the hole.



The amount of water in the bucket is $A \times h(t)$;

The rate at which water is leaving the bucket is $a \times v(t)$;

Hence

$$\frac{\mathrm{d}Ah(t)}{\mathrm{d}t} = -av(t).$$

In fluid mechanics it is shown that the velocity of the water as it passes through the hole only depends on the height $h(t)$ of the water, and that, for some constant K,

$$v(t) = \sqrt{Kh(t)}.$$

The last two equations together give a differential equation for $h(t)$, namely,

$$\frac{\mathrm{d}h(t)}{\mathrm{d}t} = -\frac{a}{A}\sqrt{Kh(t)}.$$

To make things a bit easier we assume that the constants are such that $\frac{a}{A}\sqrt{K} = 2$. Then $h(t)$ satisfies

(39) $$h'(t) = -2\sqrt{h(t)}.$$

This equation is separable, and when you solve it you get

$$\frac{\mathrm{d}h}{2\sqrt{h}} = -1 \implies \sqrt{h(t)} = -t + C.$$

This formula can't be valid for *all* values of t, for if you take $t > C$, the RHS becomes negative and can't be equal to the square root in the LHS. But when $t \leq C$ we do get a solution,

$$h(t; C) = (C - t)^2.$$

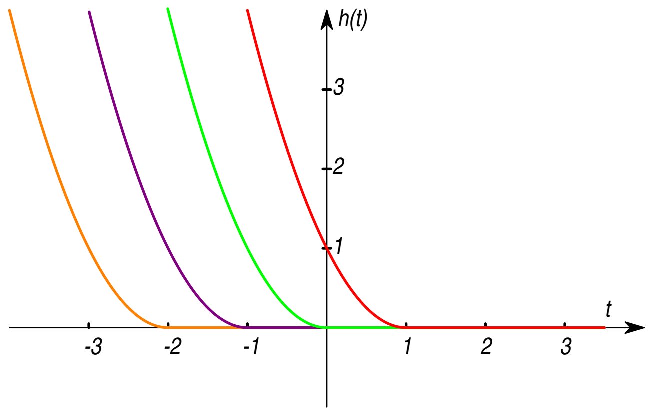


Figure 14: Several solutions $h(t; C)$ of the Leaking Bucket Equation (39). Note how they all have the same values when $t \geq 1$.

This solution describes a bucket which is losing water until at time C it is empty. Motivated by the physical interpretation of our solution it is natural to assume that the bucket stays empty when $t > C$, so that the solution with integration constant C is given by

$$h(t) = \begin{cases} (C-t)^2 & \text{when } t \leq C \\ 0 & \text{for } t > C. \end{cases}$$

We now come to the question: is the Leaky Bucket Equation deterministic? The answer is: NO. If you let C be any negative number, then $h(t; C)$ describes the water level of a bucket which long ago had water, but emptied out at time $C < 0$. In particular, for all these solutions of the diffeq (39) you have $h(0) = 0$, and knowing the value of $h(t)$ at $t = 0$ in this case therefore doesn't tell you what $h(t)$ is at other times.

Once you put it in terms of the physical interpretation it is actually quite obvious why this system can't be deterministic: it's because you can't answer the question *"If you know that the bucket once had water and that it is empty now, then how much water did it hold one hour ago?"*

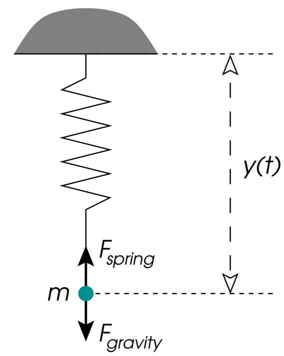
▶

35. **Higher order equations**

After looking at first order differential equations we now turn to higher order equations.

◀ **35.1 Example: Spring with a weight.**

A body of mass m is suspended by a spring. There are two forces on the body: gravity and the tension in the spring. Let F be the sum of these two forces. Newton's law says that the motion of the weight satisfies $F = ma$ where a is the acceleration. The force of gravity is mg where g=32ft/sec^2; the quantity mg is called the ***weight*** of the body. We assume ***Hooke's law*** which says that the tension in the spring is proportional to the amount by which the spring is stretched; the constant or proportionality is called the ***spring constant***. We write k for this spring constant.



The total force acting on the body is therefore

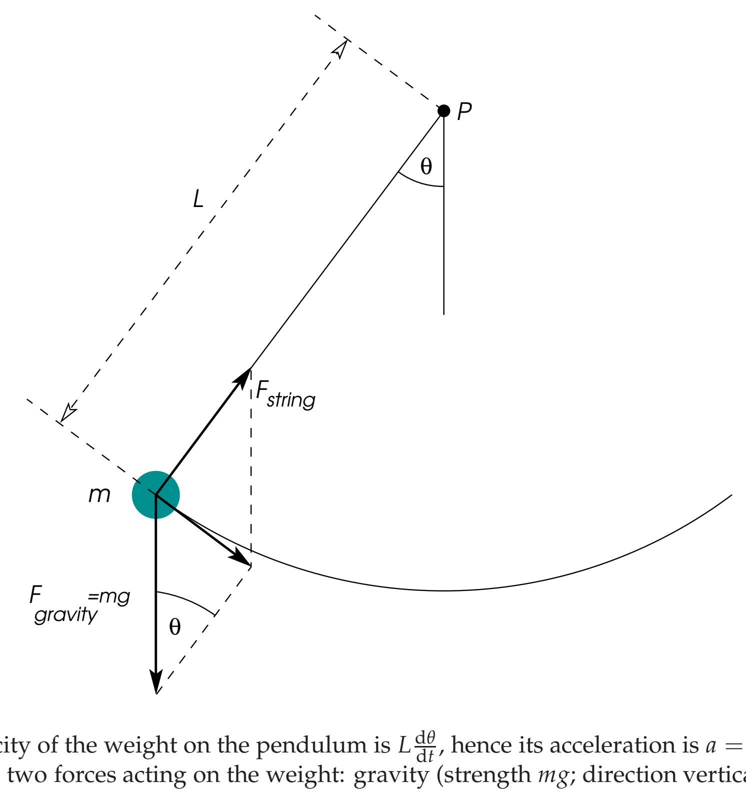
$$F = mg - ky(t).$$

According to ***Newton's first/second/third law*** the acceleration a of the body satisfies $F = ma$. Since the acceleration a is the second derivative of position y we get the following differential equation for $y(t)$

$$m\frac{d^2y}{dt^2} = mg - ky(t). \tag{40}$$

▶

◀ **35.2 Example: the pendulum.**



The velocity of the weight on the pendulum is $L\frac{d\theta}{dt}$, hence its acceleration is $a = Ld^2q/dt^2$. There are two forces acting on the weight: gravity (strength mg; direction vertically down) and the tension in the string (strength: whatever it takes to keep the weight on the circle of radius L and center P; direction parallel to the string). Together they leave a force of size $F_{\text{gravity}} \cdot \sin\theta$ which accelerates the weight. By Newton's "$F = ma$" law you get

$$mL\frac{d^2\theta}{dt^2} = -mg\sin\theta(t),$$

or, canceling ms,

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta(t) = 0. \tag{41}$$

▶

36. Constant Coefficient Linear Homogeneous Equations

36.1. *Differential operators*

In this section we study the homogeneous linear differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = 0 \tag{42}$$

where the coefficients $a_1, \ldots, a_n$ are constants.

◀ **36.1 Examples.** The three equations

$$\frac{dy}{dx} - y = 0,$$
$$y'' - y = 0, \qquad y'' + y = 0$$
$$y^{(\text{iv})} - y = 0$$

are homogeneous linear differential equations with constant coefficients. Their degrees are 1, 2, 2, and 4. ▶

It will be handy to have an abbreviation for the Left Hand Side in (42), so we agree to write $\mathcal{L}[y]$ for the result of substituting a function y in the LHS of (42). In other words, for any given function $y = y(x)$ we set

$$\mathcal{L}[y](x) \stackrel{\text{def}}{=} y^{(n)}(x) + a_1 y^{(n-1)}(x) + \cdots + a_{n-1}y'(x) + a_n y(x).$$

We call $\mathcal{L}$ an ***operator***. An operator is like a function in that you give it an input, it does a computation and gives you an output. The difference is that ordinary functions take a number as their input, while the operator $\mathcal{L}$ takes a function $y(x)$ as its input, and gives another function (the LHS of (42)) as its output. Since the computation of $\mathcal{L}[y]$ involves taking derivatives of y, the operator $\mathcal{L}$ is called a ***differential operator.***

◀ **36.2 Example.** The differential equations in the previous example correspond to the differential operators

$$\mathcal{L}_1[y] = y' - y,$$
$$\mathcal{L}_2[y] = y'' - y, \quad \mathcal{L}_3[y] = y'' + y$$
$$\mathcal{L}_4[y] = y^{(\text{iv})} - y.$$

So one has

$$\mathcal{L}_3[\sin 2x] = \frac{d^2 \sin 2x}{dx^2} - \sin 2x = -4\sin 2x - \sin 2x = -5\sin 2x.$$

▶

36.2. *The superposition principle*

The following theorem is the most important property of linear differential equations.

Theorem 36.3 (Superposition Principle). *For any two functions y_1 and y_2 we have*

$$\mathcal{L}[y_1 + y_2] = \mathcal{L}[y_1] + \mathcal{L}[y_2].$$

For any function y and any constant c we have

$$\mathcal{L}[cy] = c\mathcal{L}[y].$$

The proof, which is rather straightforward once you know what to do, will be given in lecture. It follows from this theorem that if $y_1, \dots, y_k$ are given functions, and $c_1, \dots, c_k$ are constants, then

$$\mathcal{L}[c_1 y_1 + \cdots + c_k y_k] = c_1 \mathcal{L}[y_1] + \cdots + c_k \mathcal{L}[y_k].$$

The importance of the superposition principle is that it allows you to take old solutions to the homogeneous equation and make new ones. Namely, if $y_1, \dots, y_k$ are solutions to the homogeneous equation $\mathcal{L}[y] = 0$, then so is $c_1 y_1 + \cdots + c_k y_k$ for any choice of constants c_1, $\dots$, c_k.

◀ **36.4 Example.** Consider the equation

$$y'' - 4y = 0.$$

My cousin Bruce says that the two functions $y_1(x) = e^{2x}$ and $y_2(x) = e^{-2x}$ both are solutions to this equations. You can check that Bruce is right just by substituting his solutions in the equation.

The Superposition Principle now implies that

$$y(x) = c_1 e^{2x} + c_2 e^{-2x}$$

also is a solution, for any choice of constants c_1, c_2. ▶

36.3. *The characteristic polynomial*

This example contains in it the general method for solving linear constant coefficient ODEs. Suppose we want to solve the equation (42), i.e.

$$\mathcal{L}[y] \stackrel{\text{def}}{=} y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0.$$

Then the first thing to do is to see if there are any exponential functions $y = e^{rx}$ which satisfy the equation. Since

$$\frac{\mathrm{d}e^{rx}}{\mathrm{d}x} = re^{rx}, \quad \frac{\mathrm{d}^2 e^{rx}}{\mathrm{d}x^2} = r^2 e^{rx}, \quad \frac{\mathrm{d}^3 e^{rx}}{\mathrm{d}x^3} = r^3 e^{rx}, \quad \text{etc.} \dots$$

we see that

$$\mathcal{L}[e^{rx}] = \big(r^n + a_1 r^{n-1} + \cdots a_{n-1} r + a_n\big) e^{rx}. \tag{43}$$

The polynomial

$$P(r) = r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n.$$

is called the ***characteristic polynomial***.

We see that $y = e^{rx}$ is a solution of $\mathcal{L}[y] = 0$ if and only if $P(r) = 0$.

◀ **36.5 Example.** We look for all exponential solutions of the equation

$$y'' - 4y = 0.$$

Substitution of $y = e^{rx}$ gives

$$y'' - 4y = r^2 e^{rx} - 4e^{rx} = \big(r^2 - 4\big) e^{rx}.$$

The exponential e^{rx} can't vanish, so $y'' - 4y = 0$ will hold exactly when $r^2 - 4 = 0$, i.e. when $r = \pm 2$. Therefore the only exponential functions which satisfy $y'' - 4y = 0$ are $y_1(x) = e^{2x}$ and $y_2(x) = e^{-2x}$. ▶

Theorem 36.6. *Suppose the polynomial $P(r)$ has n distinct roots $r_1, r_2, \ldots, r_n$. Then the general solution of $\mathcal{L}[y] = 0$ is*

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x}$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Proof. We have just seen that the functions $y_1(x) = e^{r_1 x}$, $y_2(x) = e^{r_2 x}$, $y_3(x) = e^{r_3 x}$, etc. are solutions of the equation $\mathcal{L}[y] = 0$. In Math 320 (or 319, or…) you prove that these are all the solutions (it also follows from the method of variation of parameters that there aren't any other solutions).

□

36.4. *Complex roots and repeated roots*

If the characteristic polynomial has n distinct real roots then Theorem 36.6 tells you what the general solution to the equation $\mathcal{L}[y] = 0$ is. In general a polynomial equation like $P(r) = 0$ can have repeated roots, and it can have complex roots.

◀ **36.7 Example.** Solve $y'' + 2y' + y = 0$.

The characteristic polynomial is $P(r) = r^2 + 2r + 1 = (r + 1)^2$, so the only root of the characteristic equation $r^2 + 2r + 1 = 0$ is $r = -1$ (it's a repeated root). This means that for this equation we only get *one* exponential solution, namely $y(x) = e^{-x}$.

It turns out that for this equation there is another solution which is not exponential. It is $y_2(x) = xe^{-x}$. You can check that it really satisfies the equation $y'' + 2y' + y = 0$. ▶

When there are repeated roots there are other solutions: if $P(r) = 0$, then $t^j e^{rt}$ is a solution if j is a nonnegative integer less than the multiplicity of r. Also, if any of the roots are complex, the phrase *general solution* should be understood to mean *general complex solution* and the coefficients c_j should be complex. If the equation is real, the real and imaginary part of a complex solution are again solutions. We only describe the case $n = 2$ in detail.

Theorem 36.8. *Consider the differential equation*

$$\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0 \tag{†}$$

and suppose that r_1 and r_2 are the solutions of the characteristic equation of $r^2 + a_1 r + a_2 = 0$. Then

(i): *If r_1 and r_2 are distinct and real, the general solution of (†) is*

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

(ii): *If $r_1 = r_2$, the general solution of (†) is*

$$y = c_1 e^{r_1 x} + c_2 x e^{r_1 x}.$$

(iii): *If $r_1 = \alpha + \beta i$ and $r_2 = \alpha - \beta i$, the general solution of (†) is*

$$y = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x).$$

In each case c_1 and c_2 are arbitrary constants.

Case (i) and case (iii) can be subsumed into a single case using complex notation:

$$e^{(\alpha\pm\beta i)x} = e^{\alpha x}\cos\beta x \pm ie^{\alpha x}\sin\beta x,$$

$$e^{\alpha x}\cos\beta x = \frac{e^{(\alpha+\beta i)x} + e^{(\alpha-\beta i)x}}{2}, \qquad e^{\alpha x}\sin\beta x = \frac{e^{(\alpha+\beta i)x} - e^{(\alpha-\beta i)x}}{2i}.$$

37. Inhomogeneous Linear Equations

In this section we study the inhomogeneous linear differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = k(x)$$

where the coefficients $a_1, \ldots, a_n$ are constants and the function $k(x)$ is a given function. In the operator notation this equation may be written

$$\mathcal{L}[y] = k(x).$$

The following theorem says that once we know one particular solution y_p of the inhomogeneous equation $\mathcal{L}[y] = k(x)$ we can find all the solutions y to the inhomogeneous equation $\mathcal{L}[y] = k(x)$ by finding all the solutions y_h to the homogeneous equation $\mathcal{L}[y] = 0$.

Theorem 37.1 (Another Superposition Principle). *Assume $\mathcal{L}[y_p] = k(x)$. Then $\mathcal{L}[y] = k(x)$ if and only if $y = y_p + y_h$ where $\mathcal{L}[y_h] = 0$.*

Proof. Suppose $\mathcal{L}[y_p] = k(x)$ and $y = y_p + y_h$. Then

$$\mathcal{L}[y] = \mathcal{L}[y_p + y_h] = \mathcal{L}[y_p] + \mathcal{L}[y_h] = k(x) + \mathcal{L}[y_h].$$

Hence $\mathcal{L}[y] = k(x)$ if and only if $\mathcal{L}[y_h] = 0$. □

38. Variation of Constants

There is a method to find the general solution of a linear inhomogeneous equation of arbitrary order, *provided you already know the solutions to the homogeneous equation.* We won't explain this method here, but merely show you the answer you get in the case of second order equations.

If $y_1(x)$ and $y_2(x)$ are solutions to the homogeneous equation

$$y''(x) + a(x)y'(x) + b(x)y(x) = 0$$

for which

$$W(x) \stackrel{\text{def}}{=} y_1(x)y_2'(x) - y_1'(x)y_2(x) \neq 0,$$

then the general solution of the inhomogeneous equation

$$y''(x) + a(x)y'(x) + b(x)y(x) = f(x)$$

is given by

$$y(x) = -y_1(x)\int \frac{y_2(\xi)f(\xi)}{W(\xi)}\mathrm{d}\xi + y_2(x)\int \frac{y_1(\xi)f(\xi)}{W(\xi)}\mathrm{d}\xi.$$

For more details you should take a more advanced course like MATH 319 or 320.

38.1. *Undetermined Coefficients*

The easiest way to find a particular solution y_p to the inhomogeneous equation is the method of undetermined coefficients or "educated guessing." Unlike the method of "variation of constants" which was (hardly) explained in the previous section, this method does not work for all equations. But it does give you the answer for a few equations which show up often enough to make it worth knowing the method.

The basis of the "method" is this: it turns out that many of the second order equations with you run into have the form

$$y'' + ay' + by = f(t),$$

where a and b are constants, and where the righthand side $f(t)$ comes from a fairly short list of functions. For all $f(t)$ in this list you memorize (yuck!) a particular solution y_p. With the particular solution in hand you can then find the general solution by adding it to the general solution of the homogeneous equation.

Here is the list:

$f(t) =$ **polynomial in** t**:** In this case you try $y_p(t) =$ some other polynomial in t with the same degree as $f(t)$.
Exceptions: if $r = 0$ is a root of the characteristic equation, then you must try a polynomial $y_p(t)$ of degree one higher than $f(t)$;
if $r = 0$ is a double root then the degree of $y_p(t)$ must be two more than the degree of $f(t)$.

$f(t) = e^{at}$**:** try $y_p(t) = Ae^{at}$.
Exceptions: if $r = a$ is a root of the characteristic equation, then you must try $y_p(t) = Ate^{at}$;
if $r = a$ is a double root then try $y_p(t) = At^2e^{at}$.

$f(t) = \sin bt$ **or** $f(t) = \cos bt$**:** In both cases, try $y_p(t) = A\cos bt + B\sin bt$.
Exceptions: if $r = bi$ is a root of the characteristic equation, then you should try $y_p(t) = t(A\cos bt + B\sin bt)$.

$f(t) = e^{at}\sin bt$ **or** $f(t) = e^{at}\cos bt$**:** Try $y_p(t) = e^{at}(A\cos bt + B\sin bt)$.
Exceptions: if $r = a + bi$ is a root of the characteristic equation, then you should try $y_p(t) = te^{at}(A\cos bt + B\sin bt)$.

◀ **38.1 Example.** Find the general solution to the following equations

$$y'' + xy' - y = 2e^x \tag{44}$$

$$y'' - 2y' + y = \sqrt{1 + x^2} \tag{45}$$

The first equation does not have constant coefficients so the method doesn't apply. Sorry, but we can't solve this equation in this course.[10]

The second equation does have constant coefficients, so we can solve the homogeneous equation ($y'' - 2y' + y = 0$), but the righthand side does not appear in our list. Again, the method doesn't work. ▶

◀ **38.2 A more upbeat example.** To find a particular solution of

$$y'' - y' + y = 3t^2$$

we note that (1) the equation is linear with constant coefficients, and (2) the right hand side is a polynomial, so it's in our list of "right hand sides for which we know what to guess." We try a polynomial of the same degree as the right hand side, namely 2. We don't know

[10]Who says you can't solve this equation? For equation (44) you *can* find a solution by computing its Taylor series! For more details you should again take a more advanced course (like MATH 319), or, in this case, give it a try yourself.

which polynomial, so *we leave its coefficients undetermined* (whence the name of the method.) I.e. we try

$$y_p(t) = A + Bt + Ct^2.$$

To see if this is a solution, we compute

$$y_p'(t) = B + 2Ct, \qquad y_p''(t) = 2C,$$

so that

$$y_p'' - y_p' + y_p = (A - B + 2C) + (B - 2C)t + Ct^2.$$

Thus $y_p'' - y_p' + y_p = 3t^2$ if and only if

$$A - B + 2C = 0, \quad B - 2C = 0, \quad C = 3.$$

Solving these equations leads to $C = 3$, $B = 2C = 6$ and $A = B - 2C = 0$. We conclude that

$$y_p(t) = 6t + 3t^2$$

is a particular solution. ▶

◀ 38.3 Another example, which is rather long, but that's because it is meant to cover several cases.

Find the general solution to the equation

$$y'' + 3y' + 2y = t + t^3 - e^t + 2e^{-2t} - e^{-t}\sin 2t.$$

Solution: First we find the characteristic equation,

$$r^2 + 3r + 2 = (r + 2)(r + 1) = 0.$$

The characteristic roots are $r_1 = -1$, and $r_2 = -2$. The general solution to the homogeneous equation is

$$y_h(t) = C_1 e^{-t} + C_2 e^{-2t}.$$

We now look for a particular solutions. Initially it doesn't look very good as the righthand side does not appear in our list. However, the righthand side is a sum of five terms, each of which is in our list.

Abbreviate $\mathcal{L}[y] = y'' + 3y' + 2y$. Then we will find functions $y_1, \ldots, y_4$ for which one has

$$\mathcal{L}[y_1] = t + t^3, \quad \mathcal{L}[y_2] = -e^t, \quad \mathcal{L}[y_3] = 2e^{-2t}, \quad \mathcal{L}[y_4] = -e^{-t}\sin 2t.$$

Then, by the Superposition Principle (Theorem 36.3) you get that $y_p \stackrel{\text{def}}{=} y_1 + y_2 + y_3 + y_4$ satisfies

$$\mathcal{L}[y_p] = \mathcal{L}[y_1] + \mathcal{L}[y_2] + \mathcal{L}[y_3] + \mathcal{L}[y_4] = t + t^3 - e^t + 2e^{-2t} - e^{-t}\sin 2t.$$

So y_p (once we find it) is a particular solution.

Now let's find $y_1, \ldots, y_4$.

$y_1(t)$**:** the righthand side $t + t^3$ is a polynomial, and $r = 0$ is not a root of the characteristic equation, so we try a polynomial of the same degree. Try

$$y_1(t) = A + Bt + Ct^2 + Dt^3.$$

Here A, B, C, D are the undetermined coefficients that give the method its name. You compute

$$\begin{aligned}\mathcal{L}[y_1] &= y_1'' + 3y_1' + 2y_1\\ &= (2C + 6Dt) + 3(B + 2Ct + 3Dt^2) + 2(A + Bt + Ct^2 + Dt^3)\\ &= (2C + 3B + 2A) + (2B + 6C + 6D)t + (2C + 9D)t^2 + 2Dt^3.\end{aligned}$$

So to get $\mathcal{L}[y_1] = t + t^3$ we must impose the equations

$$2D = 1, \quad 2C + 9D = 0, \quad 2B + 6C + 6D = 1, \quad 2C + 6B + 2A = 0.$$

You can solve these equations one-by-one, with result

$$D = \tfrac{1}{2}, \quad C = -\tfrac{9}{4}, \quad B = -\tfrac{23}{4}, \quad A = \tfrac{87}{8},$$

and thus

$$y_1(t) = \tfrac{87}{8} - \tfrac{23}{4}t - \tfrac{9}{4}t^2 + \tfrac{1}{2}t^3.$$

$y_2(t)$: We want $y_2(t)$ to satisfy $\mathcal{L}[y_2] = -e^t$. Since $e^t = e^{at}$ with $a = 1$, and $a = 1$ is not a characteristic root, we simply try $y_2(t) = Ae^t$. A quick calculation gives

$$\mathcal{L}[y_2] = Ae^t + 3Ae^t + 2Ae^t = 6Ae^t.$$

To achieve $\mathcal{L}[y_2] = -e^t$ we therefore need $6A = -1$, i.e. $A = -\frac{1}{6}$. Thus

$$y_2(t) = -\tfrac{1}{6}e^t.$$

$y_3(t)$: We want $y_3(t)$ to satisfy $\mathcal{L}[y_3] = -e^{-2t}$. Since $e^{-2t} = e^{at}$ with $a = -2$, and $a = -2$ is a characteristic root, we can't simply try $y_3(t) = Ae^{-2t}$. Instead you have to try $y_3(t) = Ate^{-2t}$. Another calculation gives

$$\begin{aligned}
\mathcal{L}[y_3] &= (4t-4)Ae^{-2t} + 3(-2t+2)Ae^{-2t} + 2Ate^{-2t} \qquad \text{(factor out } Ae^{-2t}\text{)}\\
&= \big[(4+3(-2)+2)t + (-4+3)\big]Ae^{-2t}\\
&= -Ae^{-2t}.
\end{aligned}$$

Note that all the terms with te^{-2t} cancel: this is no accident, but a consequence of the fact that $a = -2$ is a characteristic root.

To get $\mathcal{L}[y_3] = 2e^{-2t}$ we see we have to choose $A = -2$. We find

$$y_3(t) = -2te^{-2t}.$$

$y_4(t)$: Finally, we need a function $y_4(t)$ for which one has $\mathcal{L}[y_4] = -e^{-t}\sin 2t$. The list tells us to try

$$y_4(t) = e^{-t}\big(A\cos 2t + B\sin 2t\big).$$

(Since $-1 + 2i$ is not a root of the characteristic equation we are not in one of the exceptional cases.)

Diligent computation yields

$$\begin{aligned}
y_4(t) &= & Ae^{-t}\cos 2t &\quad + \quad Be^{-t}e^{-t}\sin 2t\\
y_4'(t) &= & (-A+2B)e^{-t}\cos 2t &\quad + \quad (-B-2A)e^{-t}\sin 2t\\
y_4''(t) &= & (-3A-4B)e^{-t}\cos 2t &\quad + \quad (-3B+4A)e^{-t}\sin 2t
\end{aligned}$$

so that

$$\mathcal{L}[y_4] = (-4A+2B)e^{-t}\cos 2t + (-2A-4B)e^{-t}\sin 2t.$$

We want this to equal $-e^{-t}\sin 2t$, so we have to find A, B with

$$-4A + 2B = 0, \quad -2A - 4B = -1.$$

The first equation implies $B = 2A$, the second then gives $-10A = -1$, so $A = \frac{1}{10}$ and $B = \frac{2}{10}$. We have found

$$y_4(t) = \tfrac{1}{10}e^{-t}\cos 2t + \tfrac{2}{10}e^{-t}\sin 2t.$$

After all these calculations we get the following impressive particular solution of our differential equation,

$$y_p(t) = \tfrac{87}{8} - \tfrac{23}{4}t - \tfrac{9}{4}t^2 + \tfrac{1}{2}t^3 - \tfrac{1}{6}e^t - 2te^{-2t} + \tfrac{1}{10}e^{-t}\cos 2t + \tfrac{2}{10}e^{-t}\sin 2t$$

and the even more impressive *general* solution to the equation,

$$\begin{aligned} y(t) &= y_h(y) + y_p(t) \\ &= C_1 e^{-t} + C_2 e^{-2t} \\ &\quad + \tfrac{87}{8} - \tfrac{23}{4} t - \tfrac{9}{4} t^2 + \tfrac{1}{2} t^3 \\ &\qquad - \tfrac{1}{6} e^t - 2te^{-2t} + \tfrac{1}{10} e^{-t}\cos 2t + \tfrac{2}{10} e^{-t} \sin 2t. \end{aligned}$$

You shouldn't be put off by the fact that the result is a pretty long formula, and that the computations took up two pages. The approach is to (i) break up the right hand side into terms which are in the list at the beginning of this section, (ii) to compute the particular solutions for each of those terms and (iii) to use the Superposition Principle (Theorem 36.3) to add the pieces together, resulting in a particular solution for the whole right hand side you started with. ▶

39. Applications of Second Order Linear Equations

39.1. *Spring with a weight*

In example 35.1 we showed that the height $y(t)$ a mass m suspended from a spring with constant k satisfies

$$my''(t) + ky(t) = mg, \quad \text{or} \quad y''(t) + \frac{k}{m} y(t) = g. \tag{46}$$

This is a Linear Inhomogeneous Equation whose homogeneous equation, $y'' + \frac{k}{m} y = 0$ has

$$y_h(t) = C_1 \cos \omega t + C_2 \sin \omega t$$

as general solution, where $\omega = \sqrt{k/m}$. The right hand side is a constant, which is a polynomial of degree zero, so the method of "educated guessing" applies, and we can find a particular solution by trying a constant $y_p = A$ as particular solution. You find that $y_p'' + \frac{k}{m} y_p = \frac{k}{m} A$, which will equal g if $A = \frac{mg}{k}$. Hence the general solution to the "spring with weight equation" is

$$y(t) = \frac{mg}{k} + C_1 \cos \omega t + C_2 \sin \omega t.$$

To solve the initial value problem $y(0) = y_0$ and $y'(0) = v_0$ we solve for the constants C_1 and C_2 and get

$$y(t) = \frac{mg}{k} + \frac{v_0}{\omega} \sin(\omega t) + \big(y_0 - \frac{mg}{k}\big) \cos(\omega t).$$

which you could rewrite as

$$y(t) = \frac{mg}{k} + Y \cos(\omega t - \phi)$$

for certain numbers Y, ϕ.

The weight in this model just oscillates up and down forever: this motion is called a ***simple harmonic oscillation,*** and the equation (46) is called the equation of the ***Harmonic Oscillator.***

39.2. *The pendulum equation*

In example 35.2 we saw that the angle $\theta(t)$ subtended by a swinging pendulum satisfies the ***pendulum equation***,

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta(t) = 0. \tag{41}$$

This equation is *not linear* and cannot be solved by the methods you have learned in this course. However, if the oscillations of the pendulum are small, i.e. if θ is small, then we can approximate $\sin\theta$ by θ. Remember that the error in this approximation is the remainder term in the Taylor expansion of $\sin\theta$ at $\theta = 0$. According to Lagrange this is

$$\sin\theta = \theta + R_3(\theta), \qquad R_3(\theta) = \cos\tilde{\theta}\ \frac{\theta^3}{3!} \text{ with } |\tilde{\theta}| \le \theta.$$

When θ is small, e.g. if $|\theta| \le 10^\circ \approx 0.175$ radians then compared to θ the error is at most

$$\left|\frac{R_3(\theta)}{\theta}\right| \le \frac{(0.175)^2}{3!} \approx 0.005,$$

in other words, the error is no more than half a percent.

So for small angles we will assume that $\sin\theta \approx \theta$ and hence $\theta(t)$ *almost* satisfies the equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta(t) = 0. \tag{47}$$

In contrast to the pendulum equation (41), this equation is linear, and we could solve it right now.

The procedure of replacing inconvenient quantities like $\sin\theta$ by more manageable ones (like θ) in order to end up with linear equations is called ***linearization.*** Note that the solutions to the linearized equation (47), which we will derive in a moment, are not solutions of the Pendulum Equation (41). However, if the solutions we find have small angles (have $|\theta|$ small), then the Pendulum Equation and its linearized form (47) are almost the same, and "you would think that their solutions should also be almost the same." I put that in quotation marks, because (1) it's not a very precise statement and (2) if it were more precise, you would have to prove it, which is not easy, and not a topic for this course (or even MATH 319 – take MATH 419 or 519 for more details.)

Let's solve the linearized equation (47). Setting $\theta = e^{rt}$ you find the characteristic equation

$$r^2 + \frac{g}{L} = 0$$

which has two complex roots, $r_\pm = \pm i\sqrt{\frac{g}{L}}$. Therefore, the general solution to (47) is

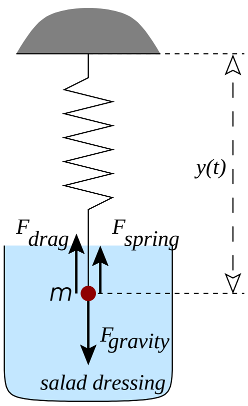
$$\theta(t) = A\cos\Big(\sqrt{\frac{g}{L}}t\Big) + B\sin\Big(\sqrt{\frac{g}{L}}t\Big),$$

and you would expect the general solution of the Pendulum Equation (41) to be almost the same. So you see that a pendulum will oscillate, and that the period of its oscillation is given by

$$T = 2\pi\sqrt{\frac{L}{g}}.$$

Once again: because we have used a linearization, you should expect this statement to be valid only for small oscillations. When you study the Pendulum Equation instead of its linearization (47), you discover that the period T of oscillation actually depends on the amplitude of the oscillation: the bigger the swings, the longer they take.

39.3. *The effect of friction*



A real weight suspended from a real spring will of course not oscillate forever. Various kinds of friction will slow it down and bring it to a stop. As an example let's assume that air drag is noticeable, so, as the weight moves the surrounding air will exert a force on the weight (To make this more likely, assume the weight is actually moving in some viscous liquid like salad oil.) This drag is stronger as the weight moves faster. A simple model is to assume that the friction force is proportional to the velocity of the weight,

$$F_{\text{friction}} = -hy'(t).$$

This adds an extra term to the oscillator equation (46), and gives

$$my''(t) = F_{\text{grav}} + F_{\text{friction}} = -ky(t) + mg - hy'(t)$$

i.e.

$$my''(t) + hy'(t) + ky(t) = mg. \tag{48}$$

This is a second order linear homogeneous differential equation with constant coefficients. A particular solution is easy to find, $y_p = mg/k$ works again.

To solve the homogeneous equation you try $y = e^{rt}$, which leads to the characteristic equation

$$mr^2 + hr + k = 0,$$

whose roots are

$$r_\pm = \frac{-h \pm \sqrt{h^2 - 4mk}}{2m}$$

If friction is large, i.e. if $h > \sqrt{4km}$, then the two roots $r_\pm$ are real, and all solutions are of exponential type,

$$y(t) = \frac{mg}{k} + C_+ e^{r_+ t} + C_- e^{r_- t}.$$

Both roots $r_\pm$ are negative, so all solutions satisfy

$$\lim_{t\to\infty} y(t) = 0.$$

If friction is weak, more precisely, if $h < \sqrt{4mk}$ then the two roots $r_\pm$ are complex numbers,

$$r_\pm = -\frac{h}{2m} \pm i\omega, \quad \text{with } \omega = \frac{\sqrt{4km - h^2}}{2m}.$$

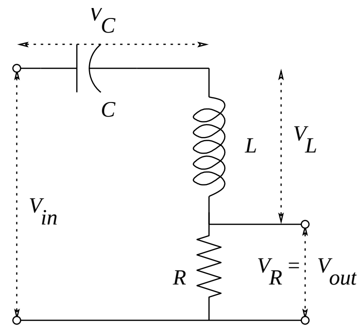
The general solution in this case is

$$y(t) = \frac{mg}{k} + e^{-\frac{h}{2m}t}\Big(A\cos\omega t + B\sin\omega t\Big).$$

These solutions also tend to zero as $t \to \infty$, but they oscillate infinitely often.

39.4. *Electric circuits*

Many equations in physics and engineering have the form (48). For example in the electric circuit in the diagram a time varying voltage $V_{in}(t)$ is applied to a resistor R, an inductance L and a capacitor C. This causes a current $I(t)$ to flow through the circuit. How much is this current, and how much is, say, the voltage across the resistor?



Electrical engineers will tell you that the total voltage $V_{in}(t)$ must equal the sum of the voltages $V_R(t)$, $V_L(t)$ and $V_C(t)$ across the three components. These voltages are related to the current $I(t)$ which flows through the three components as follows:

$$V_R(t) = RI(t)$$
$$\frac{\mathrm{d}V_C(t)}{\mathrm{d}t} = \frac{1}{C}I(t)$$
$$V_L(t) = L\frac{\mathrm{d}I(t)}{\mathrm{d}t}.$$

Surprisingly, these little electrical components know calculus! (Here R, C and L are constants depending on the particular components in the circuit. They are measured in "Ohm," "Farad," and "Henry.")

Starting from the equation

$$V_{in}(t) = V_R(t) + V_L(t) + V_C(t)$$

you get

$$\begin{aligned} V_{in}'(t) &= V_R'(t) + V_L'(t) + V_C'(t) \\ &= RI'(t) + LI''(t) + \frac{1}{C}I(t) \end{aligned}$$

In other words, for a given input voltage the current $I(t)$ satisfies a second order inhomogeneous linear differential equation

$$L\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + R\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{1}{C}I = V_{in}'(t). \tag{49}$$

Once you know the current $I(t)$ you get the output voltage $V_{\text{out}}(t)$ from

$$V_{\text{out}}(t) = RI(t).$$

In general you can write down a differential equation for any electrical circuit. As you add more components the equation gets more complicated, but if you stick to resistors, inductances and capacitors the equations will always be linear, albeit of very high order.

40. PROBLEMS

General Questions

288. Classify each of the following as homogeneous linear, inhomogeneous linear, or nonlinear and specify the order. For each linear equation say whether or not the coefficients are constant.

$(\boldsymbol{i})\ y'' + y = 0$ $(\boldsymbol{ii})\ xy'' + yy' = 0$

$(\boldsymbol{iii})\ xy'' - y' = 0$ $(\boldsymbol{iv})\ xy'' + yy' = x$

$(\boldsymbol{v})\ xy'' - y' = x$ $(\boldsymbol{vi})\ y' + y = xe^x$.

289. $(\boldsymbol{i})$ Show that $y = x^2 + 5$ is a solution of $xy'' - y' = 0$.
$(\boldsymbol{ii})$ Show that $y = C_1x^2 + C_2$ is a solution of $xy'' - y' = 0$.

290. $(\boldsymbol{i})$ Show that $y = (\tan(c_1x + c_2))/c_1$ is a solution of $yy'' = 2(y')^2 - 2y'$.
$(\boldsymbol{ii})$ Show that $y_1 = \tan(x)$ and $y_2 = 1$ are solutions of this equation, but that $y_1 + y_2$ is not.
$(\boldsymbol{iii})$ Is the equation linear homogeneous?

Separation of Variables

291. The differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{4 - y^2}{4}$$

is called the *Logistic Equation*.

(a) Find the solutions y_0, y_1, y_2, and y_3 which satisfy $y_0(0) = 0$, $y_1(0) = 1$, $y_2(0) = 2$ and $y_3(0) = 3$.

(b) Find $\lim_{t\to\infty} y_k(t)$ for $k = 1, 2, 3$.

(c) Find $\lim_{t\to-\infty} y_k(t)$ for $k = 1, 2, 3$.

(d) Graph the four solutions $y_0, \ldots, y_3$.

* * *

In each of the following problems you should find the function y of x which satisfies the conditions (A is an unspecified constant: you should at least indicate for which values of A your solution is valid.)

292. $\dfrac{\mathrm{d}y}{\mathrm{d}x} + x^2y = 0, y(1) = 5.$

293. $\dfrac{\mathrm{d}y}{\mathrm{d}x} + (1 + 3x^2)y = 0, y(1) = 1.$

294. $\dfrac{\mathrm{d}y}{\mathrm{d}x} + x\cos^2 y = 0, y(0) = \frac{\pi}{3}.$

295. $\dfrac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{1+x}{1+y} = 0, y(0) = A.$

296. $\dfrac{\mathrm{d}y}{\mathrm{d}x} + 1 - y^2 = 0, y(0) = A.$

297. $\dfrac{\mathrm{d}y}{\mathrm{d}x} + 1 + y^2 = 0, y(0) = A.$

298. $\dfrac{\mathrm{d}y}{\mathrm{d}x} - (\cos x)y = e^{\sin x}, y(0) = A.$

299. $y^2\dfrac{\mathrm{d}y}{\mathrm{d}x} + x^3 = 0, y(0) = A.$

300. Read Example 34.2 on "Leaky bucket dating" again. In that example we assumed that $\frac{a}{A}\sqrt{K} = 2$.

(a) Solve diffeq for $h(t)$ without assuming $\frac{a}{A}\sqrt{K} = 2$. Abbreviate $C = \frac{a}{A}\sqrt{K}$.

(b) If in an experiment one found that the bucket empties in 20 seconds after being filled to height 20 cm, then how much is the constant C?

Linear Homogeneous

301. (a) Show that $y = 4e^x + 7e^{2x}$ is a solution of $y'' - 3y' + 2y = 0$.
(b) Show that $y = C_1e^x + C_2e^{2x}$ is a solution of $y'' - 3y' + 2y = 0$.
(c) Find a solution of $y'' - 3y' + 2y = 0$ such that $y(0) = 7$ and $y'(0) = 9$.

302. (a) Find all solutions of $\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0$.
(b) Find all solutions of $\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = e^{-x}$.
(c) Find y if $\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = e^{-x}$ and $y = 7$ when $x = 0$.

303. (a) Find all real solutions of

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 10y = 0.$$

(b) Find y if

$$y'' - 6y' + 10y = 0,$$

and in addition y satisfies the initial conditions $y(0) = 7$, and $y'(0) = 11$.

* * *

Find the general solution $y = y(x)$ of the following differential equations

304. $\dfrac{d^4y}{dx^4} = y$

305. $\dfrac{d^4y}{dx^4} + y = 0$

306. $\dfrac{d^4y}{dx^4} - \dfrac{d^2y}{dx^2} = 0$

307. $\dfrac{d^4y}{dx^4} + \dfrac{d^2y}{dx^2} = 0$

308. $\dfrac{d^3y}{dx^3} + y = 0$

309. $\dfrac{d^3y}{dx^3} - y = 0$

310. $y^{(4)}(t) - 2y''(t) - 3y(t) = 0$

311. $y^{(4)}(t) + 4y''(t) + 3y(t) = 0.$

312. $y^{(4)}(t) + 2y''(t) + 2y(t) = 0.$

313. $y^{(4)}(t) + y''(t) - 6y(t) = 0.$

314. $y^{(4)}(t) - 8y''(t) + 15y(t) = 0.$

315. $f'''(x) - 125f(x) = 0.$

316. $u^{(5)}(x) - 32u(x) = 0.$

317. $u^{(5)}(x) + 32u(x) = 0.$

318. $y'''(t) - 5y''(t) + 6y'(t) - 2y(t) = 0.$

319. $h^{(4)}(t) - h^{(3)}(t) + 4h''(t) - 4h(t) = 0.$

320. $z'''(x) - 5z''(x) + 4z(x) = 0.$

* * *

Solve each of the following initial value problems. Your final answer should not use complex numbers, but you may use complex numbers to find it.

321. $y'' + 9y = 0, y(0) = 0, y'(0) = -3.$

322. $y'' + 9y = 0, y(0) = -3, y'(0) = 0.$

323. $y'' - 5y' + 6y = 0, y(0) = 0, y'(0) = 1.$

324. $y'' + 5y' + 6y = 0, y(0) = 1, y'(0) = 0.$

325. $y'' + 5y' + 6y = 0, y(0) = 0, y'(0) = 1.$

326. $y'' - 6y' + 5y = 0, y(0) = 1, y'(0) = 0.$

327. $y'' - 6y' + 5y = 0, y(0) = 0, y'(0) = 1.$

328. $y'' + 6y' + 5y = 0, y(0) = 1, y'(0) = 0.$

329. $y'' + 6y' + 5y = 0, y(0) = 0, y'(0) = 1.$

330. $y'' - 4y' + 5y = 0, y(0) = 1, y'(0) = 0.$

331. $y'' - 4y' + 5y = 0, y(0) = 0, y'(0) = 1.$

332. $y'' + 4y' + 5y = 0, y(0) = 1, y'(0) = 0.$

333. $y'' + 4y' + 5y = 0, y(0) = 0, y'(0) = 1.$

334. $y'' - 5y' + 6y = 0, y(0) = 1, y'(0) = 0.$

335. $f'''(t) + f''(t) - f'(t) + 15f(t) = 0$, with initial conditions $f(0) = 0, f'(0) = 1, f''(0) = 0.$

Linear Inhomogeneous

336. Find particular solutions of

$$y'' - 3y' + 2y = e^{3x}$$
$$y'' - 3y' + 2y = e^{x}$$
$$y'' - 3y' + 2y = 4e^{3x} + 5e^{x}$$

* * *

Find the general solution $y(t)$ of the following differential equations

337. $\dfrac{d^2y}{dt^2} - y = 2$

338. $\dfrac{d^2y}{dt^2} - y = 2e^t$

339. $\dfrac{d^2y}{dt^2} + 9y = \cos 3t$

340. $\dfrac{d^2y}{dt^2} + 9y = \cos t$

341. $\dfrac{d^2y}{dt^2} + y = \cos t$

342. $\dfrac{d^2y}{dt^2} + y = \cos 3t.$

343. Find y if

(a)	$\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 0$	$y(0) = 2,$	$y'(0) = 3$
(b)	$\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = e^{-x}$	$y(0) = 0,$	$y'(0) = 0$
(c)	$\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = xe^{-x}$	$y(0) = 0,$	$y'(0) = 0$
(d)	$\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = e^{-x} + xe^{-x}$	$y(0) = 2,$	$y'(0) = 3.$

Hint: Use the Superposition Principle to save work.

344. (***i***) Find the general solution of

$$z'' + 4z' + 5z = e^{it}$$

using complex exponentials.

(***ii***) Solve

$$z'' + 4z' + 5z = \sin t$$

using your solution to question (i).

(***iii***) Find a solution for the equation

$$z'' + 2z' + 2z = 2e^{-(1-i)t}$$

in the form $z(t) = u(t)e^{-(1-i)t}$.

(***iv***) Find a solution for the equation

$$x'' + 2x' + 2x = 2e^{-t}\cos t.$$

Hint: Take the real part of the previous answer.

(***v***) Find a solution for the equation

$$y'' + 2y' + 2y = 2e^{-t}\sin t.$$

Applications

345. A population of bacteria grows at a rate proportional to its size. Write and solve a differential equation which expresses this. If there are 1000 bacteria after one hour and 2000 bacteria after two hours, how many bacteria are there after three hours?

346. Rabbits in Madison have a birth rate of 5% per year and a death rate (from old age) of 2% per year. Each year 1000 rabbits get run over and 700 rabbits move in from Sun Prairie.

(***i***) Write a differential equation which describes Madison's rabbit population at time t.

(***ii***) If there were 12,000 rabbits in Madison in 1991, how many are there in 1994?

347. According to ***Newton's law of cooling*** the rate dT/dt at which an object cools is proportional to the difference $T - A$ between its temperature T and the ambient temperature A. The differential equation which expresses this is

$$\frac{dT}{dt} = k(T - A)$$

where $k < 0$ and A are constants.

(***i***) Solve this equation and show that every solution satisfies

$$\lim_{t \to \infty} T = A.$$

(***ii***) A cup of coffee at a temperature of 180^oF sits in a room whose temperature is 75^0F. In five minutes its temperature has dropped to 150°F. When will its temperature be 90°F? What is the limit of the temperature as $t \to \infty$?

348. Retaw is a mysterious living liquid; it grows at a rate of 5% of its volume per hour. A scientist has a tank initially holding y_0 gallons of retaw and removes retaw from the tank continuously at the rate of 3 gallons per hour.

(***i***) Find a differential equation for the number $y(t)$ of gallons of retaw in the tank at time t.

(***ii***) Solve this equation for y as a function of t. (The initial volume y_0 will appear in your answer.)

(***iii***) What is $\lim_{t \to \infty} y(t)$ if $y_0 = 100$?

(***iv***) What should the value of y_0 be so that $y(t)$ remains constant?

349. A 1000 gallon vat is full of 25% solution of acid. Starting at time $t = 0$ a 40% solution of acid is pumped into the vat at 20 gallons per minute. The solution is kept well mixed and drawn off at 20 gallons per minute so as to maintain the total value of 1000 gallons. Derive an expression for the acid concentration at times $t > 0$. As $t \to \infty$ what percentage solution is approached?

350. The volume of a lake is $V = 10^9$ cubic feet. Pollution P runs into the lake at 3 cubic feet per minute, and clean water runs in at 21 cubic feet per minute. The lake drains at a rate of 24 cubic feet per minute so its volume is constant. Let C be the concentration of pollution in the lake; i.e. $C = P/V$.

(***i***) Give a differential equation for C.

(***ii***) Solve the differential equation. Use the initial condition $C = C_0$ when $t = 0$ to evaluate the constant of integration.

(***iii***) There is a critical value C^* with the property that for any solution $C = C(t)$ we have

$$\lim_{t \to \infty} C = C^*.$$

Find C^*. If $C_0 = C^*$, what is $C(t)$?

351. A philanthropist endows a chair. This means that she donates an amount of money B_0 to the university. The university invests the money (it earns interest) and pays the salary of a professor. Denote the interest rate on the investment by r (e.g. if $r = .06$, then the investment earns interest at a rate of 6% per year) the salary of the professor by a (e.g. $a = \$50,000$ per year), and the balance in the investment account at time t by B.

(***i***) Give a differential equation for B.

(***ii***) Solve the differential equation. Use the initial condition $B = B_0$ when $t = 0$ to evaluate the constant of integration.

(***iii***) There is a critical value B^* with the property that (1) if $B_0 < B^*$, then there is a $t > 0$ with $B(t) = 0$ (i.e. the account runs out of money) while (2) if $B_0 > B^*$, then $\lim_{t \to \infty} B = \infty$. Find B^*.

(***iv***) This problem is like the pollution problem except for the signs of r and a. Explain.

352. A citizen pays social security taxes of a dollars per year for T_1 years, then retires, then receives payments of b dollars per year for T_2 years, then dies. The account which receives and dispenses the money earns interest at a rate of r% per year and has no money at time $t = 0$ and no money at the time $t = T_1 + T_2$ of death. Find two differential equations for the balance $B(t)$ at time t; one valid for $0 \le t \le T_1$, the other valid for $T_1 \le t \le T_1 + T_2$. Express the ratio b/a in terms of T_1, T_2, and r. Reasonable values for T_1, T_2, and r are $T_1 = 40$, $T_2 = 20$, and $r = 5\% = .05$. This model ignores inflation. Notice that $0 < dB/\mathrm{d}t$ for $0 < t < T_1$, that $dB/\mathrm{d}t < 0$ for $T_1 < t < T_1 + T_2$, and that the account earns interest *even for* $T_1 < t < T_1 + T_2$.

353. A 300 gallon tank is full of milk containing 2% butterfat. Milk containing 1% butterfat is pumped in a 10 gallons per minute starting at 10:00 AM and the well mixed milk is drained off at 15 gallons per minute. What is the percent butterfat in the milk in the tank 5 minutes later at 10:05 AM? Hint: How much milk is in the tank at time t? How much butterfat is in the milk at time $t = 0$?

354. A sixteen pound weight is suspended from the lower end of a spring whose upper end is attached to a rigid support. The weight extends the spring by half a foot. It is struck by a sharp blow which gives it an initial downward velocity of eight feet per second. Find its position as a function of time.

355. A sixteen pound weight is suspended from the lower end of a spring whose upper end is attached to a rigid support. The weight extends the spring by half a foot. The weight is pulled down one feet and released. Find its position as a function of time.

356. The equation for the displacement $y(t)$ from equilibrium of a spring subject to a forced vibration of frequency ω is

$$\frac{d^2y}{\mathrm{d}t^2} + 4y = \sin(\omega t). \tag{50}$$

(***i***) Find the solution $y = y(\omega, t)$ of (50) for $\omega \ne 2$ if $y(0) = 0$ and $y'(0) = 0$.

(***ii***) What is $\lim_{\omega \to 2} y(\omega, t)$?

(***iii***) Find the solution $y(t)$ of

$$\frac{d^2y}{\mathrm{d}t^2} + 4y = \sin(2t) \tag{51}$$

if $y(0) = 0$ and $y'(0) = 0$. (Hint: Compare with (50).)

357. Suppose that an undamped spring is subjected to an external periodic force so that its position y at time t satisfies the differential equation

$$\frac{d^2y}{\mathrm{d}t^2} + \omega_0^2 y = c\sin(\omega t).$$

(***i***) Show that the general solution is

$$y = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t + \frac{c}{\omega^2 - \omega_0^2}\sin\omega t.$$

when $\omega_0 \ne \omega$.

(***ii***) Solve the equation when $\omega = \omega_0$.

(***iii***) Show that in part (i) the solution remains bounded as $t \to \infty$ but in part (ii) this is not so. (This phenomenon is called ***resonance***. To see an example of resonance try Googling "Tacoma Bridge Disaster.")

358. Have look at the electrical circuit equation (49) from §39.4.

(***i***) Find the general solution of (49), assuming that $V_{\text{in}}(t)$ does not depend on time t. What is $\lim_{t\to\infty} I(t)$?

(***ii***) Assume for simplicity that $L = C = 1$, and that the resistor has been short circuited, i.e. that $R = 0$. If the input voltage is a sinusoidal wave,

$$V_{\text{in}}(t) = A\sin\omega t, \qquad (\omega \ne 1)$$

then find a particular solution, and then the general solution.

(***iii***) Repeat problem (ii) with $\omega = 1$.

(***iv***) Suppose again that $L = C = 1$, but now assume that $R > 0$. Find the general solution when $V_{\text{in}}(t)$ is constant.

(***v***) Still assuming $L = C = 1$, $R > 0$ find a particular solution of the equation when the input voltage is a sinusoidal wave

$$V_{\text{in}}(t) = A \sin \omega t.$$

Vectors

41. Introduction to vectors

Definition 41.1. *A vector is a column of two, three, or more numbers, written as*

$$\vec{\boldsymbol{a}} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad or \quad \vec{\boldsymbol{a}} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad or \quad \vec{\boldsymbol{a}} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

in general.

The ***length of a vector*** $\vec{\boldsymbol{a}} = \left(\begin{smallmatrix} a_1 \\ a_2 \\ a_3 \end{smallmatrix}\right)$ *is defined by*

$$\|\vec{\boldsymbol{a}}\| = \left\| \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \right\| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

We will always deal with either the two or three dimensional cases, in other words, the cases $n = 2$ or $n = 3$, respectively. For these cases there is a geometric description of vectors which is very useful. In fact, the two and three dimensional theories have their origins in mechanics and geometry. In higher dimensions the geometric description fails, simply because we cannot visualize a four dimensional space, let alone a higher dimensional space. Instead of a geometric description of vectors there is an abstract theory called ***Linear Algebra*** which deals with "vector spaces" of any dimension (even infinite!). This theory of vectors in higher dimensional spaces is very useful in science, engineering and economics. You can learn about it in courses like MATH 320 or 340/341.

41.1. *Basic arithmetic of vectors*

You can add and subtract vectors, and you can multiply them with arbitrary real numbers. this section tells you how.

The ***sum of two vectors*** is defined by

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}, \tag{52}$$

and

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}.$$

The ***zero vector*** is defined by

$$\vec{\boldsymbol{0}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad or \quad \vec{\boldsymbol{0}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

It has the property that

$$\vec{\boldsymbol{a}} + \vec{\boldsymbol{0}} = \vec{\boldsymbol{0}} + \vec{\boldsymbol{a}} = \vec{\boldsymbol{a}}$$

no matter what the vector $\vec{\boldsymbol{a}}$ is.

You can multiply a vector $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ with a real number t according to the rule

$$t\vec{a} = \begin{pmatrix} ta_1 \\ ta_2 \\ ta_3 \end{pmatrix}.$$

In particular, "minus a vector" is defined by

$$-\vec{a} = (-1)\vec{a} = \begin{pmatrix} -a_1 \\ -a_2 \\ -a_3 \end{pmatrix}.$$

The difference of two vectors is defined by

$$\vec{a} - \vec{b} = \vec{a} + (-\vec{b}).$$

So, to subtract two vectors you subtract their components,

$$\vec{a} - \vec{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} - \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{pmatrix}$$

◀ **41.2 Some GOOD examples.**

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -3 \\ \pi \end{pmatrix} = \begin{pmatrix} -1 \\ 3+\pi \end{pmatrix} \qquad 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 12 \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} 2 \\ -12 \\ 3-\sqrt{2} \end{pmatrix} \qquad a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$0 \cdot \begin{pmatrix} 12\sqrt{39} \\ \pi^2 - \ln 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{0} \qquad \begin{pmatrix} t + t^2 \\ 1 - t^2 \end{pmatrix} = (1+t)\begin{pmatrix} t \\ 1-t \end{pmatrix}$$

▶

◀ **41.3 Two very, very BAD examples.** Vectors must have the same size to be added, therefore

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \textbf{undefined!!!}$$

Vectors and numbers are different things, so an equation like

$$\vec{a} = 3 \quad \text{is nonsense!}$$

This equation says that some vector ($\vec{a}$) is equal to some number (in this case: 3). ***Vectors and numbers are never equal!*** ▶

41.2. *Algebraic properties of vector addition and multiplication*

Addition of vectors and multiplication of numbers and vectors were defined in such a way that the following always hold for any vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ (of the same size) and any real numbers s, t

(53) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ [vector addition is commutative]

(54) $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$ [vector addition is associative]

(55) $t(\vec{a} + \vec{b}) = t\vec{a} + t\vec{b}$ [first distributive property]

(56) $(s+t)\vec{a} = s\vec{a} + t\vec{a}$ [second distributive property]

◀ **41.4 Prove (53).** Let $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ be two vectors, and consider both possible ways of adding them:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} b_1 + a_1 \\ b_2 + a_2 \\ b_3 + a_3 \end{pmatrix}$$

We know (or we have assumed long ago) that addition of real numbers is commutative, so that $a_1 + b_1 = b_1 + a_1$, etc. Therefore

$$\vec{a} + \vec{b} = \begin{pmatrix} a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \end{pmatrix} = \begin{pmatrix} b_1+a_1 \\ b_2+a_2 \\ b_3+a_3 \end{pmatrix} = \vec{b} + \vec{a}.$$

This proves (53). ▶

◀ **41.5 Example.** If $\vec{v}$ and $\vec{w}$ are two vectors, we define

$$\vec{a} = 2\vec{v} + 3\vec{w}, \quad \vec{b} = -\vec{v} + \vec{w}.$$

Problem: Compute $\vec{a} + \vec{b}$ and $2\vec{a} - 3\vec{b}$ in terms of $\vec{v}$ and $\vec{w}$.

Solution:

$$\vec{a} + \vec{b} = (2\vec{v} + 3\vec{w}) + (-\vec{v} + \vec{w}) = (2-1)\vec{v} + (3+1)\vec{w} = \vec{v} + 4\vec{w}$$

$$2\vec{a} - 3\vec{b} = 2(2\vec{v} + 3\vec{w}) - 3(-\vec{v} + \vec{w}) = 4\vec{w} + 6\vec{w} + 3\vec{v} - 3\vec{w} = 7\vec{v} + 3\vec{w}.$$

Problem: Find s, t so that $s\vec{a} + t\vec{b} = \vec{v}$.

Solution: Simplifying $s\vec{a} + t\vec{b}$ you find

$$s\vec{a} + t\vec{b} = s(2\vec{v} + 3\vec{w}) + t(-\vec{v} + \vec{w}) = (2s - t)\vec{v} + (3s + t)\vec{w}.$$

One way to ensure that $s\vec{a} + t\vec{b} = \vec{v}$ holds is therefore to choose s and t to be the solutions of

$$2s - t = 1$$
$$3s + t = 0$$

The second equation says $t = -3s$. The first equation then leads to $2s + 3s = 1$, i.e. $s = \frac{1}{5}$. Since $t = -3s$ we get $t = -\frac{3}{5}$. The solution we have found is therefore

$$\tfrac{1}{5}\vec{a} - \tfrac{3}{5}\vec{b} = \vec{v}.$$

▶

41.3. *Geometric description of vectors*

Vectors originally appeared in mechanics, where they represented forces: a force acting on some object has a ***magnitude*** and a ***direction***. Thus a force can be thought of as an arrow, where the length of the arrow indicates how strong the force is (how hard it pushes or pulls).

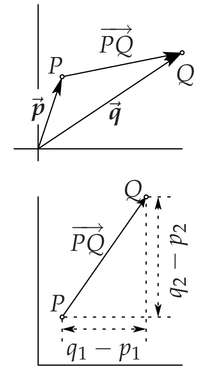
So we will think of vectors as ***arrows***: if you specify two points P and Q, then the arrow pointing from P to Q is a vector and we denote this vector by $\overrightarrow{PQ}$.

The precise mathematical definition is as follows:

Definition 41.6. *For any pair of points P and Q whose coordinates are (p_1, p_2, p_3) and (q_1, q_2, q_3) one defines a vector $\overrightarrow{PQ}$ by*

$$\overrightarrow{PQ} = \begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 \\ q_3 - p_3 \end{pmatrix}.$$

*If the initial point of an arrow is the origin O, and the final point is any point Q, then the vector $\overrightarrow{OQ}$ is called the **position vector** of the point Q.*



two pictures of
the vector $\overrightarrow{PQ} = \vec{q} - \vec{p}$

If $\vec{p}$ and $\vec{q}$ are the position vectors of P and Q, then one can write $\overrightarrow{PQ}$ as

$$\overrightarrow{PQ} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} - \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \vec{q} - \vec{p}.$$

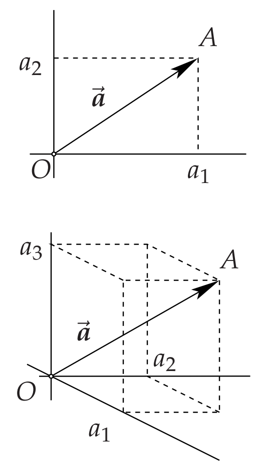
For plane vectors we define $\overrightarrow{PQ}$ similarly, namely, $\overrightarrow{PQ} = \left(\begin{smallmatrix} q_1 - p_1 \\ q_2 - p_2 \end{smallmatrix}\right)$. The old formula for the distance between two points P and Q in the plane

$$\text{distance from } P \text{ to } Q = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2}$$

says that the length of the vector $\overrightarrow{PQ}$ is just the distance between the points P and Q, i.e.

$$\text{distance from } P \text{ to } Q = \left\| \overrightarrow{PQ} \right\|.$$

This formula is also valid if P and Q are points in space.



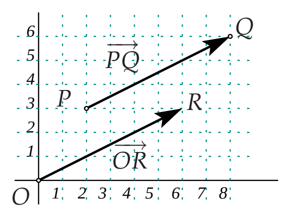
position vectors in the
plane and in space

◀ **41.7 Example.** The point P has coordinates $(2, 3)$; the point Q has coordinates $(8, 6)$. The vector $\overrightarrow{PQ}$ is therefore

$$\overrightarrow{PQ} = \begin{pmatrix} 8 - 2 \\ 6 - 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}.$$

This vector is the position vector of the point R whose coordinates are $(6, 3)$. Thus

$$\overrightarrow{PQ} = \overrightarrow{OR} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}.$$



The distance from P to Q is the length of the vector $\overrightarrow{PQ}$, i.e.

$$\text{distance } P \text{ to } Q = \left\| \begin{pmatrix} 6 \\ 3 \end{pmatrix} \right\| = \sqrt{6^2 + 3^2} = 3\sqrt{5}.$$

▶

◀ **41.8 Example.** Find the distance between the points A and B whose position vectors are $\vec{a} = \left(\begin{smallmatrix} 1 \\ 1 \\ 0 \end{smallmatrix}\right)$ and $\vec{b} = \left(\begin{smallmatrix} 0 \\ 1 \\ 1 \end{smallmatrix}\right)$ respectively.

Solution: One has

$$\text{distance } A \text{ to } B = \|\overrightarrow{AB}\| = \|\vec{b} - \vec{a}\| = \left\| \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\| = \sqrt{(-1)^2 + 0^2 + 1^2} = \sqrt{2}$$

▶

41.4. *Geometric interpretation of vector addition and multiplication*

Suppose you have two vectors $\vec{a}$ and $\vec{b}$. Consider them as position vectors, i.e. represent them by vectors that have the origin as initial point:

$$\vec{a} = \overrightarrow{OA}, \quad \vec{b} = \overrightarrow{OB}.$$

Then the origin and the three endpoints of the vectors $\vec{a}$, $\vec{b}$ and $\vec{a} + \vec{b}$ form a parallelogram. See figure 15.

To multiply a vector $\vec{a}$ with a real number t you multiply its length with $|t|$; if $t < 0$ you reverse the direction of $\vec{a}$.

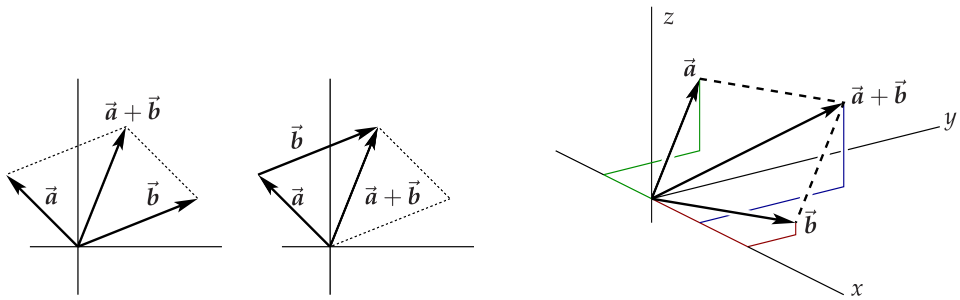


Figure 15: Two ways of adding plane vectors, and an addition of space vectors

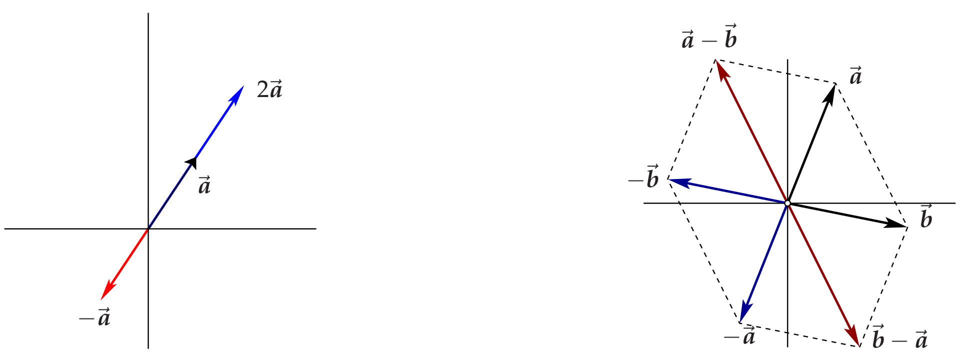


Figure 16: Multiples of a vector, and the difference of two vectors.

◀ **41.9 Example.** In example 41.5 we assumed two vectors $\vec{v}$ and $\vec{w}$ were given, and then defined $\vec{a} = 2\vec{v} + 3\vec{w}$ and $\vec{b} = -\vec{v} + \vec{w}$. In figure 17 the vectors $\vec{a}$ and $\vec{b}$ are constructed geometrically from some arbitrarily chosen $\vec{v}$ and $\vec{w}$. We also found algebraically in example 41.5 that $\vec{a} + \vec{b} = \vec{v} + 4\vec{w}$. The third drawing in figure 17 illustrates this. ▶

42. Parametric equations for lines and planes

Given two ***distinct*** points A and B we consider the line segment AB. If X is any given point on AB then we will now find a formula for the position vector of X.

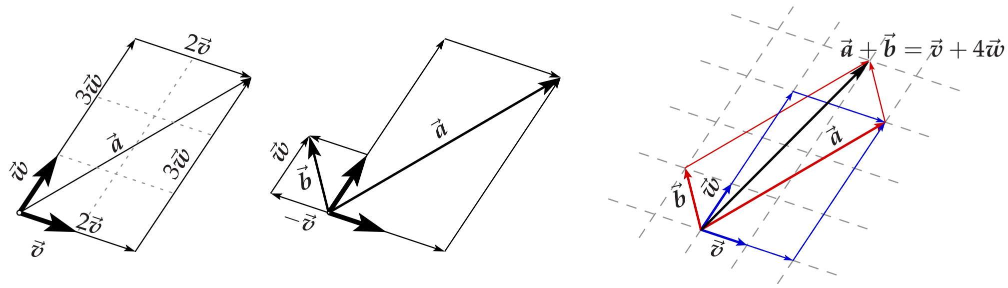
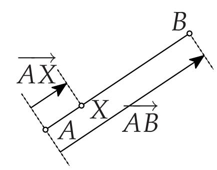


Figure 17: Picture proof that $\vec{a} + \vec{b} = \vec{v} + 4\vec{w}$ in example 41.9.

Define t to be the ratio between the lengths of the line segments AX and AB,

$$t = \frac{\text{length } AX}{\text{length } AB}.$$



Then the vectors $\overrightarrow{AX}$ and $\overrightarrow{AB}$ are related by $\overrightarrow{AX} = t\overrightarrow{AB}$. Since AX is shorter than AB we have $0 < t < 1$.

The position vector of the point X on the line segment AB is

$$\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX} = \overrightarrow{OA} + t\overrightarrow{AB}.$$

If we write $\vec{a}, \vec{b}, \vec{x}$ for the position vectors of A, B, X, then we get

$$\vec{x} = (1-t)\vec{a} + t\vec{b} = \vec{a} + t(\vec{b} - \vec{a}). \tag{57}$$

This equation is called the ***parametric equation for the line through** A **and** B***.** In our derivation the parameter t satisfied $0 \le t \le 1$, but there is nothing that keeps us from substituting negative values of t, or numbers $t > 1$ in (57). The resulting vectors $\vec{x}$ are position vectors of points X which lie on the line ℓ through A and B.

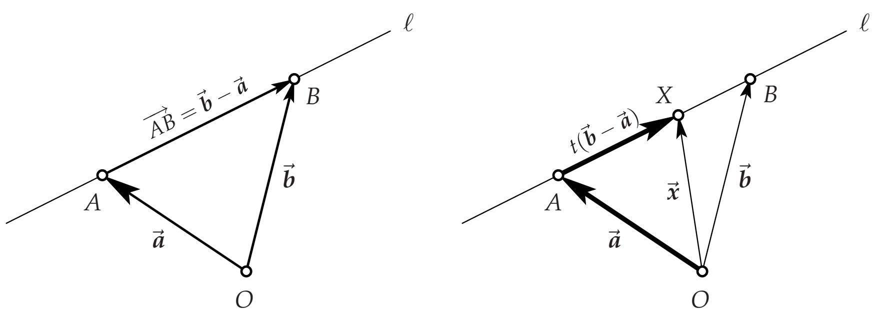
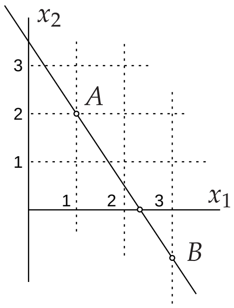


Figure 18: Constructing points on the line through A and B

◀ **42.1 Find the parametric equation for the line ℓ through the points $A(2,1)$ and $B(3,-1)$, and determine where ℓ intersects the x_1 axis.**

Solution: The position vectors of A, B are $\vec{a} = \binom{1}{2}$ and $\vec{b} = \binom{3}{-1}$, so the position vector of an arbitrary point on ℓ is given by

$$\vec{x} = \vec{a} + t(\vec{b} - \vec{a}) = \begin{pmatrix}1\\2\end{pmatrix} + t\begin{pmatrix}3-1\\-1-2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} + t\begin{pmatrix}2\\-3\end{pmatrix} = \begin{pmatrix}1+2t\\2-3t\end{pmatrix}$$



where t is an arbitrary real number.

This vector points to the point $X = (1+2t, 2-3t)$. By definition, a point lies on the x_1-axis if its x_2 component vanishes. Thus if the point

$$X = (1+2t, 2-3t)$$

lies on the x_1-axis, then $2-3t=0$, i.e. $t=\frac{2}{3}$. The intersection point X of ℓ and the x_1-axis is therefore $X|_{t=2/3} = (1+2\cdot\frac{2}{3}, 0) = (\frac{5}{3}, 0)$.

▶

◀ **42.2 Midpoint of a line segment.** If M is the midpoint of the line segment AB, then the vectors $\overrightarrow{AM}$ and $\overrightarrow{MB}$ are both parallel and have the same direction and length (namely, half the length of the line segment AB). Hence they are equal: $\overrightarrow{AM} = \overrightarrow{MB}$. If $\vec{a}$, $\vec{m}$, and $\vec{b}$ are the position vectors of A , M and B, then this means

$$\vec{m} - \vec{a} = \overrightarrow{AM} = \overrightarrow{MB} = \vec{b} - \vec{m}.$$

Add $\vec{m}$ and $\vec{a}$ to both sides, and divide by 2 to get

$$\vec{m} = \tfrac{1}{2}\vec{a} + \tfrac{1}{2}\vec{b} = \frac{\vec{a}+\vec{b}}{2}.$$

▶

42.1. *Parametric equations for planes in space**

You can specify a plane in three dimensional space by naming a point A on the plane $\mathcal{P}$, and two vectors $\vec{v}$ and $\vec{w}$ parallel to the plane $\mathcal{P}$, but not parallel to each other. Then any point on the plane $\mathcal{P}$ has position vector $\vec{x}$ given by

$$\vec{x} = \vec{a} + s\vec{v} + t\vec{w}. \tag{58}$$

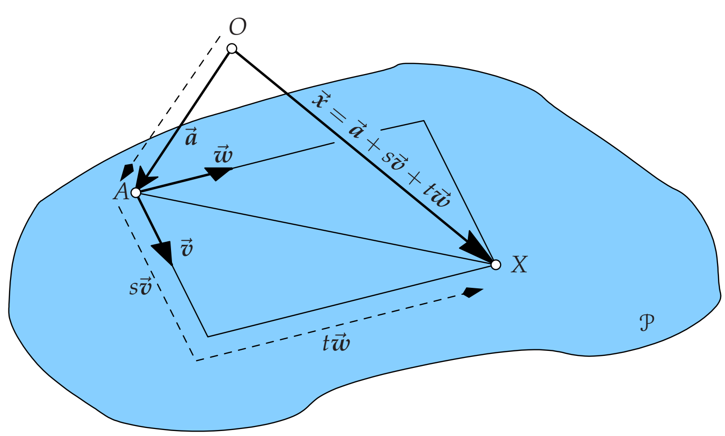


Figure 19: Generating points on a plane $\mathcal{P}$

The following construction explains why (58) will give you any point on the plane through A, parallel to $\vec{v}, \vec{w}$.

Let A, $\vec{v}$, $\vec{w}$ be given, and suppose we want to express the position vector of some other point X on the plane $\mathcal{P}$ in terms of $\vec{a} = \overrightarrow{OA}$, $\vec{v}$, and $\vec{w}$.

First we note that

$$\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX}.$$

Next, you draw a parallelogram in the plane $\mathcal{P}$ whose sides are parallel to the vectors $\vec{v}$ and $\vec{w}$, and whose diagonal is the line segment AX. The sides of this parallelogram represent vectors which are multiples of $\vec{v}$ and $\vec{w}$ and which add up to $\overrightarrow{AX}$. So, if one side of the parallelogram is $s\vec{v}$ and the other $t\vec{w}$ then we have $\overrightarrow{AX} = s\vec{v} + t\vec{w}$. With $\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX}$ this implies (58).

43. **Vector Bases**

43.1. *The Standard Basis Vectors*

The notation for vectors which we have been using so far is not the most traditional. In the late 19th century GIBBS and HEAVYSIDE adapted HAMILTON's theory of Quaternions to deal with vectors. Their notation is still popular in texts on electromagnetism and fluid mechanics.

Define the following three vectors:

$$\vec{\boldsymbol{i}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{\boldsymbol{j}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{\boldsymbol{k}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then *every vector can be written as a linear combination of* $\vec{\boldsymbol{i}}$, $\vec{\boldsymbol{j}}$ *and* $\vec{\boldsymbol{k}}$, namely as follows:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_1\vec{\boldsymbol{i}} + a_2\vec{\boldsymbol{j}} + a_3\vec{\boldsymbol{k}}.$$

Moreover, *there is only one way to write a given vector as a linear combination of* $\{\vec{\boldsymbol{i}}, \vec{\boldsymbol{j}}, \vec{\boldsymbol{k}}\}$. This means that

$$a_1\vec{\boldsymbol{i}} + a_2\vec{\boldsymbol{j}} + a_3\vec{\boldsymbol{k}} = b_1\vec{\boldsymbol{i}} + b_2\vec{\boldsymbol{j}} + b_3\vec{\boldsymbol{k}} \iff \begin{cases} a_1 = b_1 \\ a_2 = b_2 \\ a_3 = b_3 \end{cases}$$

For plane vectors one defines

$$\vec{\boldsymbol{i}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{\boldsymbol{j}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and just as for three dimensional vectors one can write every (plane) vector $\vec{\boldsymbol{a}}$ as a linear combination of $\vec{\boldsymbol{i}}$ and $\vec{\boldsymbol{j}}$,

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1\vec{\boldsymbol{i}} + a_2\vec{\boldsymbol{j}}.$$

Just as for space vectors, there is only one way to write a given vector as a linear combination of $\vec{\boldsymbol{i}}$ and $\vec{\boldsymbol{j}}$.

43.2. *A Basis of Vectors (in general)**

The vectors $\vec{\boldsymbol{i}}, \vec{\boldsymbol{j}}, \vec{\boldsymbol{k}}$ are called the ***standard basis vectors***. They are an example of what is called a "basis". Here is the definition in the case of space vectors:

Definition 43.1. *A triple of space vectors* $\{\vec{u}, \vec{v}, \vec{w}\}$ *is a **basis** if every space vector* $\vec{a}$ *can be written as a linear combination of* $\{\vec{u}, \vec{v}, \vec{w}\}$*, i.e.*

$$\vec{a} = a_u\vec{u} + a_v\vec{v} + a_w\vec{w},$$

and if there is only one way to do so for any given vector $\vec{a}$ *(i.e. the vector* $\vec{a}$ *determines the coefficients* a_u, a_v, a_w*).*

For plane vectors the definition of a basis is almost the same, except that a basis consists of two vectors rather than three:

Definition 43.2. *A pair of plane vectors* $\{\vec{u}, \vec{v}\}$ *is a **basis** if every plane vector* $\vec{a}$ *can be written as a linear combination of* $\{\vec{u}, \vec{v}\}$*, i.e.* $\vec{a} = a_u\vec{u} + a_v\vec{v}$*, and if there is only one way to do so for any given vector* $\vec{a}$ *(i.e. the vector* $\vec{a}$ *determines the coefficients* a_u, a_v*).*

44. Dot Product

Definition 44.1. *The "inner product" or "dot product" of two vectors is given by*

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \bullet \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3.$$

Note that the dot-product of two vectors is a number!

The dot product of two plane vectors is (predictably) defined by

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \bullet \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1b_1 + a_2b_2.$$

An important property of the dot product is its relation with the length of a vector:

$$\|\vec{a}\|^2 = \vec{a} \bullet \vec{a}. \tag{59}$$

44.1. *Algebraic properties of the dot product*

The dot product satisfies the following rules,

$$\vec{a} \bullet \vec{b} = \vec{b} \bullet \vec{a} \tag{60}$$

$$\vec{a} \bullet (\vec{b} + \vec{c}) = \vec{a} \bullet \vec{b} + \vec{a} \bullet \vec{c} \tag{61}$$

$$(\vec{b} + \vec{c}) \bullet \vec{a} = \vec{b} \bullet \vec{a} + \vec{c} \bullet \vec{a} \tag{62}$$

$$t(\vec{a} \bullet \vec{b}) = (t\vec{a}) \bullet \vec{b} \tag{63}$$

which hold for all vectors $\vec{a}, \vec{b}, \vec{c}$ and any real number t.

◀ **44.2 Example.** Simplify $\|\vec{a} + \vec{b}\|^2$.

One has

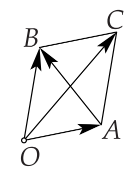
$$\begin{aligned} \|\vec{a} + \vec{b}\|^2 &= (\vec{a} + \vec{b}) \bullet (\vec{a} + \vec{b}) \\ &= \vec{a} \bullet (\vec{a} + \vec{b}) + \vec{b} \bullet (\vec{a} + \vec{b}) \\ &= \vec{a} \bullet \vec{a} + \underbrace{\vec{a} \bullet \vec{b} + \vec{b} \bullet \vec{a}}_{=2\vec{a}\bullet\vec{b} \text{ by } (60)} + \vec{b} \bullet \vec{b} \\ &= \|\vec{a}\|^2 + 2\vec{a} \bullet \vec{b} + \|\vec{b}\|^2 \end{aligned}$$

▶

44.2. *The diagonals of a parallelogram*

Here is an example of how you can use the algebra of the dot product to prove something in geometry.

Suppose you have a parallelogram one of whose vertices is the origin. Label the vertices, starting at the origin and going around counterclockwise, O, A, C and B. Let $\vec{a} = \overrightarrow{OA}$, $\vec{b} = \overrightarrow{OB}$, $\vec{c} = \overrightarrow{OC}$. One has



$$\overrightarrow{OC} = \vec{c} = \vec{a} + \vec{b}, \quad \text{and} \quad \overrightarrow{AB} = \vec{b} - \vec{a}.$$

These vectors correspond to the diagonals OC and AB

Theorem 44.3. *In a parallelogram $OACB$ the sum of the squares of the lengths of the two diagonals equals the sum of the squares of the lengths of all four sides.*

Proof. The squared lengths of the diagonals are

$$\|\overrightarrow{OC}\|^2 = \|\vec{a} + \vec{b}\|^2 = \|\vec{a}\|^2 + 2\vec{a} \bullet \vec{b} + \|\vec{b}\|^2$$
$$\|\overrightarrow{AB}\|^2 = \|\vec{a} - \vec{b}\|^2 = \|\vec{a}\|^2 - 2\vec{a} \bullet \vec{b} + \|\vec{b}\|^2$$

Adding both these equations you get

$$\|\overrightarrow{OC}\|^2 + \|\overrightarrow{AB}\|^2 = 2\left(\|\vec{a}\|^2 + \|\vec{b}\|^2\right).$$

The squared lengths of the sides are

$$\|\overrightarrow{OA}\|^2 = \|\vec{a}\|^2, \quad \|\overrightarrow{AB}\|^2 = \|\vec{b}\|^2, \quad \|\overrightarrow{BC}\|^2 = \|\vec{a}\|^2, \quad \|\overrightarrow{OC}\|^2 = \|\vec{b}\|^2.$$

Together these also add up to $2\left(\|\vec{a}\|^2 + \|\vec{b}\|^2\right)$. □

44.3. *The dot product and the angle between two vectors*

Here is the most important interpretation of the dot product:

Theorem 44.4. *If the angle between two vectors $\vec{a}$ and $\vec{b}$ is θ, then one has*

$$\vec{a} \bullet \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cos\theta.$$

Proof. We need *the law of cosines* from high-school trigonometry. Recall that for a triangle OAB with angle θ at the point O, and with sides OA and OB of lengths a and b, the length c of the opposing side AB is given by

$$c^2 = a^2 + b^2 - 2ab\cos\theta. \tag{64}$$

In trigonometry this is proved by dropping a perpendicular line from B onto the side OA. The triangle OAB gets divided into two right triangles, one of which has AB as hypotenuse. Pythagoras then implies

$$c^2 = (b\sin\theta)^2 + (a - b\cos\theta)^2.$$

After simplification you get (64).

To prove the theorem you let O be the origin, and then observe that the length of the side AB is the length of the vector $\overrightarrow{AB} = \vec{b} - \vec{a}$. Here $\vec{a} = \overrightarrow{OA}$, $\vec{b} = \overrightarrow{OB}$, and hence

$$c^2 = \|\vec{b} - \vec{a}\|^2 = (\vec{b} - \vec{a}) \bullet (\vec{b} - \vec{a}) = \|\vec{b}\|^2 + \|\vec{a}\|^2 - 2\vec{a} \bullet \vec{b}.$$

Compare this with (64), keeping in mind that $a = \|\vec{a}\|$ and $b = \|\vec{b}\|$: you are led to conclude that $-2\vec{a} \bullet \vec{b} = -2ab\cos\theta$, and thus $\vec{a} \bullet \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cos\theta$. □

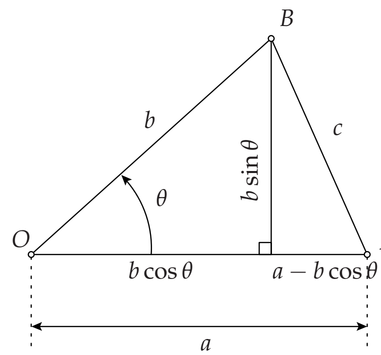
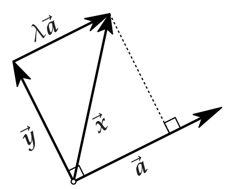


Figure 20: Proof of the law of cosines



44.4. *Orthogonal projection of one vector onto another*

The following construction comes up very often. Let $\vec{a} \neq \vec{0}$ be a given vector. Then for any other vector $\vec{x}$ there is a number λ such that

$$\vec{x} = \lambda\vec{a} + \vec{y}$$

where $\vec{y} \perp \vec{a}$. In other words, you can write any vector $\vec{x}$ as the sum of one vector parallel to $\vec{a}$ and another vector orthogonal to $\vec{a}$. The two vectors $\lambda\vec{a}$ and $\vec{y}$ are called the ***parallel*** and ***orthogonal components*** of the vector $\vec{x}$ (with respect to $\vec{a}$), and sometimes the following notation is used

$$\vec{x}^{\parallel} = \lambda\vec{a}, \qquad \vec{x}^{\perp} = \vec{y}.,$$

so that

$$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}.$$

There are moderately simple formulas for $\vec{x}^{\parallel}$ and $\vec{x}^{\perp}$, but it is better to remember the following derivation of these formulas.

Assume that the vectors $\vec{a}$ and vx are given. Then we look for a number λ such that $\vec{y} = \vec{x} - \lambda\vec{a}$ is perpendicular to $\vec{a}$. Recall that $\vec{a} \perp (\vec{x} - \lambda\vec{a})$ if and only if

$$\vec{a} \bullet (\vec{x} - \lambda\vec{a}) = 0.$$

Expand the dot product and you get this equation for λ

$$\vec{a} \bullet \vec{x} - \lambda\vec{a} \bullet \vec{a} = 0,$$

whence

$$\lambda = \frac{\vec{a} \bullet \vec{x}}{\vec{a} \bullet \vec{a}} = \frac{\vec{a} \bullet \vec{x}}{\|\vec{a}\|^2} \tag{65}$$

To compute the parallel and orthogonal components of $\vec{x}$ w.r.t. $\vec{a}$ you first compute λ according to (65), which tells you that the parallel coponent is given by

$$\vec{x}^{\parallel} = \lambda\vec{a} = \frac{\vec{a} \bullet \vec{x}}{\vec{a} \bullet \vec{a}}\,\vec{a}.$$

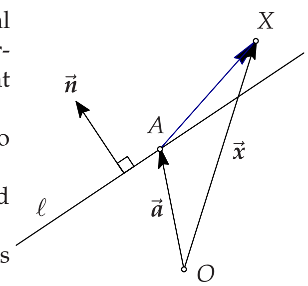
The orthogonal component is then "the rest," i.e. by definition $\vec{x}^{\perp} = \vec{x} - \vec{x}^{\parallel}$, so

$$\vec{x}^{\perp} = \vec{x} - \vec{x}^{\parallel} = \vec{x} - \frac{\vec{a} \bullet \vec{x}}{\vec{a} \bullet \vec{a}}\,\vec{a}.$$

44.5. *Defining equations of lines*

In § 42 we saw how to generate points on a line given two points on that line by means of a "parametrization." I.e. given points A and B on the line ℓ the point whose position vector is $\vec{x} = \vec{a} + t(\vec{b} - \vec{a})$ will be on ℓ for any value of the "parameter" t.

In this section we will use the dot-product to give a different description of lines in the plane (and planes in three dimensional space.) We will derive an equation for a line. Rather than generating points on the line ℓ this equation tells us if any given point X in the plane is on the line or not.



Is X on ℓ?

Here is the derivation of the equation of a line in the plane. To produce the equation you need two ingredients:
1. One particular point on the line (let's call this point A, and write $\vec{a}$ for its position vector),
2. a ***normal vector*** $\vec{n}$ for the line, i.e. a nonzero vector which is perpendicular to the line.
Now let X be any point in the plane, and consider the line segment AX.

- Clearly, X will be on the line if and only if AX is parallel to ℓ [11]
- Since ℓ is perpendicular to $\vec{n}$, the segment AX and the line ℓ will be parallel if and only if $AX \perp \vec{n}$.
- $AX \perp \vec{n}$ holds if and only if $\overrightarrow{AX} \bullet \vec{n} = 0$.

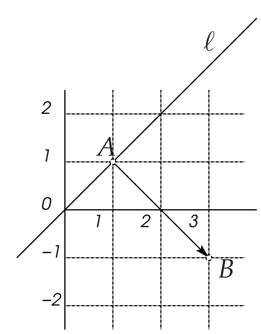
So in the end we see that X lies on the line ℓ if and only if the following vector equation is satisfied:

$$\overrightarrow{AX} \bullet \vec{n} = 0 \quad \text{or} \quad (\vec{x} - \vec{a}) \bullet \vec{n} = 0 \tag{66}$$

This equation is called a ***defining equation for the line*** ℓ.

Any given line has many defining equations. Just by changing the length of the normal you get a different equation, which still describes the same line.

◀ **44.5 Problem.** Find a defining equation for the line ℓ which goes through $A(1,1)$ and is perpendicular to the line segment AB where B is the point $(3,-1)$.



Solution. We already know a point on the line, namely A, but we still need a normal vector. The line is required to be perpendicular to AB, so $\vec{n} = \overrightarrow{AB}$ is a normal vector:

$$\vec{n} = \overrightarrow{AB} = \begin{pmatrix} 3-1 \\ (-1)-1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

Of course any multiple of $\vec{n}$ is also a normal vector, for instance

$$\vec{m} = \tfrac{1}{2}\vec{n} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is a normal vector.

[11] From plane Euclidean geometry: parallel lines either don't intersect or they coincide.

With $\vec{a} = \binom{1}{1}$ we then get the following equation for ℓ

$$\vec{n} \bullet (\vec{x} - \vec{a}) = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} x_1 - 1 \\ x_2 - 1 \end{pmatrix} = 2x_1 - 2x_2 = 0.$$

If you choose the normal $\vec{m}$ instead, you get

$$\vec{m} \bullet (\vec{x} - \vec{a}) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} x_1 - 1 \\ x_2 - 1 \end{pmatrix} = x_1 - x_2 = 0.$$

Both equations $2x_1 - 2x_2 = 0$ and $x_1 - x_2 = 0$ are equivalent. ▶

44.6. *Distance to a line*

Let ℓ be a line in the plane and assume a point A on the line as well as a vector $\vec{n}$ perpendicular to ℓ are known. Using the dot product one can easily compute the distance from the line to any other given point P in the plane. Here is how:

Draw the line m through A perpendicular to ℓ, and drop a perpendicular line from P onto m. let Q be the projection of P onto m. The distance from P to ℓ is then equal to the length of the line segment AQ. Since AQP is a right triangle one has

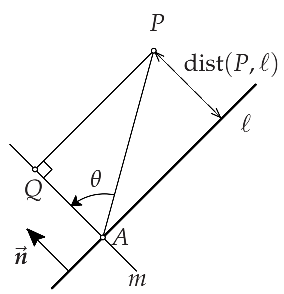
$$AQ = AP \cos\theta.$$

Here θ is the angle between the normal $\vec{n}$ and the vector $\overrightarrow{AP}$. One also has

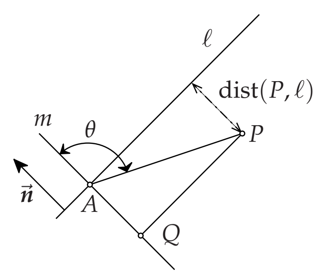
$$\vec{n} \bullet (\vec{p} - \vec{a}) = \vec{n} \bullet \overrightarrow{AP} = \|\overrightarrow{AP}\| \; \|\vec{n}\| \cos\theta = AP \, \|\vec{n}\| \cos\theta.$$

Hence we get

$$\operatorname{dist}(P, \ell) = \frac{\vec{n} \bullet (\vec{p} - \vec{a})}{\|\vec{n}\|}.$$



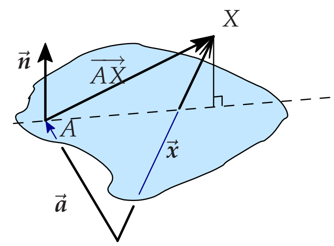
$\theta < \dfrac{\pi}{2}$



$\theta > \dfrac{\pi}{2}$

This argument from a drawing contains a hidden assumption, namely that the point P lies on the side of the line ℓ pointed to by the vector $\vec{n}$. If this is not the case, so that $\vec{n}$ and $\overrightarrow{AP}$ point to opposite sides of ℓ, then the angle between them exceeds 90°, i.e. $\theta > \pi/2$. In this case $\cos\theta < 0$, and one has $AQ = -AP\cos\theta$. the distance formula therefore has to be modified to

$$\operatorname{dist}(P, \ell) = -\frac{\vec{n} \bullet (\vec{p} - \vec{a})}{\|\vec{n}\|}.$$



44.7. *Defining equation of a plane*

Just as we have seen how you can form the defining equation for a line in the plane from just one point on the line and one normal vector to the line, you can also form the defining equation for a plane in space, again knowing only one point on the plane, and a vector perpendicular to it.

If A is a point on some plane $\mathcal{P}$ and $\vec{n}$ is a vector perpendicular to $\mathcal{P}$, then any other point X lies on $\mathcal{P}$ if and only if $\overrightarrow{AX} \perp \vec{n}$. In other words, in terms of the position vectors $\vec{a}$ and $\vec{x}$ of A and X,

$$\text{the point } X \text{ is on } \mathcal{P} \iff \vec{n} \bullet (\vec{x} - \vec{a}) = 0.$$

Arguing just as in § 44.6 you find that the distance of a point X in space to the plane $\mathcal{P}$ is

$$\text{dist}(X, \mathcal{P}) = \pm \frac{\vec{n} \bullet (\vec{x} - \vec{a})}{\|\vec{n}\|}. \tag{67}$$

Here the sign is "+" if X and the normal $\vec{n}$ are on the same side of the plane $\mathcal{P}$; otherwise the sign is "−".

◀ **44.6 Find the defining equation for the plane $\mathcal{P}$ through the point $A(1, 0, 2)$ which is perpendicular to the vector $\begin{pmatrix}1\\2\\1\end{pmatrix}$.**

Solution: We know a point (A) and a normal vector $\vec{n} = \begin{pmatrix}1\\2\\1\end{pmatrix}$ for $\mathcal{P}$. Then any point X with coordinates (x_1, x_2, x_3), or, with position vector $\vec{x} = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$, will lie on the plane $\mathcal{P}$ if and only if

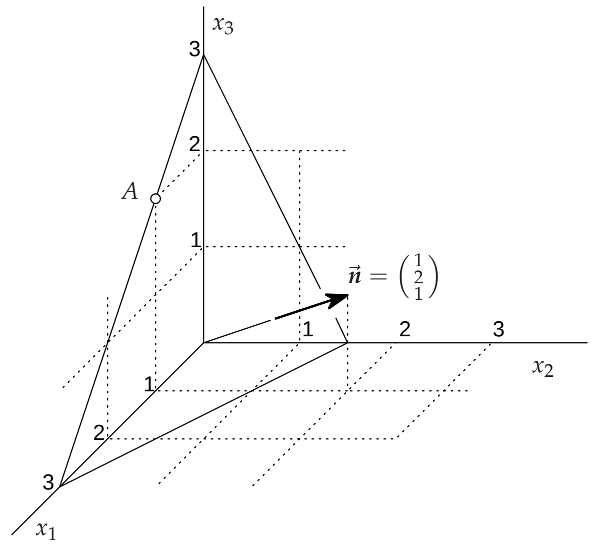
$$\begin{aligned}
\vec{n} \bullet (\vec{x} - \vec{a}) = 0 &\iff \begin{pmatrix}1\\2\\1\end{pmatrix} \bullet \left\{ \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} - \begin{pmatrix}1\\0\\2\end{pmatrix} \right\} = 0 \\
&\iff \begin{pmatrix}1\\2\\1\end{pmatrix} \bullet \begin{pmatrix}x_1 - 1\\x_2\\x_3 - 2\end{pmatrix} = 0 \\
&\iff 1 \cdot (x_1 - 1) + 2 \cdot (x_2) + 1 \cdot (x_3 - 2) = 0 \\
&\iff x_1 + 2x_2 + x_3 - 3 = 0.
\end{aligned}$$

▶

◀ **44.7 Let $\mathcal{P}$ be the plane from the previous example. Which of the points $P(0, 0, 1)$, $Q(0, 0, 2)$, $R(-1, 2, 0)$ and $S(-1, 0, 5)$ lie on $\mathcal{P}$? Compute the distances from the points P, Q, R, S to the plane $\mathcal{P}$. Separate the points which do not lie on $\mathcal{P}$ into two group of points which lie on the same side of $\mathcal{P}$.**

Solution: We apply (67) to the position vectors $\vec{p}, \vec{q}, \vec{r}, \vec{s}$ of the points P, Q, R, S. For each calculation we need

$$\|\vec{n}\| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}.$$



The third component of the given normal $\vec{n} = \begin{pmatrix}1\\2\\1\end{pmatrix}$ is positive, so $\vec{n}$ points "upwards." Therefore, if a point lies on the side of $\mathcal{P}$ pointed to by $\vec{n}$, we shall say that the point lies *above the plane.*

P: $\vec{p} = \begin{pmatrix}0\\0\\1\end{pmatrix}$, $\vec{p} - \vec{a} = \begin{pmatrix}-1\\0\\-1\end{pmatrix}$, $\vec{n} \bullet (\vec{p} - \vec{a}) = 1 \cdot (-1) + 2 \cdot (0) + 1 \cdot (-1) = -2$

$$\frac{\vec{n} \bullet (\vec{p} - \vec{a})}{\|\vec{n}\|} = -\frac{2}{\sqrt{6}} = -\frac{1}{3}\sqrt{6}.$$

This quantity is negative, so P lies below $\mathcal{P}$. Its distance to $\mathcal{P}$ is $\frac{1}{3}\sqrt{6}$.

Q: $\vec{q} = \begin{pmatrix}0\\0\\2\end{pmatrix}$, $\vec{p} - \vec{a} = \begin{pmatrix}-1\\0\\0\end{pmatrix}$, $\vec{n} \bullet (\vec{p} - \vec{a}) = 1 \cdot (-1) + 2 \cdot (0) + 1 \cdot (0) = -1$

$$\frac{\vec{n} \bullet (\vec{p} - \vec{a})}{\|\vec{n}\|} = -\frac{1}{\sqrt{6}} = -\frac{1}{6}\sqrt{6}.$$

This quantity is negative, so Q also lies below $\mathcal{P}$. Its distance to $\mathcal{P}$ is $\frac{1}{6}\sqrt{6}$.

R: $\vec{r} = \begin{pmatrix}-1\\2\\0\end{pmatrix}$, $\vec{p} - \vec{a} = \begin{pmatrix}-2\\2\\-2\end{pmatrix}$, $\vec{n} \bullet (\vec{p} - \vec{a}) = 1 \cdot (-2) + 2 \cdot (2) + 1 \cdot (-2) = 0$

$$\frac{\vec{n} \bullet (\vec{p} - \vec{a})}{\|\vec{n}\|} = 0.$$

Thus R lies on the plane $\mathcal{P}$, and its distance to $\mathcal{P}$ is of course 0.

S: $\vec{s} = \begin{pmatrix}-1\\0\\5\end{pmatrix}$, $\vec{p} - \vec{a} = \begin{pmatrix}-2\\0\\3\end{pmatrix}$, $\vec{n} \bullet (\vec{p} - \vec{a}) = 1 \cdot (-1) + 2 \cdot (0) + 1 \cdot (3) = 2$

$$\frac{\vec{n} \bullet (\vec{p} - \vec{a})}{\|\vec{n}\|} = \frac{2}{\sqrt{6}} = \frac{1}{3}\sqrt{6}.$$

This quantity is positive, so S lies above $\mathcal{P}$. Its distance to $\mathcal{P}$ is $\frac{1}{3}\sqrt{6}$.

We have found that P and Q lie below the plane, R lies on the plane, and S is above the plane. ▶

◀ **44.8 Where does the line through the points $B(2,0,0)$ and $C(0,1,2)$ intersect the plane $\mathcal{P}$ from example 44.6?**

Solution: Let ℓ be the line through B and C. We set up the parametric equation for ℓ. According to §42, (57) every point X on ℓ has position vector $\vec{x}$ given by

$$\vec{x} = \vec{b} + t(\vec{c} - \vec{b}) = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0-2 \\ 1-0 \\ 2-0 \end{pmatrix} = \begin{pmatrix} 2-2t \\ t \\ 2t \end{pmatrix} \tag{68}$$

for some value of t.

The point X whose position vector $\vec{x}$ is given above lies on the plane $\mathcal{P}$ if $\vec{x}$ satisfies the defining equation of the plane. In example 44.6 we found this defining equation. It was

$$\vec{n} \bullet (\vec{x} - \vec{a}) = 0, \text{ i.e. } x_1 + 2x_2 + x_3 - 3 = 0. \tag{69}$$

So to find the point of intersection of ℓ and $\mathcal{P}$ you substitute the parametrization (68) in the defining equation (69):

$$0 = x_1 + 2x_2 + x_3 - 3 = (2-2t) + 2(t) + (2t) - 3 = 2t - 1.$$

This implies $t = \frac{1}{2}$, and thus the intersection point has position vector

$$\vec{x} = \vec{b} + \tfrac{1}{2}(\vec{c} - \vec{b}) = \begin{pmatrix} 2-2t \\ t \\ 2t \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ 1 \end{pmatrix},$$

i.e. ℓ and $\mathcal{P}$ intersect at $X(1, \frac{1}{2}, 1)$. ▶

45. Cross Product

45.1. *Algebraic definition of the cross product*

Here is the definition of the cross-product of two vectors. The definition looks a bit strange and arbitrary at first sight – it really makes you wonder who thought of this. We will just put up with that for now and explore the properties of the cross product. Later on we will see a geometric interpretation of the cross product which will show that this particular definition is really useful. We will also find a few tricks that will help you reproduce the formula without memorizing it.

Definition 45.1. *The "outer product" or "cross product" of two vectors is given by*

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

Note that the cross-product of two vectors is again a vector!

◀ **45.2 Example.** If you set $\vec{b} = \vec{a}$ in the definition you find the following important fact: *The cross product of any vector with itself is the zero vector:*

$$\vec{a} \times \vec{a} = \vec{0} \quad \text{for } \textbf{\textit{any}} \text{ vector } \vec{a}.$$

▶

◀ **45.3 Example.** Let $\vec{a} = \left(\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right)$, $\vec{b} = \left(\begin{smallmatrix} -2 \\ 1 \\ 0 \end{smallmatrix}\right)$ and compute the cross product of these vectors.

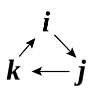
Solution:

$$\vec{a} \times \vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \cdot 0 - 3 \cdot 1 \\ 3 \cdot (-2) - 1 \cdot 0 \\ 1 \cdot 1 - 2 \cdot (-2) \end{pmatrix} = \begin{pmatrix} -3 \\ -6 \\ 5 \end{pmatrix}$$

▶

$\times$	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{i}$	$\vec{0}$	$\vec{k}$	$-\vec{j}$
$\vec{j}$	$-\vec{k}$	$\vec{0}$	$\vec{i}$
$\vec{k}$	$\vec{j}$	$-\vec{i}$	$\vec{0}$

In terms of the standard basis vectors you can check the *multiplication table*. An easy way to remember the multiplication table is to put the vectors $\vec{i}, \vec{j}, \vec{k}$ clockwise in a circle. Given two of the three vectors their product is either plus or minus the remaining vector. To determine the sign you step from the first vector to the second, to the third: if this makes you go clockwise you have a plus sign, if you have to go counterclockwise, you get a minus.



The products of $\vec{i}, \vec{j}$ and $\vec{k}$ are all you need to know to compute the cross product. Given two vectors $\vec{a}$ and $\vec{b}$ write them as $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$, and multiply as follows

$$
\begin{aligned}
\vec{a} \times \vec{b} =& (a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \times (b_1\vec{i} + b_2\vec{j} + b_3\vec{k}) \\
=& \quad a_1\vec{i} \times (b_1\vec{i} + b_2\vec{j} + b_3\vec{k}) \\
& + a_2\vec{j} \times (b_1\vec{i} + b_2\vec{j} + b_3\vec{k}) \\
& + a_3\vec{k} \times (b_1\vec{i} + b_2\vec{j} + b_3\vec{k}) \\
=& \quad a_1b_1\vec{i}\times\vec{i} + a_1b_2\vec{i}\times\vec{j} + a_1b_3\vec{i}\times\vec{k} + \\
& \quad a_2b_1\vec{j}\times\vec{i} + a_2b_2\vec{j}\times\vec{j} + a_2b_3\vec{j}\times\vec{k} + \\
& \quad a_3b_1\vec{k}\times\vec{i} + a_3b_2\vec{k}\times\vec{j} + a_3b_3\vec{k}\times\vec{k} \\
=& \quad a_1b_1\vec{0} + a_1b_2\vec{k} - a_1b_3\vec{j} \\
& -a_2b_1\vec{k} + a_2b_2\vec{0} + a_2b_3\vec{i} + \\
& \quad a_3b_1\vec{j} - a_3b_2\vec{i} + a_3b_3\vec{0} \\
=& (a_2b_3 - a_3b_2)\vec{i} + (a_3b_1 - a_1b_3)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}
\end{aligned}
$$

This is a useful way of remembering how to compute the cross product, particularly when many of the components a_i and b_j are zero.

◀ **45.4 Example.** Compute $\vec{k} \times (p\vec{i} + q\vec{j} + r\vec{k})$:

$$\vec{k} \times (p\vec{i} + q\vec{j} + r\vec{k}) = p(\vec{k}\times\vec{i}) + q(\vec{k}\times\vec{j}) + r(\vec{k}\times\vec{k}) = -q\vec{i} + p\vec{j}.$$

▶

There is another way of remembering how to find $\vec{a} \times \vec{b}$. It involves the "triple product" and determinants. See § 45.3.

45.2. *Algebraic properties of the cross product*

Unlike the dot product, the cross product of two vectors behaves much less like ordinary multiplication. To begin with, the product is ***not commutative*** – instead one has

(70) $$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \quad \text{for } \textbf{\textit{all}} \text{ vectors } \vec{a} \text{ and } \vec{b}.$$

This property is sometimes called "anti-commutative."

Since the crossproduct of two vectors is again a vector you can compute the cross product of three vectors $\vec{a}, \vec{b}, \vec{c}$. You now have a choice: do you first multiply $\vec{a}$ and $\vec{b}$, or $\vec{b}$ and $\vec{c}$, or $\vec{a}$ and $\vec{c}$? With numbers it makes no difference (e.g. $2 \times (3 \times 5) = 2 \times 15 = 30$ and $(2 \times 3) \times 5 = 6 \times 5 =$ also 30) but with the cross product of vectors it does matter: the cross product is ***not associative,*** i.e.

$\vec{i} \times (\vec{i} \times \vec{j}) = \vec{i} \times \vec{k} = -\vec{j}$
$(\vec{i} \times \vec{i}) \times \vec{j} = \vec{0} \times \vec{j} = \vec{0}$
so "$\times$" is not associative

$$\vec{a}\times(\vec{b}\times\vec{c}) \neq (\vec{a}\times\vec{b})\times\vec{c} \quad \text{for } \textbf{\textit{most}} \text{ vectors } \vec{a},\vec{b},\vec{c}.$$

The ***distributive law*** does hold, i.e.

$$\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}, \quad \text{and} \quad (\vec{b}+\vec{c})\times\vec{a} = \vec{b}\times\vec{a}+\vec{c}\times\vec{a}$$

is true for all vectors $\vec{a},\vec{b},\vec{c}$.

Also, an associative law, where one of the factors is a number and the other two are vectors, does hold. I.e.

$$t(\vec{a}\times\vec{b}) = (t\vec{a})\times\vec{b} = \vec{a}\times(t\vec{b})$$

holds for all vectors $\vec{a},\vec{b}$ and any number t. We were already using these properties when we multiplied $(a_1\vec{i}+a_2\vec{j}+a_3\vec{k})\times(b_1\vec{i}+b_2\vec{j}+b_3\vec{k})$ in the previous section.

Finally, the cross product is only defined for space vectors, not for plane vectors.

45.3. *The triple product and determinants*

Definition 45.5. *The triple product of three given vectors $\vec{a},\vec{b}$, and $\vec{c}$ is defined to be*

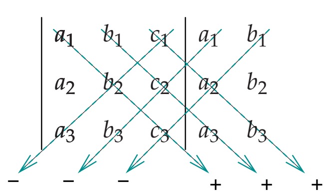
$$\vec{a}\bullet(\vec{b}\times\vec{c}).$$

In terms of the components of $\vec{a},\vec{b}$, and $\vec{c}$ one has

$$\begin{aligned}\vec{a}\bullet(\vec{b}\times\vec{c}) &= \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\bullet\begin{pmatrix}b_2c_3-b_3c_2\\b_3c_1-b_1c_3\\b_1c_2-b_2c_1\end{pmatrix}\\ &= a_1b_2c_3-a_1b_3c_2+a_2b_3c_1-a_2b_1c_3+a_3b_1c_2-a_3b_2c_1.\end{aligned}$$

This quantity is called a ***determinant,*** and is written as follows

(71) $$\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix} = a_1b_2c_3-a_1b_3c_2+a_2b_3c_1-a_2b_1c_3+a_3b_1c_2-a_3b_2c_1$$



There's a useful shortcut for computing such a determinant: after writing the determinant, append a fourth and a fifth column which are just copies of the first two columns of the determinant. The determinant then is the sum of six products, one for each dotted line in the drawing. Each term has a sign: if the factors are read from top-left to bottom-right, the term is positive, if they are read from top-right to bottom left the term is negative.

This shortcut is also very useful for computing the crossproduct. To compute the cross product of two given vectors $\vec{a}$ and $\vec{b}$ you arrange their components in the following determinant

(72) $$\vec{a}\times\vec{b} = \begin{vmatrix}\vec{i} & a_1 & b_1\\ \vec{j} & a_2 & b_2\\ \vec{k} & a_3 & b_3\end{vmatrix} = (a_2b_3-a_3b_2)\vec{i}+(a_3b_1-a_1b_3)\vec{j}+(a_1b_2-a_2b_1)\vec{k}.$$

This is not a normal determinant since some of its entries are vectors, but if you ignore that odd circumstance and simply compute the determinant according to the definition (71), you get (72).

An important property of the triple product is that it is much more symmetric in the factors $\vec{a},\vec{b},\vec{c}$ than the notation $\vec{a}\bullet(\vec{b}\times\vec{c})$ suggests.

Theorem 45.6. *For any triple of vectors $\vec{a}, \vec{b}, \vec{c}$ one has*

$$\vec{a} \bullet (\vec{b} \times \vec{c}) = \vec{b} \bullet (\vec{c} \times \vec{a}) = \vec{c} \bullet (\vec{a} \times \vec{b}),$$

and

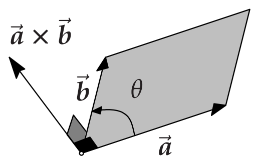
$$\vec{a} \bullet (\vec{b} \times \vec{c}) = -\vec{b} \bullet (\vec{a} \times \vec{c}) = -\vec{c} \bullet (\vec{b} \times \vec{a}).$$

In other words, if you exchange two factors in the product $\vec{a} \bullet (\vec{b} \times \vec{c})$ it changes its sign. If you "rotate the factors," i.e. if you replace $\vec{a}$ by $\vec{b}$, $\vec{b}$ by $\vec{c}$ and $\vec{c}$ by $\vec{a}$, the product doesn't change at all.

45.4. *Geometric description of the cross product*

Theorem 45.7.

$$\vec{a} \times \vec{b} \perp \vec{a}, \vec{b}$$



Proof. We use the triple product:

$$\vec{a} \bullet (\vec{a} \times \vec{b}) = \vec{b} \bullet (\vec{a} \times \vec{a}) = \vec{0}$$

since $\vec{a} \times \vec{a} = \vec{0}$ for any vector $\vec{a}$. It follows that $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$.

Similarly, $\vec{b} \bullet (\vec{a} \times \vec{b}) = \vec{a} \bullet (\vec{b} \times \vec{b}) = \vec{0}$ shows that $\vec{a} \bullet \vec{b}$ is perpendicular to $\vec{b}$. □

Theorem 45.8.

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \, \|\vec{b}\| \, \sin\theta$$

Proof. Bruce just slipped us a piece of paper with the following formula on it:

$$\|\vec{a} \times \vec{b}\|^2 + (\vec{a} \bullet \vec{b})^2 = \|\vec{a}\|^2 \|\vec{b}\|^2.$$

After setting $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and diligently computing both sides we find that this formula actually holds for any pair of vectors $\vec{a}, \vec{b}$! The (long) computation which implies this identity will be presented in class (maybe).

If we assume that Bruce's identity holds then we get

$$\|\vec{a} \times \vec{b}\|^2 = \|\vec{a}\|^2 \|\vec{b}\|^2 - (\vec{a} \bullet \vec{b})^2 = \|\vec{a}\|^2 \|\vec{b}\|^2 - \|\vec{a}\|^2 \|\vec{b}\|^2 \cos^2\theta = \|\vec{a}\|^2 \|\vec{b}\|^2 \sin^2\theta$$

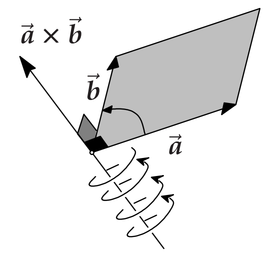
since $1 - \cos^2\theta = \sin^2\theta$. The theorem is proved. □

These two theorems *almost* allow you to construct the cross product of two vectors geometrically. If $\vec{a}$ and $\vec{b}$ are two vectors, then their cross product satisfies the following description:

(1) If $\vec{a}$ and $\vec{b}$ are parallel, then the angle θ between them vanishes, and so their cross product is the zero vector. Assume from here on that $\vec{a}$ and $\vec{b}$ are not parallel.
(2) $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$. In other words, since $\vec{a}$ and $\vec{b}$ are not parallel, they determine a plane, and their cross product is a vector perpendicular to this plane.
(3) the length of the cross product $\vec{a} \times \vec{b}$ is $\|\vec{a}\| \cdot \|\vec{b}\| \sin\theta$.

There are only two vectors that satisfy conditions 2 and 3: to determine which one of these is the cross product you must apply the ***Right Hand Rule*** (screwdriver rule, corkscrew rule, etc.) for $\vec{a}, \vec{b}, \vec{a} \times \vec{b}$: if you turn a screw whose axis is perpendicular to $\vec{a}$ and $\vec{b}$ in the direction from $\vec{a}$ to $\vec{b}$, the screw moves in the direction of $\vec{a} \times \vec{b}$.

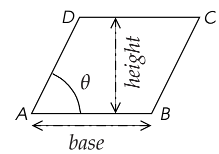
Alternatively, without seriously injuring yourself, you should be able to make a fist with your ***right*** hand, and then stick out your thumb, index and middle fingers so that your thumb is $\vec{a}$, your index finger is $\vec{b}$ and your middle finger is $\vec{a} \times \vec{b}$. Only people with the most flexible joints can do this with their left hand.



46. A few applications of the cross product

46.1. *Area of a parallelogram*

Let $ABCD$ be a parallelogram. Its area is given by "height times base," a formula which should be familiar from high school geometry.



If the angle between the sides AB and AD is θ, then the height of the parallelogram is $\|\overrightarrow{AD}\| \sin\theta$, so that the area of $ABCD$ is

$$\text{area of } ABCD = \|\overrightarrow{AB}\| \cdot \|\overrightarrow{AD}\| \sin\theta = \|\overrightarrow{AB} \times \overrightarrow{AD}\| . \tag{73}$$

The area of the triangle ABD is of course half as much,

$$\text{area of triangle } ABD = \tfrac{1}{2} \|\overrightarrow{AB} \times \overrightarrow{AD}\| .$$

These formulae are valid even when the points $A, B, C,$ and D are points in space. Of course they must lie in one plane for otherwise $ABCD$ couldn't be a parallelogram.

◀ **46.1 Example.** Let the points $A(1,0,2)$, $B(2,0,0)$, $C(3,1,-1)$ and $D(2,1,1)$ be given.

Show that ABCD is a parallelogram, and compute its area.

Solution: $ABCD$ will be a parallelogram if and only if $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD}$. In terms of the position vectors $\vec{a}$,$\vec{b}$, $\vec{c}$ and $\vec{d}$ of A, B, C, D this boils down to

$$\vec{c} - \vec{a} = (\vec{b} - \vec{a}) + (\vec{d} - \vec{a}), \quad \text{i.e.} \quad \vec{a} + \vec{c} = \vec{b} + \vec{d}.$$

For our points we get

$$\vec{a} + \vec{c} = \begin{pmatrix}1\\0\\2\end{pmatrix} + \begin{pmatrix}3\\1\\-1\end{pmatrix} = \begin{pmatrix}4\\1\\1\end{pmatrix}, \qquad vb + \vec{d} = \begin{pmatrix}2\\0\\0\end{pmatrix} + \begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}4\\1\\1\end{pmatrix}.$$

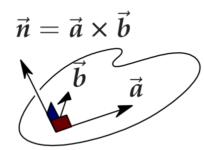
So $ABCD$ is indeed a parallelogram. Its area is the length of

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{pmatrix}2-1\\0\\0-2\end{pmatrix} \times \begin{pmatrix}2-2\\1-0\\1-0\end{pmatrix} = \begin{pmatrix}1\\0\\-2\end{pmatrix} \times \begin{pmatrix}1\\-1\\-1\end{pmatrix} = \begin{pmatrix}-2\\-1\\-1\end{pmatrix}.$$

So the area of $ABCD$ is $\sqrt{(-2)^2 + (-1)^2 + (-1)^2} = \sqrt{6}$. ▶

46.2. *Finding the normal to a plane*

If you know two vectors $\vec{a}$ and $\vec{b}$ which are parallel to a given plane $\mathcal{P}$ but not parallel to each other, then you can find a normal vector for the plane $\mathcal{P}$ by computing



$$\vec{n} = \vec{a} \times \vec{b}.$$

We have just seen that the vector $\vec{n}$ must be perpendicular to both $\vec{a}$ and $\vec{b}$, and hence[12] it is perpendicular to the plane $\mathcal{P}$.

This trick is especially useful when you have three points A, B and C, and you want to find the defining equation for the plane $\mathcal{P}$ through these points. We will assume that the three points do not all lie on one line, for otherwise there are many planes through A, B and C.

To find the defining equation we need one point on the plane (we have three of them), and a normal vector to the plane. A normal vector can be obtained by computing the cross

[12] This statement needs a proof which we will skip. Instead have a look at the picture

product of two vectors parallel to the plane. Since $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are both parallel to $\mathcal{P}$, the vector $\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$ is such a normal vector.

Thus the defining equation for the plane through three given points A, B and C is

$$\vec{n} \bullet (\vec{x} - \vec{a}) = 0, \quad \text{with} \quad \vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = (\vec{b} - \vec{a}) \times (\vec{c} - \vec{a}).$$

◀ **46.2 Find the defining equation of the plane $\mathcal{P}$ through the points $A(2, -1, 0)$, $B(2, 1, -1)$ and $C(-1, 1, 1)$. Find the intersections of $\mathcal{P}$ with the three coordinate axes, and find the distance from the origin to $\mathcal{P}$.**

Solution: We have

$$\overrightarrow{AB} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \quad \text{and} \quad \overrightarrow{AC} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$$

so that

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 6 \end{pmatrix}$$

is a normal to the plane. The defining equation for $\mathcal{P}$ is therefore

$$0 = \vec{n} \bullet (\vec{x} - \vec{a}) = \begin{pmatrix} 4 \\ 3 \\ 6 \end{pmatrix} \bullet \begin{pmatrix} x_1 - 2 \\ x_2 + 1 \\ x_3 - 0 \end{pmatrix}$$

i.e.

$$4x_1 + 3x_2 + 6x_3 - 5 = 0.$$

The plane intersects the x_1 axis when $x_2 = x_3 = 0$ and hence $4x_1 - 5 = 0$, i.e. in the point $(\frac{5}{4}, 0, 0)$. The intersections with the other two axes are $(0, \frac{5}{3}, 0)$ and $(0, 0, \frac{5}{6})$.

The distance from any point with position vector $\vec{x}$ to $\mathcal{P}$ is given by

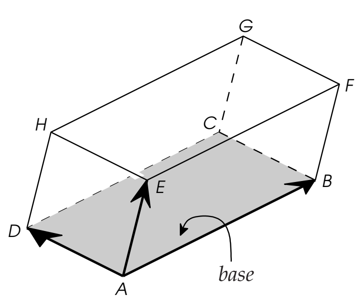
$$\text{dist} = \pm \frac{\vec{n} \bullet (\vec{x} - \vec{a})}{\|\vec{n}\|},$$

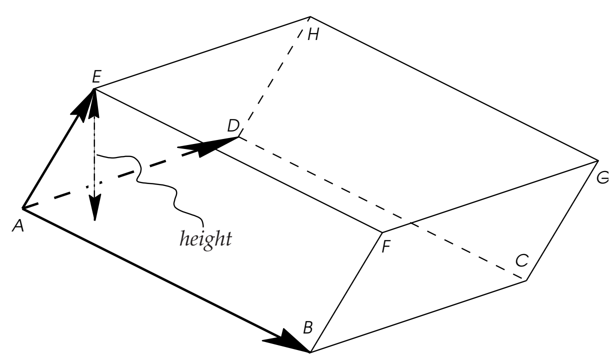
so the distance from the origin (whose position vector is $\vec{x} = \vec{0} = \left(\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix}\right)$) to $\mathcal{P}$ is

$$\text{distance origin to } \mathcal{P} = \pm \frac{\vec{a} \bullet \vec{n}}{\|\vec{n}\|} = \pm \frac{2 \cdot 4 + (-1) \cdot 3 + 0 \cdot 6}{\sqrt{4^2 + 3^2 + 6^2}} = \frac{5}{\sqrt{61}} (\approx 1.024 \cdots).$$

▶

46.3. *Volume of a parallelepiped*





A ***parallelepiped*** is a three dimensional body whose sides are parallelograms. For instance, a cube is an example of a parallelepiped; a rectangular block (whose faces are rectangles, meeting at right angles) is also a parallelepiped. Any parallelepiped has 8 vertices (corner points), 12 edges and 6 faces.

Let $\substack{ABCD \\ EFGH}$ be a parallelepiped. If we call one of the faces, say $ABCD$, the base of the parallelepiped, then the other face $EFGH$ is parallel to the base. The ***height of the parallelepiped*** is the distance from any point in $EFGH$ to the base, e.g. to compute the height of $\substack{ABCD \\ EFGH}$ one could compute the distance from the point E (or F, or G, or H) to the plane through $ABCD$.

The volume of the parallelepiped $\substack{ABCD \\ EFGH}$ is given by the formula

$$\text{Volume } \begin{matrix} ABCD \\ EFGH \end{matrix} = \text{Area of base} \times \text{height}.$$

Since the base is a parallelogram we know its area is given by

$$\text{Area of base} ABCD = \|\overrightarrow{AB} \times \overrightarrow{AD}\|$$

We also know that $\vec{n} = \overrightarrow{AB} \times \overrightarrow{AD}$ is a vector perpendicular to the plane through $ABCD$, i.e. perpendicular to the base of the parallelepiped. If we let the angle between the edge AE and the normal $\vec{n}$ be ψ, then the height of the parallelepiped is given by

$$\text{height} = \|\overrightarrow{AE}\| \cos \psi.$$

Therefore the triple product of $\overrightarrow{AB}, \overrightarrow{AD}, \overrightarrow{AE}$ is

$$\begin{aligned} \text{Volume } \begin{matrix} ABCD \\ EFGH \end{matrix} &= \text{height} \times \text{Area of base} \\ &= \|\overrightarrow{AE}\| \cos \psi \quad \|\overrightarrow{AB} \times \overrightarrow{AD}\| , \end{aligned}$$

i.e.

$$\boxed{\text{Volume } \begin{matrix} ABCD \\ EFGH \end{matrix} = \overrightarrow{AE} \bullet (\overrightarrow{AB} \times \overrightarrow{AD}).}$$

47. Notation

In the next chapter we will be using vectors, so let's take a minute to summarize the concepts and notation we have been using.

Given a point in the plane, or in space you can form its position vector. So associated to a point we have three different objects: the point, its position vector and its coordinates. here is the notation we use for these:

OBJECT	NOTATION
Point	Upper case letters, A, B, etc.
Position vector	Lowercase letters with an arrow on top. The position vector $\overrightarrow{OA}$ of the point A should be $\vec{a}$, so that letters match across changes from upper to lower case.
Coordinates of a point	The coordinates of the point A are the same as the components of its position vector $\vec{a}$: we use lower case letters with a subscript to indicate which coordinate we have in mind: (a_1, a_2).

48. PROBLEMS

Computing and drawing vectors

359. Simplify the following

$$\vec{a} = \begin{pmatrix}1\\-2\\3\end{pmatrix} + 3\begin{pmatrix}0\\1\\3\end{pmatrix};$$

$$\vec{b} = 12\begin{pmatrix}1\\1/3\end{pmatrix} - 3\begin{pmatrix}4\\1\end{pmatrix};$$

$$\vec{c} = (1+t)\begin{pmatrix}1\\1-t\end{pmatrix} - t\begin{pmatrix}1\\-t\end{pmatrix},$$

$$\vec{d} = t\begin{pmatrix}1\\0\\0\end{pmatrix} + t^2\begin{pmatrix}0\\-1\\2\end{pmatrix} - \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

360. If $\vec{a}, \vec{b}, \vec{c}$ are as in the previous problem, then which of the following expressions mean anything? Compute those expressions that are well defined.

(*i*) $\vec{a} + \vec{b}$ (*ii*) $\vec{b} + \vec{c}$ (*iii*) $\pi\vec{a}$

(*iv*) $\vec{b}^2$ (*v*) $\vec{b}/\vec{c}$ (*vi*) $\|\vec{a}\| + \|\vec{b}\|$

(*vii*) $\|\vec{b}\|^2$ (*viii*) $\vec{b}/\|\vec{c}\|$

361. Let $\vec{u}, \vec{v}, \vec{w}$ be three given vectors, and suppose

$$\vec{a} = \vec{v} + \vec{w}, \quad \vec{b} = 2\vec{u} - \vec{w}, \quad \vec{c} = \vec{u} + \vec{v} + \vec{w}.$$

(a) Simplify $\vec{p} = \vec{a} + 3\vec{b} - \vec{c}$ and $\vec{q} = \vec{c} - 2(\vec{u} + \vec{a})$.

(b) Find numbers r, s, t such that $r\vec{a} + s\vec{b} + t\vec{c} = \vec{u}$.

(c) Find numbers k, l, m such that $k\vec{a} + l\vec{b} + m\vec{c} = \vec{v}$.

362. Prove the Algebraic Properties (53), (54), (55), and (56) in section 41.2.

363. (a) Does there exist a number x such that

$$\begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}x\\x\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}?$$

(b) Make a drawing of all points P whose position vectors are given by

$$\vec{p} = \begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}x\\x\end{pmatrix}.$$

(c) Do there exist a numbers x and y such that

$$x\begin{pmatrix}1\\2\end{pmatrix} + y\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}?$$

364. Given points $A(2,1)$ and $B(-1,4)$ compute the vector $\overrightarrow{AB}$. *Is $\overrightarrow{AB}$ a position vector?*

365. Given: points $A(2,1)$, $B(3,2)$, $C(4,4)$ and $D(5,2)$. *Is $ABCD$ a parallelogram?*

366. Given: points $A(0,2,1)$, $B(0,3,2)$, $C(4,1,4)$ and D.

(a) If $ABCD$ is a parallelogram, then what are the coordinates of the point D?

(b) If $ABDC$ is a parallelogram, then what are the coordinates of the point D?

367. You are given three points in the plane: A has coordinates $(2,3)$, B has coordinates $(-1,2)$ and C has coordinates $(4,-1)$.

(a) Compute the vectors $\overrightarrow{AB}$, $\overrightarrow{BA}$, $\overrightarrow{AC}$, $\overrightarrow{CA}$, $\overrightarrow{BC}$ and $\overrightarrow{CB}$.

(b) Find the points P, Q, R and S whose position vectors are $\overrightarrow{AB}$, $\overrightarrow{BA}$, $\overrightarrow{AC}$, and $\overrightarrow{BC}$, respectively. *Make a precise drawing in figure 21.*

368. Have a look at figure 22

(a) Draw the vectors $2\vec{v} + \frac{1}{2}\vec{w}$, $-\frac{1}{2}\vec{v} + \vec{w}$, and $\frac{3}{2}\vec{v} - \frac{1}{2}\vec{w}$

(b) Find real numbers s, t such that $s\vec{v} + t\vec{w} = \vec{a}$.

(c) Find real numbers p, q such that $p\vec{v} + q\vec{w} = \vec{b}$.

(d) Find real numbers k, l, m, n such that $\vec{v} = k\vec{a} + l\vec{b}$, and $\vec{w} = m\vec{a} + n\vec{w}$.

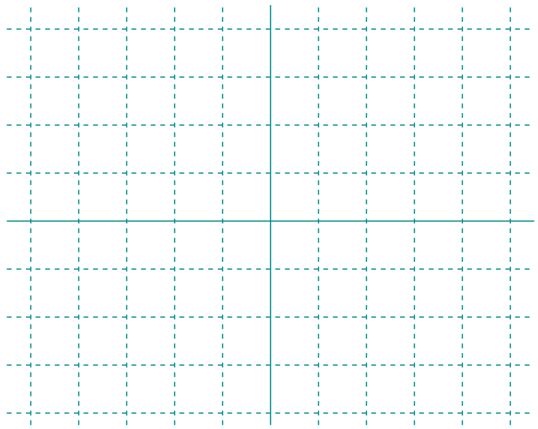


Figure 21: Your drawing for problem 367

Parametric Equations for a Line

369. In the figure below draw the points whose position vector are given by $\vec{x} = \vec{a} + t(\vec{b} - \vec{a})$ for $t = 0, 1, \frac{1}{3}, \frac{3}{4}, -1, 2$. (as always, $\vec{a} = \overrightarrow{OA}$, etc.)

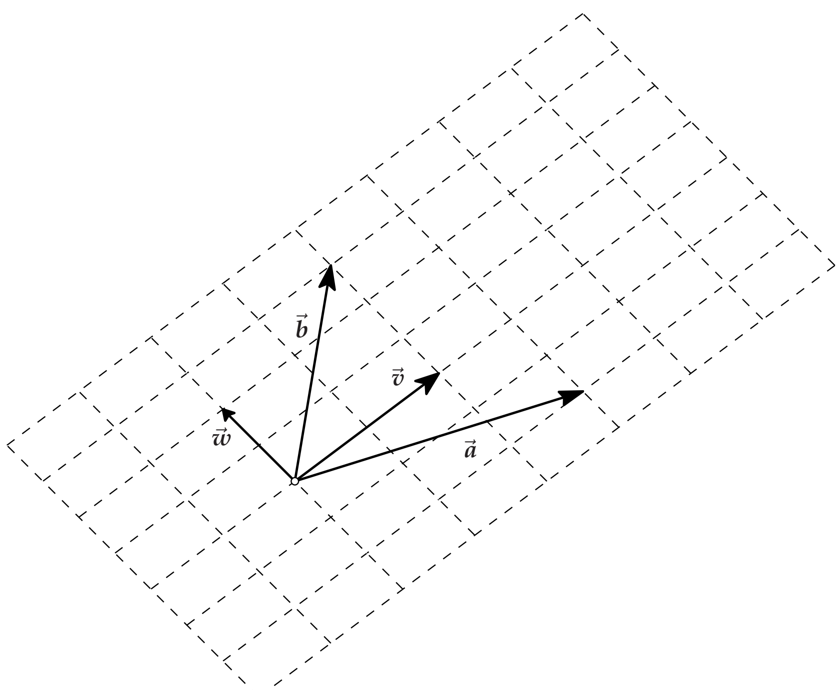
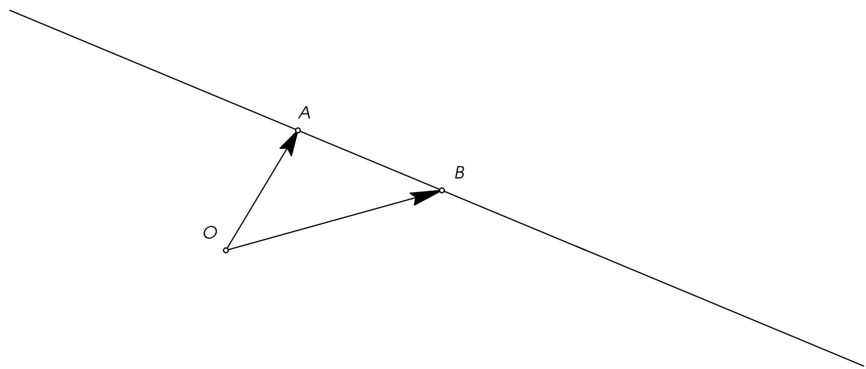


Figure 22: Drawing for problem 368



370. In the figure above also draw the points whose position vector are given by $\vec{x} = \vec{b} + s(\vec{a} - \vec{b})$ for $s = 0, 1, \frac{1}{3}, \frac{3}{4}, -1, 2$.

371. (*i*) Find a parametric equation for the line ℓ through the points $A(3, 0, 1)$ and $B(2, 1, 2)$.

(*ii*) Where does ℓ intersect the coordinate planes?

372. Consider a triangle ABC and let $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of A, B, and C.

(*i*) Compute the position vector of the midpoint P of the line segment BC. Also compute the position vectors of the midpoints Q of AC and R of AB. (Make a drawing.)

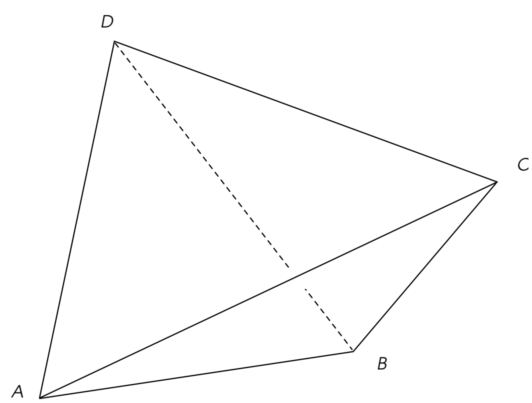
(*ii*) Let M be the point on the line segment AP which is twice as far from A as it is from P. Find the position vector of M.

(*iii*) Show that M also lies on the line segments BQ and CR.

373. Let $ABCD$ be a tetrahedron, and let $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ be the position vectors of the points A, B, C, D.

(*i*) Find position vectors of the midpoint P of AB, the midpoint Q of CD and the midpoint M of PQ.

(*ii*) Find position vectors of the midpoint R of BC, the midpoint S of AD and the midpoint N of RS.



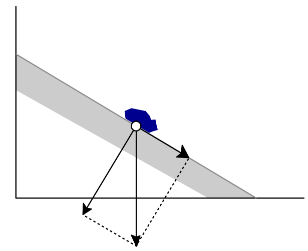
Orthogonal decomposition of one vector with respect to another

374. Given the vectors $\vec{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ find $\vec{a}^{\parallel}, \vec{a}^{\perp}, \vec{b}^{\parallel}, \vec{b}^{\perp}$ for which

$$\vec{a} = \vec{a}^{\parallel} + \vec{a}^{\perp}, \text{ with } a^{\parallel} \| \vec{b}, a^{\perp} \perp \vec{b},$$

and

$$\vec{b} = \vec{b}^{\|} + \vec{b}^{\perp}, \text{ with } b^{\|} \| \vec{a}, b^{\perp} \perp \vec{a}.$$



375. Bruce left his backpack on a hill, which in some coordinate system happens to be the line with equation $12x_1 + 5x_2 = 130$.
The force exerted by gravity on the backpack is $\vec{f}_{\text{grav}} = \binom{0}{-mg}$. Decompose this force into a part perpendicular to the hill, and a part parallel to the hill.

376. An eraser is lying on the plane $\mathcal{P}$ with equation $x_1 + 3x_2 + x_3 = 6$. Gravity pulls the eraser down, and exerts a force given by

$$\vec{f}_{\text{grav}} = \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix}.$$

(***i***) Find a normal $\vec{n}$ for the plane $\mathcal{P}$.

(***ii***) Decompose the force $\vec{f}$ into a part perpendicular to the plane $\mathcal{P}$ and a part perpendicular to $\vec{n}$.

The Dot Product

377. (***i***) Simplify $\|\vec{a} - \vec{b}\|^2$.

(***ii***) Simplify $\|2\vec{a} - \vec{b}\|^2$.

(***iii***) If $\vec{a}$ has length 3, $\vec{b}$ has length 7 and $\vec{a} \bullet \vec{b} = -2$, then compute $\|\vec{a} + \vec{b}\|$, $\|\vec{a} - \vec{b}\|$ and $\|2\vec{a} - \vec{b}\|$.

378. Simplify $(\vec{a} + \vec{b}) \bullet (\vec{a} - \vec{b})$.

379. Find the lengths of the sides, and the angles in the triangle ABC whose vertices are $A(2,1)$, $B(3,2)$, and $C(1,4)$.

380. Given: $A(1,1)$, $B(3,2)$ and a point C which lies on the line with parametric equation $\vec{c} = \binom{0}{3} + t\binom{1}{-1}$. If $\triangle ABC$ is a right triangle, then where is C? (There are three possible answers, depending on whether you assume A, B or C is the right angle.)

381. (***i***) Find the defining equation and a normal vector $\vec{n}$ for the line ℓ which is the graph of $y = 1 + \frac{1}{2}x$.

(***ii***) What is the distance from the origin to ℓ?

(***iii***) Answer the same two question for the line m which is the graph of $y = 2 - 3x$.

(***iv***) What is the angle between ℓ and m?

382. Let ℓ and m be the lines with parametrizations

$$\ell : \vec{x} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad m : \vec{x} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + s\begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

Where do they intersect, and find the angle between ℓ and m.

383. Let ℓ and m be the lines with parametrizations

$$\ell : \vec{x} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \qquad m : \vec{x} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + s \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}$$

Do ℓ and m intersect? Find the angle between ℓ and m.

384. Let ℓ and m be the lines with parametrizations

$$\ell : \vec{x} = \begin{pmatrix} 2 \\ \alpha \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \qquad m : \vec{x} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + s \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}$$

Here α is some unknown number.

If it is known that the lines ℓ and m intersect, what can you say about α?

The Cross Product

385. Compute the following cross products

(***i***) $\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ (***ii***) $\begin{pmatrix} 12 \\ -71 \\ 3\frac{1}{2} \end{pmatrix} \times \begin{pmatrix} 12 \\ -71 \\ 3\frac{1}{2} \end{pmatrix}$

(***iii***) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ (***iv***) $\begin{pmatrix} \sqrt{2} \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \end{pmatrix}$

(***v***) $\vec{i} \times (\vec{i} + \vec{j})$ (***vi***) $(\sqrt{2}\vec{i} + \vec{j}) \times \sqrt{2}\vec{j}$

(***vii***) $(2\vec{i} + \vec{k}) \times (\vec{j} - \vec{k})$ (***viii***) $(\cos\theta\vec{i} + \sin\theta\vec{k}) \times (\sin\theta\vec{i} - \cos\theta\vec{k})$

386. (***i***) Simplify $(\vec{a} + \vec{b}) \times (\vec{a} + \vec{b})$.

(***ii***) Simplify $(\vec{a} - \vec{b}) \times (\vec{a} - \vec{b})$.

(***iii***) Simplify $(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})$.

387. *True or False:* If $\vec{a} \times \vec{b} = \vec{c} \times \vec{b}$ and $\vec{b} \neq \vec{0}$ then $\vec{a} = \vec{c}$?

388. Given $A(2,0,0)$, $B(0,0,2)$ and $C(2,2,2)$. Let $\mathcal{P}$ be the plane through A, B and C.

(***i***) Find a normal vector for $\mathcal{P}$.

(***ii***) Find a defining equation for $\mathcal{P}$.

(***iii***) What is the distance from $D(0,2,0)$ to $\mathcal{P}$? What is the distance from the origin $O(0,0,0)$ to $\mathcal{P}$?

(***iv***) Do D and O lie on the same side of $\mathcal{P}$?

(***v***) Find the area of the triangle ABC.

(***vi***) Where does the plane $\mathcal{P}$ intersect the three coordinate axes?

389. (***i***) Does $D(2,1,3)$ lie on the plane $\mathcal{P}$ through the points $A(-1,0,0)$, $B(0,2,1)$ and $C(0,3,0)$?

(***ii***) The point $E(1,1,\alpha)$ lies on $\mathcal{P}$. What is α?

390. Given points $A(1,-1,1)$, $B(2,0,1)$ and $C(1,2,0)$.

(***i***) Where is the point D which makes $ABCD$ into a parallelogram?

(***ii***) What is the area of the parallelogram $ABCD$?

(***iii***) Find a defining equation for the plane $\mathcal{P}$ containing the parallelogram $ABCD$.

(***iv***) Where does $\mathcal{P}$ intersect the coordinate axes?

391. Given points $A(1,0,0)$, $B(0,2,0)$ and $D(-1,0,1)$ and $E(0,0,2)$.

(***i***) If $\mathfrak{P} = {}^{ABCD}_{EFGH}$ is a parallelepiped, then where are the points C, F, G and H?

(***ii***) Find the area of the base $ABCD$ of $\mathfrak{P}$.

(***iii***) Find the height of $\mathfrak{P}$.

(***iv***) Find the volume of $\mathfrak{P}$.

---

392. Let ${}^{ABCD}_{EFGH}$ be the cube with A at the origin, $B(1,0,0)$, $D(0,1,0)$ and $E(0,0,1)$.

(***i***) Find the coordinates of all the points A, B, C, D, E, F, G, H.

(***ii***) Find the position vectors of the midpoints of the line segments AG, BH, CE and DF. Make a drawing of the cube with these line segments.

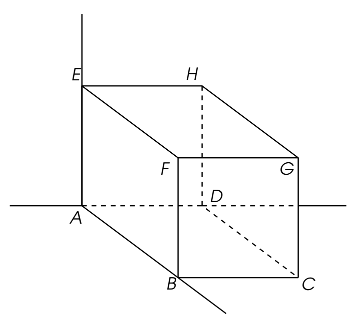
(***iii***) Find the defining equation for the plane BDE. Do the same for the plane CFH. Show that these planes are parallel.

(***iv***) Find the parametric equation for the line through AG.

(***v***) Where do the planes BDE and CFH intersect the line AG?

(***vi***) Find the angle between the planes BDE and BGH.

(***vii***) Find the angle between the planes BDE and BCH. Draw these planes.



Vector Functions and Parametrized Curves

49. Parametric Curves

Definition 49.1. *A vector function $\vec{f}$ of one variable is a function of one real variable, whose values $\vec{f}(t)$ are vectors.*

In other words for any value of t (from a ***domain*** of allowed values, usually an interval) the vector function $\vec{f}$ produces a vector $\vec{f}(t)$. Write $\vec{f}$ in components:

$$\vec{f}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}.$$

The components of a vector function $\vec{f}$ of t are themselves functions of t. They are ordinary first-semester-calculus-style functions. An example of a vector function is

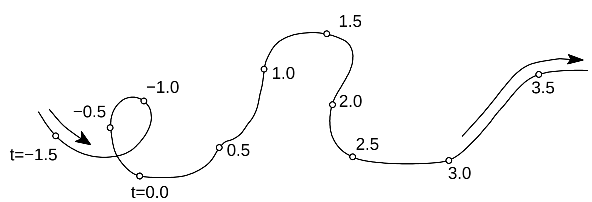
$$\vec{f}(t) = \begin{pmatrix} t - 2t^2 \\ 1 + \cos^2 \pi t \end{pmatrix}, \quad \text{so } \vec{f}(1) = \begin{pmatrix} 1 - 2(1)^2 \\ 1 + (\cos \pi)^2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

(just to mention one.)

Definition 49.2. *A **parametric curve** is a vector function $\vec{x} = \vec{x}(t)$ of one real variable t. The variable t is called the **parameter.***

Synonyms: "Parametrized curve," or "parametrization," or "vector function (of one variable)."

Logically speaking a parametrized curve is the same thing as a vector function. The name "parametrized curve" is used to remind you of a very natural and common interpretation of the concept "parametric curve." In this interpretation a vector function, or parametric curve $\vec{x}(t)$ describes the motion of a point in the plane or space. Here t stands for time, and $\vec{x}(t)$ is the position vector at time t of the moving point.



A picture of a vector function.

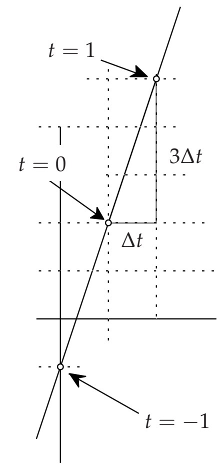
Instead of writing a parametrized curve as a vector function, one sometimes specifies the two (or three) components of the curve. Thus one will say that a parametric curve is given by

$$x_1 = x_1(t), \quad x_2 = x_2(t), \quad (\text{and } x_3 = x_3(t) \text{ if we have a space curve}).$$

50. Examples of parametrized curves

◀ **50.1 An example of Rectilinear Motion.** Here's a parametric curve:

$$\vec{x}(t) = \begin{pmatrix} 1+t \\ 2+3t \end{pmatrix}. \tag{74}$$



The components of this vector function are

$$x_1(t) = 1 + t, \qquad x_2(t) = 2 + 3t. \tag{75}$$

Both components are linear functions of time (i.e. the parameter t), so every time t increases by an amount Δt (every time Δt seconds go by) the first component increases by Δt, and the x_2 component increases by $3\Delta t$. So the point at $\vec{x}(t)$ moves horizontally to the left with speed 1, and it moves vertically upwards with speed 3.

Which curve is traced out by this vector function? In this example we can find out by eliminating the parameter, i.e. solve one of the two equations (75) for t, and substitute the value of t you find in the other equation. Here you can solve $x_1 = 1 + t$ for t, with result $t = x_1 - 1$. From there you find that

$$x_2 = 2 + 3t = 2 + 3(x_1 - 1) = 3x_1 - 1.$$

So for any t the vector $\vec{x}(t)$ is the position vector of a point on the line $x_2 = 3x_1 - 1$ (or, if you prefer the old fashioned x, y coordinates, $y = 3x - 1$).

Conclusion: This particular parametric curve traces out a straight line with equation $x_2 = 3x_1 - 1$, going from left to right. ▶

◀ **50.2 Rectilinear Motion in general.** This example generalizes the previous example. The parametric equation for a straight line from the previous chapter

$$\vec{x}(t) = \vec{a} + t\vec{v},$$

is a parametric curve. We had $\vec{v} = \vec{b} - \vec{a}$ in §42. At time $t = 0$ the object is at the point with position vector $\vec{a}$, and every second (unit of time) the object translates by $\vec{v}$. The vector $\vec{v}$ is the ***velocity vector*** of this motion.

In the first example we had $\vec{a} = \binom{1}{2}$, and $\vec{v} = \binom{1}{3}$. ▶

◀ **50.3 Going back and forth on a straight line.** Consider

$$\vec{x}(t) = \vec{a} + \sin(t)\vec{v}.$$

At each moment in time the object whose motion is described by this parametric curve finds itself on the straight line ℓ with parametric equation $\vec{x} = \vec{a} + s(\vec{b} - \vec{a})$, where $\vec{b} = \vec{a} + \vec{v}$.

However, instead of moving along the line from one end to the other, the point at $\vec{x}(t)$ keeps moving back and forth along ℓ between $\vec{a} + \vec{v}$ and $\vec{a} - \vec{v}$. ▶

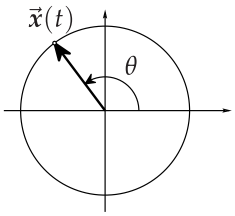
◀ **50.4 Motion along a graph.** Let $y = f(x)$ be some function of one variable (defined for x in some interval) and consider the parametric curve given by

$$\vec{x}(t) = \begin{pmatrix} t \\ f(t) \end{pmatrix} = t\vec{i} + f(t)\vec{j}.$$

At any moment in time the point at $\vec{x}(t)$ has x_1 coordinate equal to t, and $x_2 = f(t) = f(x_1)$, since $x_1 = t$. So this parametric curve describes motion on the graph of $y = f(x)$ in which the horizontal coordinate increases at a constant rate. ▶

◀ **50.5 The standard parametrization of a circle.** Consider the parametric curve

$$\vec{x}(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$



The two components of this parametrization are

$$x_1(\theta) = \cos\theta, \quad x_2(\theta) = sin\theta,$$

and they satisfy

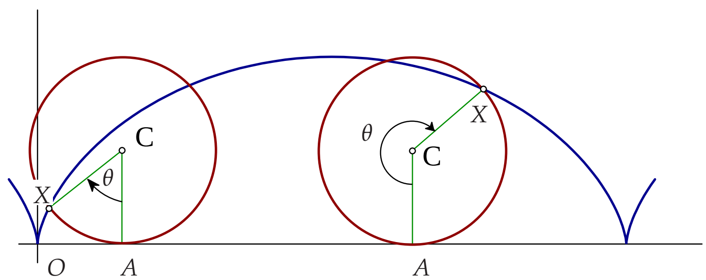
$$x_1(\theta)^2 + x_2(\theta)^2 = \cos^2\theta + \sin^2\theta = 1,$$

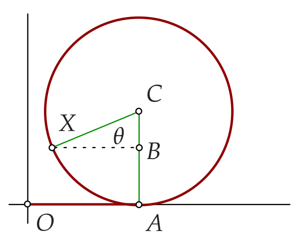
so that $\vec{x}(\theta)$ always points at a point on the unit circle.

As θ increases from $-\infty$ to $+\infty$ the point will rotate through the circle, going around infinitely often. Note that the point runs through the circle in the ***counterclockwise direction,*** which is the mathematician's favorite way of running around in circles. ▶

◀ **50.6 The Cycloid.** The Free Ferris Wheel Foundation is an organization whose goal is to empower fairground ferris wheels to roam freely and thus realize their potential. With blatant disregard for the public, members of the F^{2}WF will clandestinely unhinge ferris wheels, thereby setting them free to roll throughout the fairground and surroundings.

Suppose we were to step into the bottom of a ferris wheel at the moment of its liberation: what would happen? Where would the wheel carry us? Let our position be the point X, and let its position vector at time t be $\vec{x}(t)$. The parametric curve $\vec{x}(t)$ which describes our motion is called the cycloid.





In this example we are given a description of a motion, but no formula for the parametrization $\vec{x}(t)$. We will have to derive this formula ourselves. The key to finding $\vec{x}(t)$ is the fact that the arc AX on the wheel is exactly as long as the line segment OA on the ground (i.e. the x_1 axis). The length of the arc AX is exactly the angle θ ("arc = radius times angle in radians"), so the x_1 coordinate of A and hence the center C of the circle is θ. To find X consider the right triangle BCX. Its hypothenuse is the radius of the circle, i.e. CX has length 1. The angle at C is θ, and therefore you get

$$BX = \sin\theta, \qquad BC = \cos\theta,$$

and

$$x_1 = OA - BX = \theta - \sin\theta, \qquad x_2 = AC - BC = 1 - \cos\theta.$$

So the parametric curve defined in the beginning of this example is

$$\vec{x}(\theta) = \begin{pmatrix} \theta - \sin\theta \\ 1 - \cos\theta \end{pmatrix}.$$

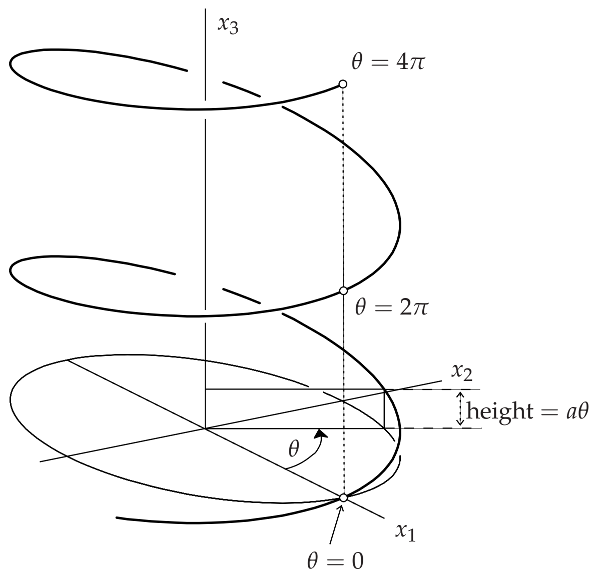
Here the angle θ is the parameter, and we can let it run from $\theta = -\infty$ to $\theta = \infty$. ▶

◀ **50.7 A three dimensional example: the Helix.** Consider the vector function

$$\vec{x}(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ a\theta \end{pmatrix}$$

where $a > 0$ is some constant.

If you ignore the x_3 component of this vector function you get the parametrization of the circle from example 50.5. So as the parameter θ runs from $-\infty$ to $+\infty$, the x_1, x_2 part of $\vec{x}(\theta)$ runs around on the unit circle infinitely often. While this happens the vertical component, i.e. $x_3(\theta)$ increases steadily from $-\infty$ to ∞ at a rate of a units per second.



▶

51. The derivative of a vector function

If $\vec{x}(t)$ is a vector function, then we define its ***derivative*** to be

$$\vec{x}'(t) = \frac{d\vec{x}}{dt} = \lim_{h\to 0} \frac{\vec{x}(t+h) - \vec{x}(t)}{h}.$$

This definition looks very much like the first-semester-calculus-definition of the derivative of a function, but for it to make sense in the context of vector functions we have to explain what the limit of a vector function is.

By definition, for a vector function $\vec{f}(t) = \binom{f_1(t)}{f_2(t)}$ one has

$$\lim_{t\to a} \vec{f}(t) = \lim_{t\to a} \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} = \begin{pmatrix} \lim_{t\to a} f_1(t) \\ \lim_{t\to a} f_2(t) \end{pmatrix}$$

In other words, to compute the limit of a vector function you just compute the limits of its components (that will be our definition.)

Let's look at the definition of the velocity vector again. Since

$$\frac{\vec{x}(t+h)-\vec{x}(t)}{h} = \frac{1}{h}\left\{\begin{pmatrix} x_1(t+h) \\ x_2(t+h)\end{pmatrix} - \begin{pmatrix} x_1(t) \\ x_2(t)\end{pmatrix}\right\} = \begin{pmatrix} \dfrac{x_1(t+h)-x_1(t)}{h} \\ \dfrac{x_2(t+h)-x_2(t)}{h}\end{pmatrix}$$

we have

$$\vec{x}'(t) = \lim_{h\to 0}\frac{\vec{x}(t+h)-\vec{x}(t)}{h} = \begin{pmatrix} \lim_{h\to 0}\dfrac{x_1(t+h)-x_1(t)}{h} \\ \lim_{h\to 0}\dfrac{x_2(t+h)-x_2(t)}{h}\end{pmatrix} = \begin{pmatrix} x_1'(t) \\ x_2'(t)\end{pmatrix}$$

So: To compute the derivative of a vector function you must differentiate its components.

◀ **51.1 Example.** Compute the derivative of

$$\vec{x}(t) = \begin{pmatrix}\cos t \\ \sin t\end{pmatrix} \quad \text{and of} \quad \vec{y}(t) = \begin{pmatrix} t-\sin t \\ 1-\cos t\end{pmatrix}.$$

Solution:

$$\vec{x}'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\cos t \\ \sin t\end{pmatrix} = \begin{pmatrix}-\sin t \\ \cos t\end{pmatrix}$$

$$\vec{y}'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} t-\sin t \\ 1-\cos t\end{pmatrix} = \begin{pmatrix} 1-\cos t \\ \sin t\end{pmatrix}.$$

▶

52. Higher derivatives and product rules

Not every function has a derivative, so it may happen that you can find $\vec{x}'(t)$ but not $\vec{x}''(t)$

If you differentiate a vector function $\vec{x}(t)$ you get another vector function, namely $\vec{x}'(t)$, and you can try to differentiate that vector function again. If you succeed, the result is called the second derivative of $\vec{x}(t)$. All this is very similar to how the second (and higher) derivative of ordinary functions were defined in 1st semester calculus. One even uses the same notation:

$$\vec{x}''(t) = \frac{\mathrm{d}\vec{x}'(t)}{\mathrm{d}t} = \frac{\mathrm{d}^2\vec{x}}{\mathrm{d}t^2} = \begin{pmatrix} x_1''(t) \\ x_2''(t)\end{pmatrix}.$$

◀ **52.1 Example.** Compute the second derivative of

$$\vec{x}(t) = \begin{pmatrix}\cos t \\ \sin t\end{pmatrix} \quad \text{and of} \quad \vec{y}(t) = \begin{pmatrix} t-\sin t \\ 1-\cos t\end{pmatrix}.$$

Solution: In example 51.1 we already found the first derivatives, so you can use those. You find

$$\vec{x}''(t) = \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}-\sin t \\ \cos t\end{pmatrix} = \begin{pmatrix}-\cos t \\ -\sin t\end{pmatrix}$$

$$\vec{y}''(t) = \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} 1-\cos t \\ \sin t\end{pmatrix} = \begin{pmatrix} \sin t \\ -\cos t\end{pmatrix}.$$

Note that our standard parametrization $\vec{x}(t)$ of the circle satisfies

$$\vec{x}''(t) = -\vec{x}(t).$$

▶

After defining the derivative in first semester calculus one quickly introduces the various rules (sum, product, quotient, chain rules) which make it possible to compute derivatives without ever actually having to use the limit-of-difference-quotient-definition. For vector functions there are similar rules which also turn out to be useful.

The ***Sum Rule*** holds. It says that if $\vec{x}(t)$ and $\vec{y}(t)$ are differentiable[13] vector functions, then so is $\vec{z}(t) = \vec{x}(t) \pm \vec{y}(t)$, and one has

$$\frac{d\vec{x}(t) \pm \vec{y}(t)}{dt} = \frac{d\vec{x}(t)}{dt} \pm \frac{d\vec{y}(t)}{dt}.$$

The ***Product Rule*** also holds, but it is more complicated, because there are several different forms of multiplication when you have vector functions. The following three versions all hold:

If $\vec{x}(t)$ and $\vec{y}(t)$ are differentiable vector functions and $f(t)$ is an ordinary differentiable function, then

$$\frac{df(t)\vec{x}(t)}{dt} = f(t)\frac{d\vec{x}(t)}{dt} + \frac{df(t)}{dt}\vec{x}(t)$$
$$\frac{d\vec{x}(t) \bullet \vec{y}(t)}{dt} = \vec{x}(t) \bullet \frac{d\vec{y}(t)}{dt} + \frac{d\vec{x}(t)}{dt} \bullet \vec{y}(t)$$
$$\frac{d\vec{x}(t) \times \vec{y}(t)}{dt} = \vec{x}(t) \times \frac{d\vec{y}(t)}{dt} + \frac{d\vec{x}(t)}{dt} \times \vec{y}(t)$$

I hope these formulae look plausible because they look like the old fashioned product rule, but even if they do, you still have to prove them before you can accept their validity. I will prove one of these in lecture. You will do some more as an exercise.

As an example of how these properties get used, consider this theorem:

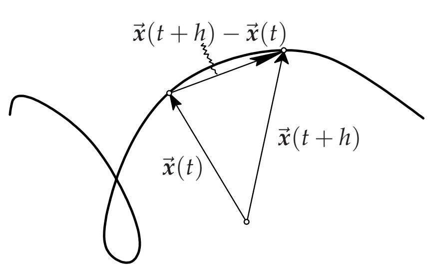
Theorem 52.2. *Let $\vec{f}(t)$ be a vector function of constant length (i.e. $\|\vec{f}(t)\|$ is constant.) Then $\vec{f}'(t) \perp \vec{f}(t)$.*

Proof. If $\|\vec{f}\|$ is constant, then so is $\vec{f}(t) \bullet \vec{f}(t) = \|\vec{f}(t)\|^2$. the derivative of a constant function is zero, so

$$0 = \frac{d}{dt}\left(\|\vec{f}(t)\|^2\right) = \frac{d}{dt}\left(\|\vec{f}(t)\| \bullet \|\vec{f}(t)\|\right) = 2\vec{f}(t) \bullet \frac{d\vec{f}(t)}{dt}.$$

So we see that $\vec{f} \bullet \vec{f}' = 0$ which means that $\vec{f}' \perp \vec{f}$. □

53. Interpretation of $\vec{x}'(t)$ as the velocity vector



[13] A vector function is differentiable if its derivative actually exists, i.e. if *all* its components are differentiable.

Let $\vec{x}(t)$ be some vector function and interpret it as describing the motion of some point in the plane (or space). At time t the point has position vector $\vec{x}(t)$; a little later, more precisely, h seconds later the point has position vector $\vec{x}(t+h)$. Its displacement is the difference vector

$$\vec{x}(t+h) - \vec{x}(t).$$

Its average velocity vector between times t and $t+h$ is

$$\frac{\text{displacement vector}}{\text{time lapse}} = \frac{\vec{x}(t+h) - \vec{x}(t)}{h}.$$

If the average velocity between times t and $t+h$ converges to one definite vector as $h \to 0$, then this limit is a reasonable candidate for ***the velocity vector at time*** t of the parametric curve $\vec{x}(t)$.

Being a vector, the velocity vector has both *magnitude* and *direction*. The length of the velocity vector is called the ***speed*** of the parametric curve. We use the following notation: we always write

$$\vec{v}(t) = \vec{x}'(t)$$

for the velocity vector, and

$$v(t) = \|\vec{v}(t)\| = \|\vec{x}'(t)\|$$

for its length, i.e. the speed.

The speed v is always a nonnegative number; the velocity is always a vector.

◀ **53.1 Velocity of linear motion.** If $\vec{x}(t) = \vec{a} + t\vec{v}$, as in examples 50.1 and 50.2, then

$$\vec{x}(t) = \begin{pmatrix} a_1 + tv_1 \\ a_2 + tv_2 \end{pmatrix}$$

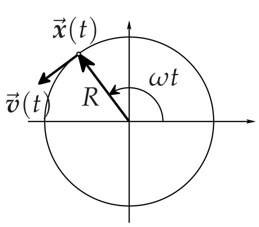
so that

$$\vec{x}'(t) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \vec{v}.$$

So when you represent a line by a parametric equation $\vec{x}(t) = \vec{a} + t\vec{v}$, the vector $\vec{v}$ is the velocity vector. The length of $\vec{v}$ is the speed of the motion.

In example 50.1 we had $\vec{v} = \binom{1}{3}$, so the speed with which the point at $\vec{x}(t) = \binom{1+t}{1+3t}$ traces out the line is $v = \|\vec{v}\| = \sqrt{1^2+3^2} = \sqrt{10}$.

▶



◀ **53.2 Motion on a circle.** Consider the parametrization

$$\vec{x}(t) = \begin{pmatrix} R\cos\omega t \\ R\sin\omega t \end{pmatrix}.$$

The point X at $\vec{x}(t)$ is on the circle centered at the origin with radius R. The segment from the origin to X makes an angle ωt with the x-axis; this angle clearly increases at a constant rate of ω radians per second.

The velocity vector of this motion is

$$\vec{v}(t) = \vec{x}'(t) = \begin{pmatrix} -\omega R\sin\omega t \\ \omega R\cos\omega t \end{pmatrix} = \omega R \begin{pmatrix} \sin\omega t \\ \cos\omega t \end{pmatrix}.$$

This vector is not constant. however, if you calculate the speed of the point X, you find

$$v = \|\vec{v}(t)\| = \omega R \left\| \begin{pmatrix} \sin\omega t \\ \cos\omega t \end{pmatrix} \right\| = \omega R.$$

So while the direction of the velocity vector $\vec{v}(t)$ is changing all the time, its magnitude is constant. In this parametrization the point X moves along the circle with constant speed $v = \omega R$. ▶

◀ **53.3 Velocity of the cycloid.** Think of the dot X on the wheel in the cycloid example 50.6. We know its position vector and velocity at time t

$$\vec{x}(t) = \begin{pmatrix} t - \sin t \\ 1 - \cos t \end{pmatrix}, \qquad \vec{x}\,'(t) = \begin{pmatrix} 1 - \cos t \\ \sin t \end{pmatrix}.$$

The speed with which X traces out the cycloid is

$$\begin{aligned} v &= \|\vec{x}\,'(t)\| \\ &= \sqrt{(1-\cos t)^2 + (\sin t)^2} \\ &= \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} \\ &= \sqrt{2(1-\cos t)}. \end{aligned}$$

You can use the double angle formula $\cos 2\alpha = 1 - 2\sin^2\alpha$ with $\alpha = \frac{t}{2}$ to simplify this to

$$v = \sqrt{4\sin^2 \tfrac{t}{2}} = 2\left|\sin\frac{t}{2}\right|.$$

The speed of the point X on the cycloid is therefore always between 0 and 2. At times $t = 0$ and other multiples of 2π we have $\vec{x}\,'(t) = \vec{\mathbf{0}}$. At these times the point X has come to a stop. At times $t = \pi + 2k\pi$ one has $v = 2$ and $\vec{x}\,'(t) = \binom{2}{0}$, i.e. the point X is moving horizontally to the right with speed 2. ▶

54. Acceleration and Force

Just as the derivative $\vec{x}\,'(t)$ of a parametric curve can be interpreted as the velocity vector $\vec{v}(t)$, the derivative of the velocity vector measures the rate of change with time of the velocity and is called the ***acceleration*** of the motion. The usual notation is

$$\vec{a}(t) = \vec{v}\,'(t) = \frac{d\vec{v}(t)}{dt} = \frac{d^2\vec{x}}{dt^2} = \vec{x}\,''(t).$$

Sir ISAAC NEWTON's law relating force and acceleration via the formula "$F = ma$" has a vector version. *If an object's motion is given by a parametrized curve $\vec{x}(t)$ then this motion is the result of a force $\vec{F}$ being exerted on the object. The force $\vec{F}$ is given by*

$$\vec{F} = m\vec{a} = m\frac{d^2\vec{x}}{dt^2}$$

where m is the mass of the object.

Somehow it is always assumed that the mass m is a positive number.

◀ **54.1 How does an object move if no forces act on it?**

If $\vec{F}(t) = \vec{\mathbf{0}}$ at all times, then, assuming $m \neq 0$ it follows from $\vec{F} = m\vec{a}$ that $\vec{a}(t) = \vec{\mathbf{0}}$. Since $\vec{a}(t) = \vec{v}\,'(t)$ you conclude that the velocity vector $\vec{v}(t)$ must be constant, i.e. that there is some fixed vector $\vec{v}$ such that

$$\vec{x}\,'(t) = \vec{v}(t) = \vec{v} \text{ for all } t.$$

This implies that

$$\vec{x}(t) = \vec{x}(0) + t\vec{v}.$$

So if no force acts on an object, then it will move with constant velocity vector along a straight line (said Newton – Archimedes long before him thought that the object would slow down and come to a complete stop unless there were a force to keep it going.) ▶

◀ **54.2 Compute the forces acting on a point on a circle.** Consider an object moving with constant angular velocity ω on a circle of radius R, i.e. consider $\vec{x}(t)$ as in example 53.2,

$$\vec{x}(t) = \begin{pmatrix} R\cos\omega t \\ R\sin\omega t \end{pmatrix} = R\begin{pmatrix} \cos\omega t \\ \sin\omega t \end{pmatrix}.$$

Then its velocity and acceleration vectors are

$$\vec{v}(t) = \omega R \begin{pmatrix} -\sin\omega t \\ \cos\omega t \end{pmatrix}$$

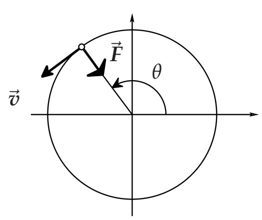
and

$$\begin{aligned} \vec{a}(t) = \vec{v}\,'(t) &= \omega^2 R \begin{pmatrix} -\cos\omega t \\ -\sin\omega t \end{pmatrix} \\ &= -\omega^2 R \begin{pmatrix} \cos\omega t \\ \sin\omega t \end{pmatrix} \end{aligned}$$

Since both $\left(\begin{smallmatrix} \cos\theta \\ \sin\theta \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -\sin\theta \\ \cos\theta \end{smallmatrix}\right)$ are unit vectors, we see that the velocity vector changes its direction but not its size: at all times you have $v = \|\vec{v}\| = \omega R$. The acceleration also keeps changing its direction, but its magnitude is always

$$a = \|\vec{a}\| = \omega^2 R = \left(\frac{v}{R}\right)^2 R = \frac{v^2}{R}.$$

The force which must be acting on the object to make it go through this motion is



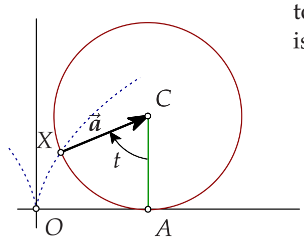
$$\vec{F} = m\vec{a} = -m\omega^2 R \begin{pmatrix} \cos\omega t \\ \sin\omega t \end{pmatrix}.$$

To conclude this example note that you can write this force as

$$\vec{F} = -m\omega^2 \vec{x}(t)$$

which tells you which way the force is directed: towards the center of the circle. ▶

◀ **54.3 How does it feel, to be on the Ferris wheel?** In other words, which force acts on us if we get carried away by a "liberated ferris wheel," as in example 50.6?



Well, you get pushed around by a force $\vec{F}$, which according to Newton is given by $\vec{F} = m\vec{a}$, where m is your mass and $\vec{a}$ is your acceleration, which we now compute:

$$\begin{aligned} \vec{a}(t) &= \vec{v}\,'(t) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} 1-\cos t \\ \sin t \end{pmatrix} \\ &= \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}. \end{aligned}$$

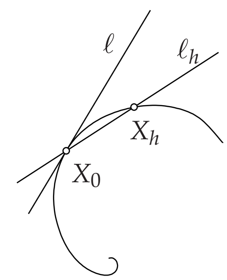
This is a unit vector: the force that's pushing you around is constantly changing its direction but its strength stays the same. If you remember that t is the angle $\angle ACX$ you see that the force $\vec{F}$ is always pointed at the center of the wheel: its direction is given by the vector $\overrightarrow{XC}$. ▶

55. Tangents and the unit tangent vector

Here we address the problem of finding the tangent line at a point on a parametric curve.

Let $\vec{x}(t)$ be a parametric curve, and let's try to find the tangent line at a particular point X_0, with position vector $\vec{x}(t_0)$ on this curve. We follow the same strategy as in 1st semester calculus: pick a point X_h on the curve near X_0, draw the line through X_0 and X_h and let $X_h \to X_0$.

The line through two points on a curve is often called a ***secant*** to the curve. So we are going to construct a tangent to the curve as a limit of secants.

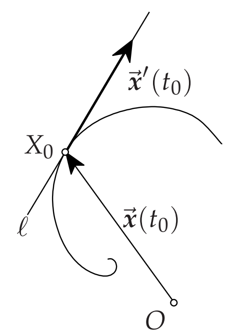


The point X_0 has position vector $\vec{x}(t_0)$, the point X_h is at $\vec{x}(t_0+h)$. Consider the line ℓ_h parametrized by

$$\vec{y}(s;h) = \vec{x}(t_0) + s\frac{\vec{x}(t_0+h) - \vec{x}(t_0)}{h}, \tag{76}$$

in which s is the parameter we use to parametrize the line.

The line ℓ_h contains both X_0 (set $s=0$) and X_h (set $s=h$), so it is the line through X_0 and X_h, i.e. a secant to the curve.



Now we let $h \to 0$, which gives

$$\vec{y}(s) \stackrel{\text{def}}{=} \lim_{h\to 0} \vec{y}(s;h) = \vec{x}(t_0) + s\lim_{h\to 0}\frac{\vec{x}(t_0+h) - \vec{x}(t_0)}{h} = \vec{x}(t_0) + s\vec{x}'(t_0),$$

In other words, the tangent line to the curve $\vec{x}(t)$ at the point with position vector $\vec{x}(t_0)$ has parametric equation

$$\vec{y}(s) = \vec{x}(t_0) + s\vec{x}'(t_0),$$

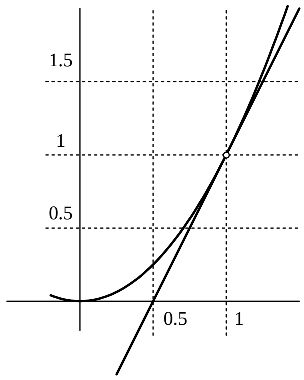
and the vector $\vec{x}'(t_0) = \vec{v}(t_0)$ is parallel to the tangent line ℓ. Because of this one calls the vector $\vec{x}'(t_0)$ a ***tangent vector*** to the curve. Any multiple $\lambda\vec{x}'(t_0)$ with $\lambda \neq 0$ is still parallel to the tangent line ℓ and is therefore also called a tangent vector.

A tangent vector of length 1 is called a ***unit tangent vector.*** If $\vec{x}'(t_0) \neq 0$ then there are exactly two unit tangent vectors. They are

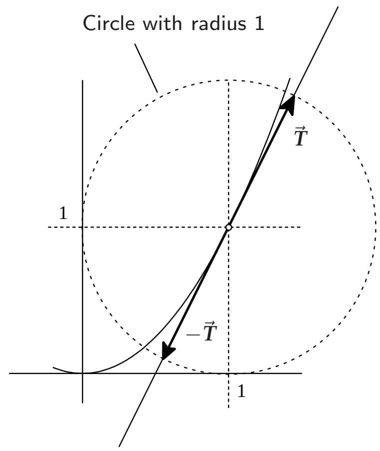
$$\vec{T}(t_0) = \pm\frac{\vec{v}(t)}{\|\vec{v}(t)\|} = \pm\frac{\vec{v}(t)}{v(t)}.$$

◀ **55.1 Example.** *Find Tangent line, and unit tangent vector at $\vec{x}(1)$, where $\vec{x}(t)$ is the parametric curve given by*

$$\vec{x}(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}, \quad \text{so that } \vec{x}'(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix}.$$



parabola with tangent line



parabola with unit tangent vectors

Solution: For $t = 1$ we have $\vec{x}'(1) = \binom{1}{2}$, so the tangent line has parametric equation

$$\vec{y}(s) = \vec{x}(1) + s\vec{x}'(1) = \begin{pmatrix}1\\1\end{pmatrix} + s\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1+s\\1+2s\end{pmatrix}.$$

In components one could write this as $y_1(s) = 1 + s$, $y_2(s) = 1 + 2s$. After eliminating s you find that on the tangent line one has

$$y_2 = 1 + 2s = 1 + 2(y_1 - 1) = 2y_1 - 1.$$

The vector $\vec{x}'(1) = \binom{1}{2}$ is a tangent vector to the parabola at $\vec{x}(1)$. To get a unit tangent vector we normalize this vector to have length one, i.e. we divide i by its length. Thus

$$\vec{T}(1) = \frac{1}{\sqrt{1^2 + 2^2}}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}\frac{1}{5}\sqrt{5}\\ \frac{2}{5}\sqrt{5}\end{pmatrix}$$

is a unit tangent vector. There is another unit tangent vector, namely

$$-\vec{T}(1) = -\begin{pmatrix}\frac{1}{5}\sqrt{5}\\ \frac{2}{5}\sqrt{5}\end{pmatrix}.$$

▶

◀ **55.2 Tangent line and unit tangent vector to Circle.** In example 50.5 and 51.1 we had parametrized the circle and found the velocity vector of this parametrization,

$$\vec{x}(\theta) = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}, \qquad \vec{x}'(\theta) = \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}.$$

If we pick a particular value of θ then the tangent line to the circle at $\vec{x}(\theta_0)$ has parametric equation

$$\vec{y}(s) = \vec{x}(\theta_0) + s\vec{x}'(\theta_0) = \begin{pmatrix}\cos\theta + s\sin\theta\\ \sin\theta - s\cos\theta\end{pmatrix}$$

This equation completely describes the tangent line, but you can try to write it in a more familiar form as a graph

$$y_2 = my_1 + n.$$

To do this you have to eliminate the parameter s from the parametric equations

$$y_1 = \cos\theta + s\sin\theta, \qquad y_2 = \sin\theta - s\cos\theta.$$

When $\sin\theta \neq 0$ you can solve $y_1 = \cos\theta + s\sin\theta$ for s, with result

$$s = \frac{y_1 - \cos\theta}{\sin\theta}.$$

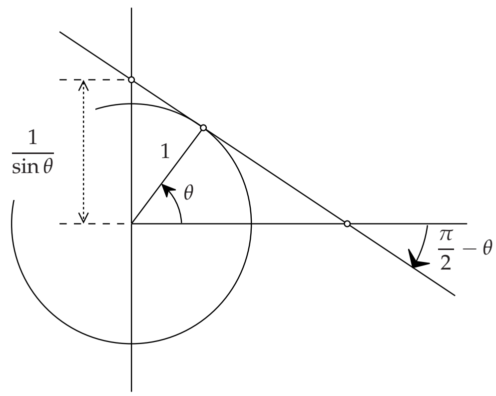
So on the tangent line you have

$$y_2 = \sin\theta - s\cos\theta = \sin\theta - \cos\theta\frac{y_1 - \cos\theta}{\sin\theta}$$

which after a little algebra turns out to be the same as

Add fractions and use $\sin^2\theta + \cos^2\theta = 1$

$$y_2 = -\cot\theta\, y_1 + \frac{1}{\sin\theta}.$$



The tangent line therefore hits the vertical axis when $y_1 = 0$, at height $n = 1/\sin\theta$, and it has slope $m = -\cot\theta$.

For this example you could have found the tangent line without using any calculus by studying the drawing above carefully.

Finally, let's find a unit tangent vector. A unit tangent is a multiple of $\vec{x}'(\theta)$ whose length is one. But the vector $\vec{x}'(\theta) = \left(\begin{smallmatrix}-\sin\theta\\ \cos\theta\end{smallmatrix}\right)$ already has length one, so the two possible unit vectors are

$$\vec{T}(\theta) = \vec{x}'(\theta) = \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix} \quad\text{and}\quad -\vec{T}(\theta) = \begin{pmatrix}\sin\theta\\ -\cos\theta\end{pmatrix}.$$

▶

56. Sketching a parametric curve

For a given parametric curve, like

$$\vec{x}(t) = \begin{pmatrix}1-t^2\\ 3t-t^3\end{pmatrix} \tag{77}$$

you might want to know what the curve looks like. The most straightforward way of getting a picture is to compute $x_1(t)$ and $x_2(t)$ for as many values of t as you feel like, and then plotting the computed points. This computation is the kind of repetitive task that computers are very good at, and there are many software packages and graphing calculators that will attempt to do the computation and drawing for you.

If the vector function has a constant whose value is not (completely) known, e.g. if we wanted to graph the parametric curve

$$\vec{x}(t) = \begin{pmatrix}1-t^2\\ 3at-t^3\end{pmatrix} \qquad (a \text{ is a constant}) \tag{78}$$

then plugging parameter values and plotting the points becomes harder, since the unknown constant a shows up in the computed points.

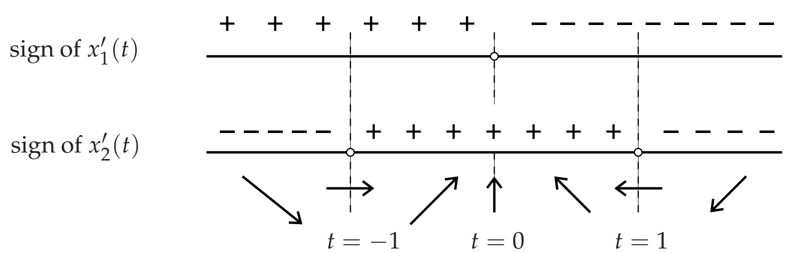
On a graphing calculator you would have to choose different values of a and see what kind of pictures you get (you would expect different pictures for different values of a).

In this section we will use the information stored in the derivative $\vec{x}'(t)$ to create a rough sketch of the graph by hand.

Let's do the specific curve (77) first. The derivative (or velocity vector) is

$$\vec{x}'(t) = \begin{pmatrix} -2t \\ 3-3t^2 \end{pmatrix}, \qquad \text{so} \begin{cases} x_1'(t) &= -2t \\ x_1'(t) &= 3(1-t^2) \end{cases}$$

We see that $x_1'(t)$ changes its sign at $t = 0$, while $x_2'(t) = 2(1-t)(1+t)$ changes its sign twice, at $t = -$ and then at $t = +1$. You can summarize this in a drawing:

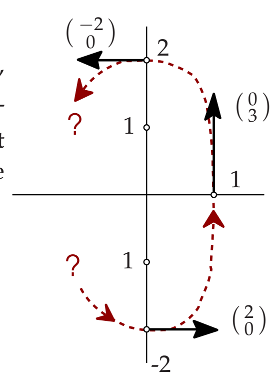


The arrows indicate the wind direction of the velocity vector $\vec{x}'(t)$ for the various values of t.

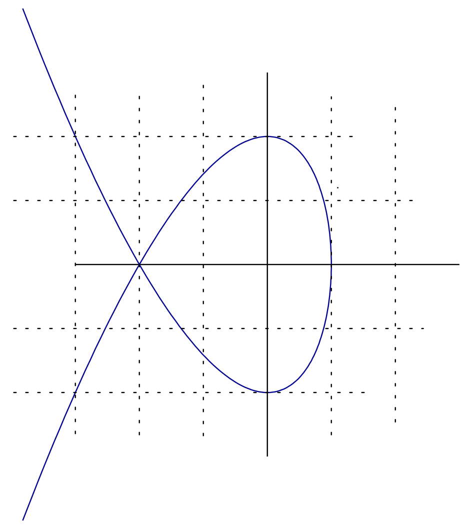
For instance, when $t < -1$ you have $x_1'(t) > 0$ and $x_2'(t) < 0$, so that the vector $\vec{x}'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} + \\ - \end{pmatrix}$ points in the direction "South-East." You see that there are three special t values at which $\vec{x}'(t)$ is either purely horizontal or vertical. Let's compute $\vec{x}(t)$ at those values

$t = -1$	$\vec{x}(-1) = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$	$\vec{x}'(-1) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$
$t = 0$	$\vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\vec{x}'(0) = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$
$t = -1$	$\vec{x}(1) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$	$\vec{x}'(1) = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$

This leads you to the following sketch:

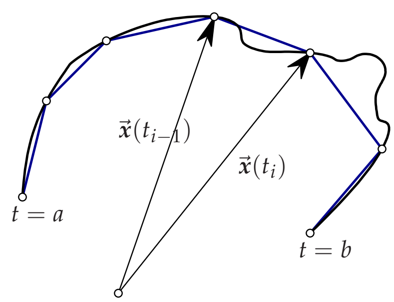


If you use a plotting program like GNUPLOT you get this picture



57. **Length of a curve**

If you have a parametric curve $\vec{x}(t)$, $a \le t \le b$, then there is a formula for the length of the curve it traces out. We'll go through a brief derivation of this formula before stating it.



To compute the length of the curve $\{\vec{x}(t) : a \le t \le b\}$ we divide it into lots of short pieces. If the pieces are short enough they will be almost straight line segments, and we know how do compute the length of a line segment. After computing the lengths of all the short line segments, you add them to get an approximation to the length of the curve. As you divide the curve into finer & finer pieces this approximation should get better & better. You can smell an integral in this description of what's coming. Here are some more details:

Divide the parameter interval into N pieces,

$$a = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = b.$$

Then we approximate the curve by the polygon with vertices at $\vec{x}(t_0) = \vec{x}(a)$, $\vec{x}(t_1)$, ..., $\vec{x}(t_N)$. The distance between to consecutive points at $\vec{x}(t_{i-1})$ and $\vec{x}(t_i)$ on this polygon is

$$\|\vec{x}(t_i) - \vec{x}(t_{i-1})\|.$$

Since we are going to take $t_{i-1} - t_i$ "very small," we can use the derivative to approximate the distance by

$$\vec{x}(t_i) - \vec{x}(t_{i-1}) = \frac{\vec{x}(t_i) - \vec{x}(t_{i-1})}{t_i - t_{i-1}}(t_i - t_{i-1}) \approx \vec{x}\,'(t_i)(t_i - t_{i-1}),$$

so that

$$\|\vec{x}(t_i) - \vec{x}(t_{i-1})\| \approx \|\vec{x}\,'(t_i)\|\,(t_i - t_{i-1}).$$

Now add all these distances and you get

$$\text{Length polygon} \approx \sum_{i=1} N \left\|\vec{x}'(t_i)\right\| \left(t_i - t_{i-1}\right) \approx \int_{t=a}^{b} \left\|\vec{x}'(t)\right\| \,\mathrm{d}t.$$

This is our formula for the length of a curve.

Just in case you think this was a proof, *it isn't!* First, we have used the symbol ≈ which stands for "approximately equal," and we said "very small" in quotation marks, so there are several places where the preceding discussion is vague. But most of all, we can't *prove* that this integral is the length of the curve, since we don't have a definition of "the length of a curve." This is an opportunity, since it leaves us free to adopt the formula we found as our formal definition of the length of a curve. Here goes:

Definition 57.1. *If $\{\vec{x}(t) : a \le t \le b\}$ is a parametric curve, then its* **length** *is given by*

$$\textit{Length} = \int_a^b \left\|\vec{x}'(t)\right\| \,\mathrm{d}t$$

provided the derivative $\vec{x}'(t)$ exists, and provided $\|\vec{x}'(t)\|$ is a Riemann-integrable function.

In this course we will not worry too much about the two caveats about differentiability and integrability at the end of the definition.

◀ **57.2 Length of a line segment.** How long is the line segment AB connecting two points $A(a_1, a_2)$ and $B(b_1, b_2)$?

Solution: Parametrize the segment by

$$\vec{x}(t) = \vec{a} + t(\vec{b} - \vec{a}), \quad (0 \le t \le 1).$$

Then

$$\left\|\vec{x}'(t)\right\| = \left\|\vec{b} - \vec{a}\right\|,$$

and thus

$$\text{Length}(AB) = \int_0^1 \left\|\vec{x}'(t)\right\| \,\mathrm{d}t = \int_0^1 \left\|\vec{b} - \vec{a}\right\| \,\mathrm{d}t = \left\|\vec{b} - \vec{a}\right\|.$$

In other words, the length of the line segment AB is the distance between the two points A and B. It looks like we already knew this, but no, we didn't: what this example shows is that the length of the line segment AB *as defined in definition 57.1* is the distance between the points A and B. So definition 57.1 gives the right answer in this example. If we had found anything else in this example we would have had to change the definition. ▶

◀ **57.3 Perimeter of a circle of radius R.** What is the length of the circle of radius R centered at the origin? This is another example where we know the answer in advance. The following computation should give us $2\pi R$ or else there's something wrong with definition 57.1.

We parametrize the circle as follows:

$$\vec{x}(t) = R\cos\theta\vec{i} + R\sin\theta\vec{j}, \quad (0 \le \theta \le 2\pi).$$

Then

$$\vec{x}'(\theta) = -R\sin\theta\vec{i} + R\cos\theta\vec{j}, \quad \text{and} \quad \left\|\vec{x}'(\theta)\right\| = \sqrt{R^2\sin^2\theta + R^2\cos^2\theta} = R.$$

The length of this circle is therefore

$$\text{Length of circle} = \int_0^{2\pi} R\mathrm{d}\theta = 2\pi R.$$

Fortunately we don't have to fix the definition! ▶

And now the bad news: The integral in the definition of the length looks innocent enough and hasn't caused us any problems in the two examples we have done so far. It is however a reliable source of very difficult integrals. To see why, you must write the integral in terms of the components $x_1(t), x_2(t)$ of $\vec{x}(t)$. Since

$$\vec{x}'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} \quad \text{and thus } \|\vec{x}'(t)\| = \sqrt{x_1'(t)^2 + x_2'(t)^2}$$

the length of the curve parametrized by $\{\vec{x}(t) : a \le t \le b\}$ is

$$\text{Length} = \int_a^b \sqrt{x_1'(t)^2 + x_2'(t)^2}\, \mathrm{d}t.$$

For most choices of $x_1(t)$, $x_2(t)$ the sum of squares under the square root cannot be simplified, and, at best, leads to a difficult integral, but more often to an impossible integral.

But, chin up, sometimes, as if by a miracle, the two squares add up to an expression whose square root can be simplified, and the integral is actually not too bad. Here is an example:

◀ **57.4 Length of the Cycloid.** After getting in at the bottom of a liberated ferris wheel we are propelled through the air along the cycloid whose parametrization is given in example 50.6,

$$\vec{x}(\theta) = \begin{pmatrix} \theta - \sin\theta \\ 1 - \cos\theta \end{pmatrix}.$$

How long is one arc of the Cycloid?

Solution: Compute $\vec{x}'(\theta)$ and you find

$$\vec{x}'(\theta) = \begin{pmatrix} 1 - \cos\theta \\ \sin\theta \end{pmatrix}$$

so that

$$\|\vec{x}'(\theta)\| = \sqrt{(1-\cos\theta)^2 + (\sin\theta)^2} = \sqrt{2 - 2\cos\theta}.$$

This doesn't look promising (this is the function we must integrate!), but just as in example 53.3 we can put the double angle formula $\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$ to our advantage:

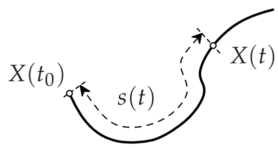
$$\|\vec{x}'(\theta)\| = \sqrt{2 - 2\cos\theta} = \sqrt{4\sin^2\frac{\theta}{2}} = 2\left|\sin\frac{\theta}{2}\right|.$$

We are concerned with only one arc of the Cycloid, so we have $0 \le \theta < 2\pi$, which implies $0 \le \frac{\theta}{2} \le \pi$, which in turn tells us that $\sin\frac{\theta}{2} > 0$ for all θ we are considering. Therefore the length of one arc of the Cycloid is

$$\begin{aligned} \text{Length} &= \int_0^{2\pi} \|\vec{x}'(\theta)\|\, \mathrm{d}\theta \\ &= \int_0^{2\pi} 2\left|\sin\frac{\theta}{2}\right| \mathrm{d}\theta \\ &= 2\int_0^{2\pi} \sin\frac{\theta}{2}\, \mathrm{d}\theta \\ &= \left[-4\cos\frac{\theta}{2}\right]_0^{2\pi} \\ &= 8. \end{aligned}$$

To visualize this answer: the height of the cycloid is 2 (twice the radius of the circle), so *the length of one arc of the Cycloid is four times its height* (Look at the drawing on page 132.) ▶

58. The arclength function



If you have a parametric curve $\vec{x}(t)$ and you pick a particular point on this curve, say, the point corresponding to parameter value t_0, then one defines the ***arclength function*** (starting at t_0) to be

τ is a dummy variable

$$s(t) = \int_{t_0}^{t} \|\vec{x}'(\tau)\| \, d\tau \tag{79}$$

Thus $s(t)$ is the length of the curve segment $\{\vec{x}(\tau) : t_0 \le \tau \le t\}$.

If you interpret the parametric curve $\vec{x}(t)$ as a description of the motion of some object, then the length $s(t)$ of the curve $\{\vec{x}(\tau) : t_0 \le \tau \le t\}$ is the distance traveled by the object since time t_0.

If you differentiate the distance traveled with respect to time you should get the speed, and indeed, by the FUNDAMENTAL THEOREM OF CALCULUS one has

$$s'(t) = \frac{d}{dt}\int_{t_0}^{t} \|\vec{x}'(\tau)\| \, d\tau = \|\vec{x}'(t)\|,$$

which we had called the speed $v(t)$ in § 53.

59. Graphs in Cartesian and in Polar Coordinates

Cartesian graphs. Most of first-semester-calculus deals with a particular kind of curve, namely, the graph of a function, "$y = f(x)$". You can regard such a curve as a special kind of parametric curve, where the parametrization is

$$\vec{x}(t) = \begin{pmatrix} t \\ f(t) \end{pmatrix}$$

and we switch notation from "(x, y)" to "(x_1, x_2)."

For this special case the velocity vector is always given by

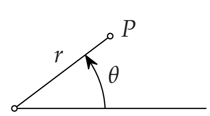
$$\vec{x}'(t) = \begin{pmatrix} 1 \\ f'(t) \end{pmatrix},$$

the speed is

$$v(t) = \|\vec{x}'(t)\| = \sqrt{1 + f'(t)^2},$$

and the length of the segment between $t = a$ and $t = b$ is

$$\text{Length} = \int_a^b \sqrt{1 + f'(t)^2} \, dt.$$



Polar graphs. Instead of choosing Cartesian coordinates (x_1, x_2) one can consider so-called ***Polar Coordinates*** in the plane. We have seen these before in the section on complex numbers: to specify the location of a point in the plane you can give its x_1, x_2 coordinates, but you could also give the absolute value and argument of the complex number $x_1 + ix_2$ (see §23.) Or, to say it without mentioning complex numbers, you can say where a point P in the plane is by saying (1) how far it is from the origin, and (2) how large the angle between the line segment OP and a fixed half line (usually the positive x-axis) is.

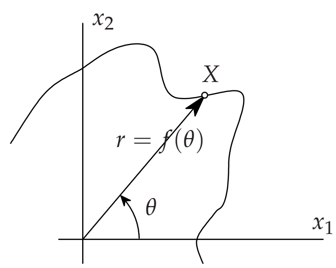
The Cartesian coordinates of a point with polar coordinates (r, θ) are

(80) $$x_1 = r\cos\theta, \qquad x_2 = r\sin\theta,$$

or, in our older notation,

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

These are the same formulas as in §23, where we had " $r = |z|$ and $\theta = \arg z$."



Often a curve is given as a graph in polar coordinates, i.e. for each angle θ there is one point (X) on the curve, and its distance r to the origin is some function $f(\theta)$ of the angle. In other words, the curve consists of all points whose polar coordinates satisfy the equation $r = f(\theta)$. You can parametrize such a curve by

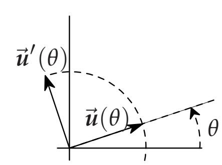
(81) $$\vec{x}(\theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} = \begin{pmatrix} f(\theta)\cos\theta \\ f(\theta)\sin\theta \end{pmatrix}.$$

or,

$$\vec{x}(\theta) = f(\theta)\cos\theta\vec{i} + f(\theta)\sin\theta\vec{j}.$$

You can apply the formulas for velocity, speed and arclength to this parametrization, but instead of doing the straightforward calculation, let's introduce some more notation. For any angle θ we define the vector

$$\vec{u}(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \cos\theta\vec{i} + \sin\theta\vec{j}.$$



The derivative of $\vec{u}$ is

$$\vec{u}'(\theta) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = -\sin\theta\vec{i} + \cos\theta\vec{j}.$$

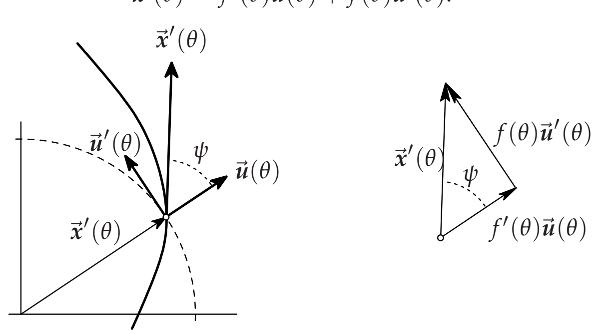
The vectors $\vec{u}(\theta)$ and $\vec{u}'(\theta)$ are perpendicular unit vectors.

Then we have

$$\vec{x}(\theta) = f(\theta)\vec{u}(\theta),$$

so by the product rule one has

$$\vec{x}'(\theta) = f'(\theta)\vec{u}(\theta) + f(\theta)\vec{u}'(\theta).$$



Since $\vec{u}(\theta)$ and $\vec{u}'(\theta)$ are perpendicular unit vectors this implies

$$v(\theta) = \|\vec{x}'(\theta)\| = \sqrt{f'(\theta)^2 + f(\theta)^2}.$$

The length of the piece of the curve between polar angles α and β is therefore

$$\text{Length} = \int_\alpha^\beta \sqrt{f'(\theta)^2 + f(\theta)^2}\, d\theta. \tag{82}$$

You can also read off that the angle ψ between the radius OX and the tangent to the curve satisfies

$$\tan\psi = \frac{f(\theta)}{f'(\theta)}.$$

60. PROBLEMS

Sketching Parametrized Curves

Sketch the curves which are traced out by the following parametrizations. Describe the motion (is the curve you draw traced out once or several times? In which direction?)

In all cases the parameter is allowed to take all values from $-\infty$ to ∞.

If a curve happens to be the graph of some function $x_2 = f(x_1)$ (or $y = f(x)$ if you prefer), then find the function $f(\cdots)$.

Is there a geometric interpretation of the parameter as an angle, or a distance, etc?

393. $\vec{x}(t) = \begin{pmatrix} 1-t \\ 2-t \end{pmatrix}$

394. $\vec{x}(t) = \begin{pmatrix} 3t+2 \\ 3t+2 \end{pmatrix}$

395. $\vec{x}(t) = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$

396. $\vec{x}(t) = \begin{pmatrix} e^t \\ t \end{pmatrix}$

397. $\vec{x}(t) = \begin{pmatrix} e^t \\ e^{-t} \end{pmatrix}$

398. $\vec{x}(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$

399. $\vec{x}(t) = \begin{pmatrix} \sin t \\ t \end{pmatrix}$

400. $\vec{x}(t) = \begin{pmatrix} \sin t \\ \cos 2t \end{pmatrix}$

401. $\vec{x}(t) = \begin{pmatrix} \sin 25t \\ \cos 25t \end{pmatrix}$

402. $\vec{x}(t) = \begin{pmatrix} 1+\cos t \\ 1+\sin t \end{pmatrix}$

403. $\vec{x}(t) = \begin{pmatrix} 2\cos t \\ \sin t \end{pmatrix}$

404. $\vec{x}(t) = \begin{pmatrix} t^2 \\ t^3 \end{pmatrix}$

* * *

Find parametric equations for the curve traced out by the X in each of the following descriptions.

405. A circle of radius 1 rolls over the x_1 axis, and X is a point on a spoke of the circle at a distance $a > 0$ from the center of the circle (the case $a = 1$ gives the cardioid.)

406. A circle of radius $r > 0$ rolls on the outside of the unit circle. X is a point on the rolling circle (These curves are called *epicycloids*.)

407. A circle of radius $0 < r < 1$ rolls on the *inside* of the unit circle. X is a point on the rolling circle.

408. Let O be the origin, A the point $(1, 0)$, and B the point on the unit circle for which the angle $\angle AOB = \theta$. Then X is the point on the tangent to the unit circle through B for which the distance BX equals the length of the circle arc AB.

409. X is the point where the tangent line at $\vec{x}(\theta)$ to the helix of example 50.7 intersects the x_1x_2 plane.

Product rules

410. If a moving object has position vector $\vec{x}(t)$ at time t, and *if it's speed is constant*, then show that the acceleration vector is always perpendicular to the velocity vector. [Hint: differentiate $v^2 = \vec{v} \bullet \vec{v}$ with respect to time and use some of the product rules from §52.]

411. If a charged particle moves in a magnetic field $\vec{B}$, then the laws of electromagnetism say that the magnetic field exerts a force on the particle and that this force is given by the following miraculous formula:

$$\vec{F} = q\vec{v} \times \vec{B}.$$

where q is the charge of the particle, and $\vec{v}$ is its velocity.

Not only does the particle know calculus (since Newton found $\vec{F} = m\vec{a}$), it also knows vector geometry!

Show that even though the magnetic field is pushing the particle around, and even though its velocity vector may be changing with time, its speed $v = \|\vec{v}\|$ remains constant.

412. NEWTON's law of gravitation states that the Earth pulls any object of mass m towards its center with a force inversely proportional to the squared distance of the object to the Earth's center.

(***i***) Show that if the Earth's center is the origin, and $\vec{r}$ is the position vector of the object of mass m, then the gravitational force is given by

$$\vec{F} = -C\frac{\vec{r}}{\|\vec{r}\|^3} \qquad (C \text{ is a positive constant.})$$

[No calculus required. You are supposed to check that this vector satisfies the description in the beginning of the problem, i.e. that it has the right length and direction.]

(***ii***) If the object is moving, then its ***angular momentum*** is defined in physics books by the formula $\vec{L} = m\vec{r} \times \vec{v}$. Show that, if the Earth's gravitational field is the only force acting on the object, then its angular momentum remains constant. [Hint: you should differentiate $\vec{L}$ with respect to time, and use a product rule.]

Curve sketching, using the tangent vector

413. Consider a triangle ABC and let $\vec{a}, \vec{b}$ and $\vec{c}$ be the position vectors of A, B and C.

(***i***) Show that the parametric curve given by

$$\vec{x}(t) = (1-t)^2\vec{a} + 2t(1-t)\vec{b} + t^2\vec{c},$$

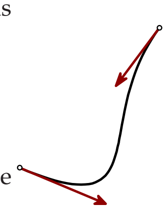
goes through the points A and C, and that at these points it is tangent to the sides of the triangle. Make a drawing.

(***ii***) At which point on this curve is the tangent parallel to the side AC of the triangle?

414. Let $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ be four given vectors. Consider the parametric curve (known as a *Bezier curve*)

$$\vec{x}(t) = (1-t)^3\vec{a} + 3t(1-t)^2\vec{b} + 3t^2(1-t)\vec{c} + t^3\vec{d}$$

where $0 \le t \le 1$.
Compute $\vec{x}(0), \vec{x}(1), \vec{x}'(0)$, and $\vec{x}'(1)$.
The characters in most fonts (like the fonts used for these notes) are made up of lots of Bezier curves.



415. Sketch the following curves by finding all points at which the tangent is either horizontal or vertical

$$(\boldsymbol{i})\ \vec{x}(t) = \begin{pmatrix} 1-t^2 \\ t+2t^2 \end{pmatrix} \qquad (\boldsymbol{ii})\ \vec{x}(t) = \begin{pmatrix} \sin t \\ \sin 2t \end{pmatrix} \qquad (\boldsymbol{iii})\ \vec{x}(t) = \begin{pmatrix} \cos t \\ \sin 2t \end{pmatrix}$$

$$(\boldsymbol{iv})\ \vec{x}(t) = \begin{pmatrix} 1-t^2 \\ 3at-t^3 \end{pmatrix} \qquad (\boldsymbol{v})\ \vec{x}(t) = \begin{pmatrix} 1-t^2 \\ 3at+t^3 \end{pmatrix} \qquad (\boldsymbol{vi})\ \vec{x}(t) = \begin{pmatrix} \cos 2t \\ \sin 3t \end{pmatrix}$$

$$(\boldsymbol{vii})\ \vec{x}(t) = \begin{pmatrix} t/(1+t^2) \\ t^2 \end{pmatrix} \qquad (\boldsymbol{viii})\ \vec{x}(t) = \begin{pmatrix} t^2 \\ \sin t \end{pmatrix} \qquad (\boldsymbol{ix})\ \vec{x}(t) = \begin{pmatrix} 1+t^2 \\ 2t^4 \end{pmatrix}$$

(in these problems, a is a positive constant.)

Lengths of curves

416. Find the length of each of the following curve segments. An "$\int$" indicates a difficult but possible integral which you should do; "$\int\int$" indicates that the resulting integral cannot reasonably be done with the methods explained in this course – you may leave an integral in your answer after simplifying it as much as you can. All other problems lead to integrals that shouldn't be too hard.

(***i***) The *cycloid* $\vec{x}(\theta) = \begin{pmatrix} R(\theta-\sin\theta) \\ R(1-\cos\theta) \end{pmatrix}$, with $0 \le \theta \le 2\pi$.

(***ii***) $[\int\int]$ The *ellipse* $\vec{x}(t) = \begin{pmatrix} \cos t \\ A\sin t \end{pmatrix}$ with $0 \le t \le 2\pi$.

(***iii***) $[\int]$ The *parabola* $\vec{x}(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$ with $0 \le t \le 1$.

(***iv***) $[\int\int]$ The *Sine graph* $\vec{x}(t) = \begin{pmatrix} t \\ \sin t \end{pmatrix}$ with $0 \le t \le \pi$.

(***v***) The *evolute of the circle* $\vec{x} = \begin{pmatrix} \cos t + t\sin t \\ \sin t - t\cos t \end{pmatrix}$ (with $0 \le t \le L$).

(***vi***) The *Catenary*, i.e. the graph of $y = \cosh x = \dfrac{e^x + e^{-x}}{2}$ for $-a \leqslant x \leqslant a$.

(***vii***) The *Cardioid*, which in polar coordinates is given by $r = 1+\cos\theta$, ($|\theta| < \pi$), so $\vec{x}(\theta) = \begin{pmatrix} (1+\cos\theta)\cos\theta \\ (1+\cos\theta)\sin\theta \end{pmatrix}$.

(***viii***) The *Helix* from example 50.7, $\vec{x}(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ a\theta \end{pmatrix}$, $0 \le \theta \le 2\pi$.

417. Below are a number of parametrized curves. For each of these curves find all points with horizontal or vertical tangents; also find all points for which the tangent is parallel to the diagonal. Finally, find the length of the piece of these curves corresponding to the indicated parameter interval (I tried hard to find examples where the integral can be done).

$$(\boldsymbol{i})\ \vec{x}(t) = \begin{pmatrix} t^{1/3} - \frac{9}{20}t^{5/3} \\ t \end{pmatrix} \qquad 0 \le t \le 1$$

$$(\boldsymbol{ii})\ \vec{x}(t) = \begin{pmatrix} t^2 \\ t^2\sqrt{t} \end{pmatrix} \qquad 1 \le t \le 2$$

$$(\boldsymbol{iii})\ \vec{x}(t) = \begin{pmatrix} t^2 \\ t - t^3/3 \end{pmatrix} \qquad 0 \le t \le \sqrt{3}$$

$$(\boldsymbol{iv})\ \vec{x}(t) = \begin{pmatrix} 8\sin t \\ 7t - \sin t \cos t \end{pmatrix} \qquad |t| \le \frac{\pi}{2}$$

$$(\boldsymbol{v})\ \vec{x}(t) = \begin{pmatrix} t \\ \sqrt{1+t} \end{pmatrix} \qquad 0 \le t \le 1$$

(The last problem is harder, but it can be done. In all the other ones the quantity under the square root that appears when you set up the integral for the length of the curve *is a perfect square.*)

418. Consider the polar graph $r = e^{k\theta}$, with $-\infty < \theta < \infty$, where k is a positive constant. This curve is called the *logarithmic spiral*.

(***i***) Find a parametrization for the polar graph of $r = e^{k\theta}$.

(***ii***) Compute the arclength function $s(\theta)$ starting at $\theta_0 = 0$.

(***iii***) Show that the angle between the radius and the tangent is the same at all points on the logarithmic spiral.

(***iv***) Which points on this curve have horizontal tangents?

419. The *Archimedean spiral* is the polar graph of $r = \theta$, where $\theta \ge 0$.

(***i***) Which points on the part of the spiral with $0 < \theta < \pi$ have a horizontal tangent? Which have a vertical tangent?

(***ii***) Find all points on the whole spiral (allowing all $\theta > 0$) which have a horizontal tangent.

(***iii***) Show that the part of the spiral with $0 < \theta < \pi$ is exactly as long as the piece of the parabola $y = \frac{1}{2}x^2$ between $x = 0$ and $x = \pi$. (It is not impossible to compute the lengths of both curves, but you don't have to to answer this problem!)

Answers and Hints

71 $\int x^n \ln x \, dx = \dfrac{x^{n+1} \ln x}{n+1} - \dfrac{x^{n+1}}{(n+1)^2} + C.$

72 $\int e^{ax} \sin bx \, dx = \dfrac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) + C.$

73 $\int e^{ax} \cos bx \, dx = \dfrac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx) + C.$

77 $\int_0^\pi \sin^{14} x dx = \frac{13 \cdot 11 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1}{14 \cdot 12 \cdot 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2}\pi$

78 $\int \cos^n x dx = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} \int \cos^{n-2} x dx$; $\int_0^{\pi/4} \cos^4 x dx = \frac{7}{16} + \frac{3}{32}\pi$

79 Hint: first integrate x^m.

80 $x \ln x - x + C$

81 $x(\ln x)^2 - 2x \ln x + 2x + C$

83 Substitute $u = \ln x$.

84 $\int_0^{\pi/4} \tan^5 x dx = \frac{1}{4}(1)^4 - \frac{1}{2}(1)^2 + \int_0^{\pi/4} \tan x dx = -\frac{1}{4} + \ln \frac{1}{2}\sqrt{2}$

90 $1 + \frac{4}{x^3-4}$

91 $1 + \frac{2x+4}{x^3-4}$

92 $1 - \frac{x^2+x+1}{x^3-4}$

93 $\frac{x^3-1}{x^2-1} = x + \frac{x-1}{x^2-1}$. You can simplify this further: $\frac{x^3-1}{x^2-1} = x + \frac{x-1}{x^2-1} = x + \frac{1}{x+1}$.

94 $x^2 + 6x + 8 = (x+3)^2 - 1 = (x+4)(x+2)$ so $\frac{1}{x^2+6x+8} = \frac{1/2}{x+2} + \frac{-1/2}{x+4}$ and $\int \frac{dx}{x^2+6x+8} = \frac{1}{2} \ln(x+2) - \frac{1}{2} \ln(x+4) + C$.

95 $\int \frac{dx}{x^2+6x+10} = \arctan(x+3) + C$.

96 $\frac{1}{5} \int \frac{dx}{x^2+4x+5} = \frac{1}{5} \arctan(x+2) + C$

97 We add

$$\begin{aligned} \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1} &= \frac{A(x+1)(x-1) + Bx(x-1) + Cx(x+1)}{x(x+1)(x-1)} \\ &= \frac{(A+B+C)x^2 + (C-B)x - A}{x(x+1)(x-1)}. \end{aligned}$$

The numerators must be equal, i.e.

$$x^2 + 3 = (A+B+C)x^2 + (C-B)x - A$$

for all x, so equating coefficients gives a system of three linear equations in three unknowns A, B, C:

$$\begin{cases} A+B+C = 1 \\ C-B = 0 \\ -A = 3 \end{cases}$$

so $A = -3$ and $B = C = 2$, i.e.

$$\frac{x^2+3}{x(x+1)(x-1)} = -\frac{3}{x} + \frac{2}{x+1} + \frac{2}{x-1}$$

and hence

$$\int \frac{x^2+3}{x(x+1)(x-1)}\,dx = -3\ln|x| + 2\ln|x+1| + 2\ln|x-1| + \text{constant}.$$

98 To solve

$$\frac{x^2+3}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1},$$

multiply by x:

$$\frac{x^2+3}{(x+1)(x-1)} = A + \frac{Bx}{x+1} + \frac{Cx}{x-1}$$

and plug in $x = 0$ to get $A = -3$; then multiply by $x+1$:

$$\frac{x^2+3}{x(x-1)} = \frac{A(x+1)}{x} + B + \frac{C(x+1)}{x-1}$$

and plug in $x = -1$ to get $B = 2$; finally multiply by $x-1$:

$$\frac{x^2+3}{x(x+1)} = \frac{A(x-1)}{x} + \frac{B(x-1)}{x+1} + C,$$

and plug in $x = 1$ to get $C = 2$.

99 Apply the method of equating coefficients to the form

$$\frac{x^2+3}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}.$$

In this problem, the Heaviside trick can still be used to find C and B; we get $B = -3$ and $C = 4$. Then

$$\frac{A}{x} - \frac{3}{x^2} + \frac{4}{x-1} = \frac{Ax(x-1) + 3(x-1) + 4x^2}{x^2(x-1)}$$

so $A = -3$. Hence

$$\int \frac{x^2+3}{x^2(x-1)}\,dx = -3\ln|x| + \frac{3}{x} + 4\ln|x-1| + \text{constant}.$$

117 $\int_0^a x\sin x dx = \sin a - a\cos a$

118 $\int_0^a x^2\cos x dx = (a^2+2)\sin a + 2a\cos a$

119 $\int_3^4 \frac{x\,dx}{\sqrt{x^2-1}} = \left[\sqrt{x^2-1}\right]_3^4 = \sqrt{15} - \sqrt{8}$

120 $\int_{1/4}^{1/3} \frac{x\,dx}{\sqrt{1-x^2}} = \left[-\sqrt{1-x^2}\right]_{1/4}^{1/3} = \frac{1}{4}\sqrt{15} - \frac{1}{3}\sqrt{8}$

121 same as previous problem after substituting $x = 1/t$

151 Use Taylor's formula : $Q(x) = 43 + 19(x-7) + \frac{11}{2}(x-7)^2$.

A different, correct, but more laborious (clumsy) solution is to say that $Q(x) = Ax^2 + Bx + C$,, compute $Q'(x) = 2Ax + B$ and $Q''(x) = 2A$. Then

$$Q(7) = 49A + 7B + C = 43, \qquad Q'(7) = 14A + B = 19, \qquad Q''(7) = 2A = 11.$$

This implies $A = 11/2$, $B = 19 - 14A = 19 - 77 = -58$, and $C = 43 - 7B - 49A = 179\frac{1}{2}$.

166 $T_\infty e^t = 1 + t + \frac{1}{2!}t^2 + \cdots + \frac{1}{n!}t^n + \cdots$

167 $T_\infty e^{\alpha t} = 1 + \alpha t + \frac{\alpha^2}{2!}t^2 + \cdots + \frac{\alpha^n}{n!}t^n + \cdots$

168 $T_\infty \sin(3t) = 3t - \frac{3^3}{3!}t^3 + \frac{3^5}{5!}t^5 + \cdots + \frac{(-1)^k 3^{2k+1}}{(2k+1)!}t^{2k+1} + \cdots$

169 $T_\infty \sinh t = t + \frac{1}{3!}t^3 + \cdots + \frac{1}{(2k+1)!}t^{2k+1} + \cdots$

170 $T_\infty \cosh t = 1 + \frac{1}{2!}t^2 + \cdots + \frac{1}{(2k)!}t^{2k} + \cdots$

171 $T_\infty \frac{1}{1+2t} = 1 - 2t + 2^2t^2 - \cdots + (-1)^n 2^n t^n + \cdots$

172 $T_\infty \frac{3}{(2-t)^2} = \frac{3}{2^2} + \frac{3\cdot 2}{2^3}t + \frac{3\cdot 3}{2^4}t^2 + \frac{3\cdot 4}{2^5}t^3 + \cdots + \frac{3\cdot(n+1)}{2^{n+2}}t^n + \cdots$ (note the cancellation of factorials)

173 $T_\infty \ln(1+t) = t - \frac{1}{2}t^2 + \frac{1}{3}t^3 + \cdots + \frac{(-1)^{n+1}}{n}t^n + \cdots$

174 $T_\infty \ln(2+2t) = T_\infty \ln[2\cdot(1+t)] = \ln 2 + \ln(1+t) = \ln 2 + t - \frac{1}{2}t^2 + \frac{1}{3}t^3 + \cdots + \frac{(-1)^{n+1}}{n}t^n + \cdots$

175 $T_\infty \ln\sqrt{1+t} = T_\infty \frac{1}{2}\ln(1+t) = \frac{1}{2}t - \frac{1}{4}t^2 + \frac{1}{6}t^3 + \cdots + \frac{(-1)^{n+1}}{2n}t^n + \cdots$

176 $T_\infty \ln(1+2t) = 2t - \frac{2^2}{2}t^2 + \frac{2^3}{3}t^3 + \cdots + \frac{(-1)^{n+1}2^n}{n}t^n + \cdots$

176 $T_\infty \ln[(1+t)(1+2t)] = T_\infty [\ln(1+t) + \ln(1+2t)] =$
$(1+2)t - \dfrac{1+2^2}{2}t^2 + \dfrac{1+2^3}{3}t^3 + \cdots + \dfrac{(-1)^{n+1}(1+2^n)}{n}t^n + \cdots$

177 $T_\infty \ln\sqrt{\left(\frac{1+t}{1-t}\right)} = T_\infty\left[\frac{1}{2}\ln(1+t) - \frac{1}{2}\ln(1-t)\right] = t + \frac{1}{3}t^3 + \frac{1}{5}t^5 + \cdots + \frac{1}{2k+1}t^{2k+1} + \cdots$

178 $T_\infty \frac{1}{1-t^2} = T_\infty\left[\frac{1/2}{1-t} + \frac{1/2}{1+t}\right] = 1 + t^2 + t^4 + \cdots + t^{2k} + \cdots$ (you could also substitute $x = -t^2$ in the geometric series $1/(1+x) = 1 - x + x^2 + \cdots$, later in this chapter we will use "little-oh" to justify this point of view.)

179 $T_\infty \frac{t}{1-t^2} = T_\infty\left[\frac{1/2}{1-t} - \frac{1/2}{1+t}\right] = t + t^3 + t^5 + \cdots + t^{2k+1} + \cdots$ (note that this function is t times the previous function so you would think its Taylor series is just t times the taylor series of the previous function. Again, "little-oh" justifies this.)

180 The pattern for the n^{th} derivative repeats every time you increase n by 4. So we indicate the the general terms for $n = 4m, 4m+1, 4m+2$ and $4m+3$:

$$T_\infty(\sin t + \cos t) = 1 + t - \frac{1}{2!}t^2 - \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \cdots + \frac{t^{4m}}{(4m)!} + \frac{t^{4m+1}}{(4m+1)!} - \frac{t^{4m+2}}{(4m+2)!} - \frac{t^{4m+3}}{(4m+3)!} + \cdots$$

181 Use a double angle formula

$$T_\infty(2\sin t\cos t) = \sin 2t = 2t - \frac{2^3}{3!}t^3 + \cdots + \frac{2^{4m+1}}{(4m+1)!}t^{4m+1} - \frac{2^{4m+3}}{(4m+3)!}t^{4m+3} + \cdots$$

182 $T_3 \tan t = t + \frac{1}{3}t^3$. There is no simple general formula for the n^{th} term in the Taylor series for $\tan x$.

183 $T_\infty\left[1 + t^2 - \frac{2}{3}t^4\right] = 1 + t^2 - \frac{2}{3}t^4$

184 $T_\infty[(1+t)^5] = 1 + 5t + 10t^2 + 10t^3 + 5t^4 + t^5$

185 $T_\infty \sqrt[3]{1+t} = 1 + \frac{1/3}{1!}t + \frac{(1/3)(1/3-1)}{2!}t^2 + \cdots + \frac{(1/3)(1/3-1)(1/3-2)\cdots(1/3-n+1)}{n!}t^n + \cdots$

186 Because of the addition formula

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

you should get the same answer for f and g, since they are the same function!

The solution is

$$T_\infty \sin(x+a) = \sin a + \cos(a)x - \frac{\sin a}{2!}x^2 - \frac{\cos a}{3!}x^3 + \cdots$$
$$\cdots + \frac{\sin a}{(4n)!}x^{4n} + \frac{\cos a}{(4n+1)!}x^{4n+1} - \frac{\sin a}{(4n+2)!}x^{4n+2} - \frac{\cos a}{(4n+3)!}x^{4n+3} + \cdots$$

189

$$f(x) = f^{(4)}(x) = \cos x, \qquad f'(x) = f^{(5)}(x) = -\sin x, \qquad f''(x) = -\cos x, \qquad f^{(3)}(x) = \sin x,$$

so

$$f(0) = f^{(4)}(0) = 1, \qquad f'(0) = f^{(3)}(0) = 0, \qquad f''(0) = -1.$$

and hence the fourth degree Taylor polynomial is

$$T_4\{\cos x\} = \sum_{k=0}^{4} \frac{f^{(k)}(0)x^k}{k!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}.$$

The error is

$$R_4\{\cos x\} = \frac{f^{(5)}(\xi)x^5}{5!} = \frac{(-\sin\xi)x^5}{5!}$$

for some unknown ξ between 0 and x. As $|\sin\xi| \le 1$ we have

$$\left|\cos x - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right)\right| = |R_4(x)| \le \frac{|x^5|}{5!} < \frac{1}{5!}$$

for $|x| < 1$.

206 The PFD of g is $g(x) = \dfrac{1}{x-2} - \dfrac{1}{x-1}$.

$g(x) = \frac{1}{2} + \left(1 - \frac{1}{2^2}\right)x + \left(1 - \frac{1}{2^3}\right)x^2 + \cdots + \left(1 - \frac{1}{2^{n+1}}\right)x^n + \cdots$.

So $g_n = 1 - 1/2^{n+1}$ and $g^{(n)}(0)$ is $n!$ times that.

207 You could repeat the computations from problem 206, and this would get you the right answer with the same amount of work. In this case you could instead note that $h(x) = xg(x)$ so that

$$h(x) = \tfrac{1}{2}x + \left(1 - \tfrac{1}{2^2}\right)x^2 + \left(1 - \tfrac{1}{2^3}\right)x^3 + \cdots + \left(1 - \tfrac{1}{2^{n+1}}\right)x^{n+1} + \cdots$$

Therefore $h_n = 1 - 1/2^n$.

The PFD of $k(x)$ is

$$k(x) = \frac{2-x}{(x-2)(x-1)} \overset{\text{cancel!}}{=} \frac{1}{1-x},$$

the Taylor series of k is just the Geometric series.

209 $T_\infty e^{at} = 1 + at + \dfrac{a^2}{2!}t^2 + \cdots + \dfrac{a^n}{n!}t^n + \cdots$.

210 $e^{1+t} = e \cdot e^t$ so $T_\infty e^{1+t} = e + et + \frac{e}{2!}t^2 + \cdots + \frac{e}{n!}t^n + \cdots$

211 Substitute $u = -t^2$ in the Taylor series for e^u.

$$T_\infty e^{-t^2} = 1 - t^2 + \frac{1}{2!}t^4 - \frac{1}{3!}t^6 + \cdots + \frac{(-1)^n}{(2n)!}t^{2n} + \cdots$$

212 PFD! The PFD of $\frac{1+t}{1-t}$ is $\frac{1+t}{1-t} = -1 + \frac{2}{1-t}$. Remembering the Geometric Series you get

$$T_\infty \frac{1+t}{1-t} = 1 + 2t + 2t^2 + 2t^3 + \cdots + 2t^n + \cdots$$

213 Substitute $u = -2t$ in the Geometric Series $1/(1-u)$. You get

$$T_\infty \frac{1}{1+2t} = 1 - 2t + 2^2t^2 - 2^3t^3 + \cdots + \cdots + (-1)^n 2^n t^n + \cdots$$

214

$$T_\infty \frac{\ln(1+x)}{x} = \frac{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots + (-1)^{n-1}\frac{1}{n}x^n + \cdots}{x}$$
$$= 1 - \frac{1}{2}x + \frac{1}{3}x^2 + \cdots + (-1)^{n-1}\frac{1}{n}x^{n-1} + \cdots$$

215

$$T_\infty \frac{e^t}{1-t} = 1 + 2t + \left(1 + 1 + \tfrac{1}{2!}\right)t^2 + \left(1 + 1 + \tfrac{1}{2!} + \tfrac{1}{3!}\right)t^3 + \cdots + \left(1 + 1 + \tfrac{1}{2!} + \cdots + \tfrac{1}{n!}\right)t^n + \cdots$$

216 $1/\sqrt{1-t} = (1-t)^{-1/2}$ so

$$T_\infty \frac{1}{\sqrt{1-t}} = 1 + \tfrac{1}{2}t + \frac{\frac{1}{2}\frac{3}{2}}{1\cdot 2}t^2 + \frac{\frac{1}{2}\frac{3}{2}\frac{5}{2}}{1\cdot 2\cdot 3}t^3 + \cdots$$

(be careful with minus signs when you compute the derivatives of $(1-t)^{-1/2}$.)

You can make this look nicer if you multiply top and bottom in the n^{th} term with 2^n:

$$T_\infty \frac{1}{\sqrt{1-t}} = 1 + \frac{1}{2}t + \frac{1\cdot 3}{2\cdot 4}t^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}t^3 + \cdots + \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}t^n + \cdots$$

217

$$T_\infty \frac{1}{\sqrt{1-t^2}} = 1 + \frac{1}{2}t^2 + \frac{1\cdot 3}{2\cdot 4}t^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}t^6 + \cdots + \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}t^{2n} + \cdots$$

218

$$T_\infty \arcsin t = t + \frac{1}{2}\frac{t^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{t^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{t^7}{7} + \cdots + \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\frac{t^{2n+1}}{2n+1} + \cdots$$

219 $T_4[e^{-t}\cos t] = 1 - t + \frac{1}{3}t^3 - \frac{1}{6}t^4$.

220 $T_4[e^{-t}\sin 2t] = t - t^2 + \dfrac{1}{3}t^3 + o(t^4)$ (the t^4 terms cancel).

221 PFD of $1/(2-t-t^2) = \frac{1}{(2+t)(1-t)} = \frac{-\frac{1}{3}}{2+t} + \frac{\frac{1}{3}}{1-t}$. Use the geometric series.

222 $\sqrt[3]{1+2t+t^2} = \sqrt[3]{(1+t)^2} = (1+t)^{2/3}$. This is very similar to problem 216. The answer follows from Newton's binomial formula.

291 $y(t) = 2\dfrac{Ae^t+1}{Ae^t-1}$

292 $y = Ce^{-x^3/3}$, $C = 5e^{1/3}$

293 $y = Ce^{-x-x^3}$, $C = e^2$

294 Implicit form of the solution $\tan y = -\frac{x^2}{2} + C$, so $C = \tan \pi/3 = \sqrt{3}$.

Solution $y(x) = \arctan\left(\sqrt{3} - x^2/3\right)$

295 Implicit form of the solution: $y + \frac{1}{2}y^2 + x + \frac{1}{2}x^2 = A + \frac{1}{2}A^2$. If you solve for y you get

$$y = -1 \pm \sqrt{A^2 + 2A + 1 - x^2 - 2x}$$

Whether you need the "+" or "−" depends on A.

296 Integration gives $\dfrac{1}{2}\ln\left|\dfrac{1+y}{1-y}\right| = x + C$.

Hence $\frac{1+A}{1-A} = \pm e^C$, and $y = \dfrac{(1+A)e^{2x} - 1 + A}{(1+A)e^{2x} + 1 - A}$.

297 $y(x) = \tan\big(\arctan(A) - x\big)$.

298 $y = xe^{\sin x} + Ae^{\sin x}$

299 Implicit form of the solution $\frac{1}{3}y^3 + x^3 = C$; $C = \frac{1}{3}A^3$. Solution is $y = \sqrt[3]{A^3 - 3x^3}$.

303 General solution: $y(t) = Ae^{3t}\cos t + Be^{3t}\sin t$. Solution with given initial values has $A = 7$, $B = -10$.

304 $y = Ae^t + Be^{-t} + C\cos t + D\sin t$

305 The characteristic roots are $r = \pm\frac{1}{2}\sqrt{2} \pm \frac{1}{2}\sqrt{2}$, so the general solution is

$$y = Ae^{\frac{1}{2}\sqrt{2}t}\cos\tfrac{1}{2}\sqrt{2}t + Be^{\frac{1}{2}\sqrt{2}t}\sin\tfrac{1}{2}\sqrt{2}t + Ce^{-\frac{1}{2}\sqrt{2}t}\cos\tfrac{1}{2}\sqrt{2}t + De^{-\frac{1}{2}\sqrt{2}t}\sin\tfrac{1}{2}\sqrt{2}t.$$

306 $y = A + Bt + C\cos t + D\sin t$

307 $y = A + Bt + C\cos t + D\sin t$.

308 The characteristic equation is $r^3 + 1 = 0$, so we must solve

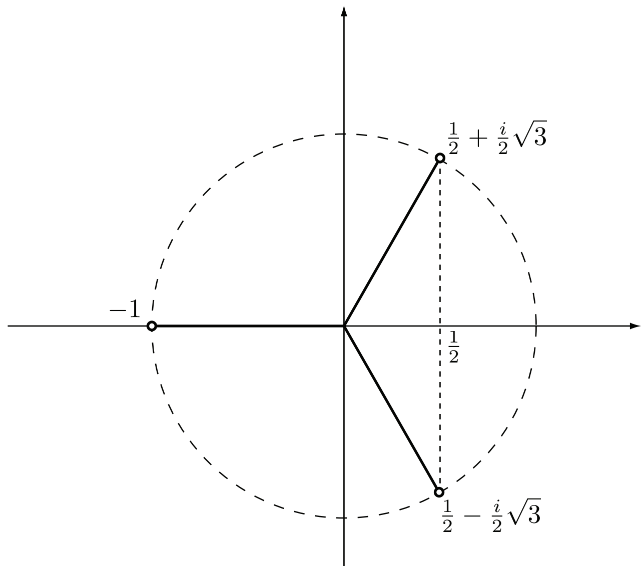
$$r^3 = -1 = e^{(\pi + 2k\pi)i}.$$

The characteristic roots are

$$r = e^{\left(\frac{\pi}{3} + \frac{2}{3}k\pi\right)i}$$

where k is an integer. The roots for $k = 0, 1, 2$ are different, and all other choices of k lead to one of these roots. They are

$$\begin{aligned} k = 0: &\qquad r = e^{\pi i/3} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + \frac{i}{2}\sqrt{3} \\ k = 1: &\qquad r = e^{\pi i} = \cos\pi + i\sin\pi = -1 \\ k = 2: &\qquad r = e^{5\pi i/3} = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - \frac{i}{2}\sqrt{3} \end{aligned}$$



The real form of the general solution of the differential equation is therefore

$$y = Ae^{-t} + Be^{\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t + Ce^{\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t$$

309 $y = Ae^{t} + Be^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t + Ce^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t$

310 $y(t) = c_1e^{\sqrt{3}t} + c_2e^{-\sqrt{3}t} + A\cos t + B\sin t$.

311 Characteristic polynomial: $r^4 + 4r^2 + 3 = (r^2+3)(r^2+1)$.

Characteristic roots: $-i\sqrt{3}, -i, i, i\sqrt{3}$.

General solution: $y(t) = A_1\cos\sqrt{3}t + B_1\sin\sqrt{3}t + A_2\cos t + B_2\sin t$.

312 Characteristic polynomial: $r^4 + 2r^2 + 2 = (r^2+1)^2 + 1$.

Characteristic roots: $r_{1,2}^2 = -1+i$, $r_{3,4}^2 = -1-i$.

Since $-1+i = \sqrt{2}e^{\pi i/4+2k\pi}$ (k an integer) the square roots of $-1+i$ are $\pm 2^{1/4}e^{\pi i/8} = 2^{1/4}\cos\frac{\pi}{8} + i2^{1/4}\sin\frac{\pi}{8}$. The angle $\pi/8$ is not one of the familiar angles so we don't simplify $\cos\pi/8, \sin\pi/8$.

Similarly, $-1-i = \sqrt{2}e^{-\pi i/4+2k\pi i}$ so the square roots of $-1-i$ are $\pm 2^{1/4}e^{-\pi i/8} = \pm 2^{1/4}\big(\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\big)$.

If you abbreviate $a = 2^{1/4}\cos\frac{\pi}{8}$ and $b = 2^{1/4}\sin\frac{\pi}{8}$, then the four characteristic roots which we have found are

$$\begin{aligned}
r_1 &= 2^{1/4}\cos\frac{\pi}{8} + i2^{1/4}\sin\frac{\pi}{8} = a+bi\\
r_2 &= 2^{1/4}\cos\frac{\pi}{8} - i2^{1/4}\sin\frac{\pi}{8} = a-bi\\
r_3 &= -2^{1/4}\cos\frac{\pi}{8} + i2^{1/4}\sin\frac{\pi}{8} = -a+bi\\
r_4 &= -2^{1/4}\cos\frac{\pi}{8} - i2^{1/4}\sin\frac{\pi}{8} = -a-bi
\end{aligned}$$

The general solution is

$$y(t) = A_1 e^{at}\cos bt + B_1 e^{at}\sin bt + A_2 e^{-at}\cos bt + B_2 e^{-at}\sin bt$$

315 Characteristic equation is $r^3 - 125 = 0$, i.e. $r^3 = 125 = 125e^{2k\pi i}$. The roots are $r = 5e^{2k\pi i/3}$, i.e.

$$5, \quad 5(-\tfrac{1}{2} + \tfrac{i}{2}\sqrt{3}) = -\tfrac{5}{2} + \tfrac{5}{2}i\sqrt{3}, \text{ and } 5(-\tfrac{1}{2} - \tfrac{i}{2}\sqrt{3}) = -\tfrac{5}{2} - \tfrac{5}{2}i\sqrt{3}.$$

The general solution is

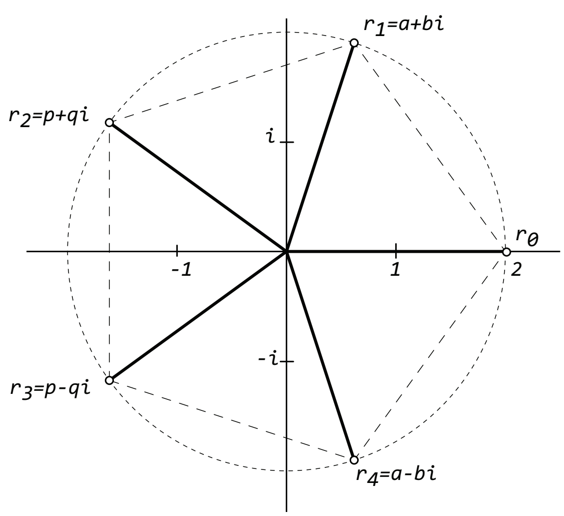
$$f(x) = c_1 e^{5x} + c_2 e^{-\frac{5}{2}x}\cos \tfrac{5}{2}\sqrt{3}x + c_3 e^{-\frac{5}{2}x}\sin \tfrac{5}{2}\sqrt{3}x.$$

316 Try $u(x) = e^{rx}$ to get the characteristic equation $r^5 = 32$ which has solutions

$$r = 2,\ 2e^{\frac{2}{5}\pi i},\ 2e^{\frac{4}{5}\pi i},\ 2e^{\frac{6}{5}\pi i},\ 2e^{\frac{8}{5}\pi i},$$

i.e.

$$\begin{aligned} r_0 &= 2 \\ r_1 &= 2\cos\tfrac{2}{5}\pi + 2i\sin\tfrac{2}{5}\pi \\ r_2 &= 2\cos\tfrac{4}{5}\pi + 2i\sin\tfrac{4}{5}\pi \\ r_3 &= 2\cos\tfrac{6}{5}\pi + 2i\sin\tfrac{6}{5}\pi \\ r_4 &= 2\cos\tfrac{8}{5}\pi + 2i\sin\tfrac{8}{5}\pi. \end{aligned}$$



Remember that the roots come in complex conjugate pairs. By making a drawing of the roots you see that r_1 and r_4 are complex conjugates of each other, and also that r_2 and r_3 are complex conjugates of each other. So the roots are

$$2, \qquad 2\cos\tfrac{2}{5}\pi \pm 2i\sin\tfrac{2}{5}\pi, \text{ and } \quad 2\cos\tfrac{4}{5}\pi \pm 2i\sin\tfrac{4}{5}\pi.$$

The general solution of the differential equation is

$$u(x) = c_1 e^{2x} + c_2 e^{ax}\cos bx + c_3 e^{ax}\sin bx + c_3 e^{px}\cos qx + c_3 e^{px}\sin qx.$$

Here we have abbreviated

$$a = 2\cos\tfrac{2}{5}\pi, b = 2\sin\tfrac{2}{5}\pi, p = 2\cos\tfrac{4}{5}\pi, q = 2\sin\tfrac{4}{5}\pi.$$

318 Characteristic polynomial is $r^3 - 5r^2 + 6r - 2 = (r-1)(r^2 - 4r + 2)$, so the characteristic roots are $r_1 = 1, r_{2,3} = 2 \pm \sqrt{2}$. General solution:

$$y(t) = c_1 e^t + c_2 e^{(2-\sqrt{2})t} + c_3 e^{(2+\sqrt{2})t}.$$

320 Characteristic polynomial is $r^3 - 5r^2 + 4 = (r-1)(r^2 - 4r - 4)$. Characteristic roots are $r_1 = 1, r_{2,3} = 2 \pm 2i\sqrt{2}$. General solution

$$z(x) = c_1 e^x + Ae^{2x}\cos(2\sqrt{2}x) + Be^{2x}\sin(2\sqrt{2}x).$$

321 General: $y(t) = A\cos 3t + B\sin 3t$. With initial conditions: $y(t) = \sin 3t$

322 General: $y(t) = A\cos 3t + B\sin 3t$. With initial conditions: $y(t) = -3\cos 3t$

323 General: $y(t) = Ae^{2t} + Be^{3t}$. With initial conditions: $y(t) = e^{3t} - e^{2t}$

324 General: $y(t) = Ae^{-2t} + Be^{-3t}$. With initial conditions: $y(t) = 3e^{-2t} - 2e^{-3t}$

325 General: $y(t) = Ae^{-2t} + Be^{-3t}$. With initial conditions: $y(t) = e^{-2t} - e^{-3t}$

326 General: $y(t) = Ae^{t} + Be^{5t}$. With initial conditions: $y(t) = \frac{5}{4}e^t - \frac{1}{4}e^{5t}$

327 General: $y(t) = Ae^{t} + Be^{5t}$. With initial conditions: $y(t) = (e^{5t} - e^t)/4$

328 General: $y(t) = Ae^{-t} + Be^{-5t}$. With initial conditions: $y(t) = \frac{5}{4}e^{-t} - \frac{1}{4}e^{-5t}$

329 General: $y(t) = Ae^{-t} + Be^{-5t}$. With initial conditions: $y(t) = \frac{1}{4}\big(e^{-t} - e^{-5t}\big)$

330 General: $y(t) = e^{2t}\big(A\cos t + B\sin t\big)$. With initial conditions: $y(t) = e^{2t}\big(\cos t - 2\sin t\big)$

331 General: $y(t) = e^{2t}\big(A\cos t + B\sin t\big)$. With initial conditions: $y(t) = e^{2t}\sin t$

332 General: $y(t) = e^{-2t}\big(A\cos t + B\sin t\big)$. With initial conditions: $y(t) = e^{-2t}\big(\cos t + 2\sin t\big)$

333 General: $y(t) = e^{-2t}\big(A\cos t + B\sin t\big)$. With initial conditions: $y(t) = e^{-2t}\sin t$

334 General: $y(t) = Ae^{2t} + Be^{3t}$. With initial conditions: $y(t) = 3e^{2t} - 2e^{3t}$

335 Characteristic polynomial: $r^3 + r^2 - r + 15 = (r+3)(r^2 - 2r + 5)$. Characteristic roots: $r_1 = -3, r_{2,3} = 1 \pm 2i$. General solution (real form) is

$$f(t) = c_1 e^{-3t} + Ae^t\cos 2t + Be^t\sin 2t.$$

The initial conditions require

$$f(0) = c_1 + A = 0, \quad f'(0) = -3c_1 + A + 2B = 1, \quad f''(0) = 9c_1 - 3A + 4B = 0.$$

Solve these equations to get $c_1 = -1/10$, $A = 1/10$, $B = 3/10$, and thus

$$f(t) = -\tfrac{1}{10}e^{-3t} + \tfrac{1}{10}e^t\cos 2t + \tfrac{3}{10}e^t\sin 2t.$$

337 $y = -2 + Ae^t + Be^{-t}$

338 $y = Ae^t + Be^{-t} + te^t$

339 $y = A\cos t + B\sin t + \frac{1}{6}t\sin t$

340 $y = A\cos 3t + B\sin 3t + \frac{1}{8}\cos t$

341 $y = A\cos t + B\sin t + \frac{1}{2}t\sin t$

342 $y = A\cos t + B\sin t - \frac{1}{8}\cos 3t$

346 Let $X(t)$ be the rabbit population size at time t. The rate at which this population grows is dX/dt rabbits per year.

$\frac{5}{100}X$ from growth at 5% per year
$\frac{2}{100}X$ from death at 2% per year
-1000 car accidents
$+700$ immigration from Sun Prairie

Together we get

$$\frac{dX}{dt} = \frac{3}{100}X - 300.$$

This equation is both separable and first order linear, so you can choose from two methods to find the general solution, which is

$$X(t) = 10,000 + Ce^{0.03t}.$$

If $X(1991) = 12000$ then

$$10,000 + Ce^{0.03\times 1991} = 12,000 \implies C = 2,000e^{-0.03\times 1991}\text{(don't simplify yet!)}$$

Hence

$$X(1994) = 10,000 + 2,000e^{-0.03\times 1991}e^{0.03\times 1994} = 10,000 + 2,000e^{0.03\times(1994-1991)} = 10,000 + 2,000e^{0.09} \approx 12,188.\ldots$$

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