cosh and sinh

The hyperbolic functions cosh and sinh are defined by

$$\cosh x = \frac{e^x + e^{-x}}{2} \tag{1}$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \tag{2}$$

We compute that the derivative of $\frac{e^x+e^{-x}}{2}$ is $\frac{e^x-e^{-x}}{2}$ and the derivative of $\frac{e^x-e^{-x}}{2}$ is $\frac{e^x+e^{-x}}{2}$, i.e.

$$\frac{d}{dx}\cosh x = \sinh x \tag{3}$$

$$\frac{d}{dx}\sinh x = \cosh x \tag{4}$$

Note that $\sinh x > 0$ for $x > 0$, and $\sinh x < 0$ for $x < 0$. However $\cosh x \geq 0$ for all x (strictly positive away from 0). $\sinh x$ is increasing for all x. $\cosh x$ is increasing for all $x > 0$ (and decreasing for $x < 0$). Note that $\cosh(x) = \cosh(-x)$ and $\sinh(-x) = -\sinh(x)$. The minimum of $\cosh x$ is attained at $x = 0$ where $\cosh(0) = 1$, thus $\cosh(x) \geq 1$ for all x.

Draw pictures:

The inverse of sinh

$\sinh x$ is (strictly) increasing and $\lim_{x\to\infty}\sinh(x)=\infty$ and $\lim_{x\to-\infty}\sinh(x)=-\infty$. We see that the range of sinh is $(-\infty,\infty)$ and sinh is invertible. Then

$$\sinh:\ (-\infty,\infty)\to(-\infty,\infty)$$
$$\sinh^{-1}:\ (-\infty,\infty)\to(-\infty,\infty)$$

Let us compute the inverse. That is, we consider the equation $\sinh(x)=y$, and express x in terms of y. This means we need to solve for x in $\frac{e^x-e^{-x}}{2}=y$. To do this we first set $w=e^x$, determine w and then take the natural logarithm of w. The equation for w becomes $w-w^{-1}=2y$ or $w^2-2yw-1=0$. By the quadratic formula there are two possibilities for w, namely $w=y+\sqrt{y^2+1}$ and $w=y-\sqrt{y^2+1}$. The first solution for w is positive, the second is negative, and since $w=e^x$ has to be positive we can discard the negative solution for w. Hence $e^x=w=y+\sqrt{y^2+1}$ and after taking the logarithm we see that $x=\ln(y+\sqrt{y^2+1})$. We have thus computed the inverse function for sinh and it is given by

$$\sinh^{-1}(y)=\ln(y+\sqrt{y^2+1}). \tag{5}$$

In some of the European literature this inverse function is denoted by Arsinh, hence $\text{Arsinh}(y)=\ln(y+\sqrt{y^2+1})$. [1]

The derivative of $\sinh^{-1}$

We could use the general formula for the derivative of inverse functions, or just the above formula for $\sinh^{-1}$, let's do the latter.

I am now writing x for the independent variable. Using the chain rule we get

$$\frac{d}{dx}\big(\ln(x+\sqrt{x^2+1})\big)=\frac{1}{x+\sqrt{x^2+1}}\frac{d}{dx}\big(x+\sqrt{x^2+1}\big)$$

We note that $\frac{d}{dx}\big(x+(x^2+1)^{1/2}\big)=1+\frac{1}{2}(x^2+1)^{-1/2}2x$ and get that the last displayed expression is equal to

$$\frac{1}{x+\sqrt{x^2+1}}\Big(1+\frac{x}{\sqrt{x^2+1}}\Big)=\frac{1}{x+\sqrt{x^2+1}}\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}=\frac{1}{\sqrt{x^2+1}}$$

Therefore we get the rule

$$\text{Arsinh}'(x)\equiv\frac{d}{dx}\sinh^{-1}(x)=\frac{1}{\sqrt{x^2+1}}. \tag{6}$$

For the integral this is

$$\int\frac{1}{\sqrt{x^2+1}}\,dx\ =\ \ln(x+\sqrt{x^2+1})\ +C\,.$$

[1] Arsinh stands for Area sinus hyperbolicus. I have not seen this in any American textbook. Use $\sinh^{-1}$ instead *but then make sure that you do not confuse it with the reciprocal of* $\sinh y$.

The inverse of cosh

As a function on the real line cosh does not have an inverse (note that $\cosh(x) = \cosh(-x)$ so that two different points in x correspond to the same value of cosh). However *if we restrict the domain to* $[0,\infty)$ then cosh is strictly increasing and invertible. The range of cosh is $[1,\infty)$ so that we have

$$\cosh:\ [0,\infty)\to[1,\infty)$$
$$\cosh^{-1}:\ [1,\infty)\to[0,\infty)$$

We compute $\cosh^{-1}(y)$ for $y \geq 1$. Thus, for each $y \geq 1$ we wish to determine an $x \geq 0$ so that $\cosh x = y$, or equivalently $(e^x + e^{-x})/2 = y$. To determine x we again first determine $w = e^x$ from the equation $w + w^{-1} = 2y$, or equivalently, $w^2 - 2yw + 1 = 0$. This quadratic equation has two solutions, namely w could be $y \pm \sqrt{y^2-1}$. The possibility of the minus sign can be discarded since a calculation[2] shows that $y - \sqrt{y^2-1}$ is < 1 for all $y \geq 1$, and therefore it can not be an e^x for some $x > 0$. Thus $w = y + \sqrt{y^2-1}$ and $e^x = w$, so $\cosh^{-1} y = x = \ln w$. We get the formula

$$\cosh^{-1}(y) = \ln(y + \sqrt{y^2-1}). \tag{7}$$

Again this inverse function is occasionally denoted by Arcosh, so we may write sometimes $\text{Arcosh}(y) = \ln(y + \sqrt{y^2+1})$.

The derivative of $\cosh^{-1}$

Again I am now writing x for the independent variable (it is now restricted to $x > 1$). The calculation is completely analogous to the calculation for the derivative of $\sinh^{-1}$. By the chain rule we get

$$\frac{d}{dx}\big(\ln(x+\sqrt{x^2-1})\big) = \frac{1}{x+\sqrt{x^2-1}}\frac{d}{dx}\big(x+\sqrt{x^2-1}\big)$$

and we now calculate that this yields $\frac{1}{\sqrt{x^2-1}}$. Hence we get the rule

$$\text{Arcosh}'(x) \equiv \frac{d}{dx}\cosh^{-1}(x) = \frac{1}{\sqrt{x^2-1}}. \tag{8}$$

This also yields (for $x > 1$)

$$\int \frac{1}{\sqrt{x^2-1}}\,dx \ = \ \ln(x+\sqrt{x^2-1}) \ + C\,.$$

[2] *Indeed* $y - \sqrt{y^2-1} = \frac{y^2 - (\sqrt{y^2-1})^2}{y+\sqrt{y^2-1}} = \frac{1}{y+\sqrt{y^2-1}} < 1$ *for* $y \geq 1$.

Problems

1. Prove the following identities.
(i) $(\cosh x)^2 - (\sinh x)^2 = 1$.
(ii) $\cosh(2x) = (\cosh x)^2 + (\sinh x)^2$.
(iii) $\sinh(2x) = 2 \sinh x \, \cosh x$.

2. Compute the integrals
(i) $\int_0^x (a^2 - t^2)^{-1/2} \, dt$ for $|x| < a$.
(ii) $\int_0^x (a^2 + t^2)^{-1/2} \, dt$ for all x.
(iii) $\int_0^x (a^2 - t^2)^{1/2} \, dt$ for $|x| < a$.
(iv) $\int_0^x (a^2 + t^2)^{1/2} \, dt$ for all x.

3. The function tanh is defined by

$$\tanh x = \frac{\sinh x}{\cosh x}$$

(i) Show that tanh is defined and differentiable for all x and show that its derivative is given by

$$\tanh'(x) = \frac{1}{\cosh^2 x}.$$

(ii) Show that the range of tanh is the interval $(-1, 1)$ and that

$$\tanh : (-\infty, \infty) \to (-1, 1)$$

is invertible.
(iii)Prove that the inverse function

$$\tanh^{-1} : (-1, 1) \to (-\infty, \infty)$$

is given by $\tanh^{-1}(y) = \frac{1}{2}\big(\ln(1 + y) - \ln(1 - y)\big)$.
(iv) Sketch the graph of tanh and the graph of $\tanh^{-1}$.