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Orthogonality of natural sheaves on moduli stacks of $SL(2)$-bundles with connections on $\mathbb{P}^1$ minus 4 points

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Abstract. A special kind of $SL(2)$-bundles with connections on $\mathbb{P}^1 \setminus \{x_1, \dots, x_4\}$ is considered. We construct an equivalence between the derived category of quasicoherent sheaves on the moduli stack of such bundles and the derived category of modules over a TDO ring on some (non-separated) curve.

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Introduction

It is well-known that for an abelian variety X there is a natural equivalence (the Fourier transform) between the derived categories of D_X-modules and $O_{X^\natural}$-modules (see [12], [13] or [18], [17]), where $X^\natural$ is the moduli space of $\natural$-extensions of X by $\mathbf{G}_m$ ($\natural$-extensions are line bundles with flat connections satisfying some additional conditions). This equivalence is defined by a natural bundle $\mathcal{P}$ on $X^\natural \times X$ with a connection along X (the Poincaré bundle), and the proof is based on the fact that $\mathcal{P}$ is an orthogonal X-family of $O_{X^\natural}$-modules and an orthogonal $X^\natural$-family of D_X-modules. Here *orthogonal* means the tensor product of two different bundles in each of these families has zero cohomology groups.

In this paper, the role of $X^\natural$ is played by the moduli space $\mathcal{M}$ of a special kind of rank 2 bundles with connections on $\mathbb{P}^1$. We construct a natural orthogonal family of bundles on $\mathcal{M}$ parametrized by $\mathbb{P}^1$, so $\mathbb{P}^1$ plays the role of X.

More precisely, $\mathcal{M}$ is the moduli stack of $SL(2)$-bundles with connections on $\mathbb{P}^1$ (see [14] for the definition of algebraic stack). These connections are supposed to have poles of order 1 at $x_1, \dots, x_4$, and the eigenvalues of their residues at $x_1, \dots, x_4$

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are fixed. The universal $\mathcal{M}$-family of $SL(2)$-bundles with connections on $\mathbb{P}^1$ defines a $\mathbb{P}^1$-family of vector bundles on $\mathcal{M}$.

To prove the $\mathbb{P}^1$-family is orthogonal, we construct a smooth compactification of $\mathcal{M}$ in terms of algebraic stacks. The compactification is based on the same idea as the compactification of a moduli space of bundles with connection constructed by C. Simpson ([19], [20]): the compactifying space is the moduli stack of a certain class of "bundles with λ-connections" introduced by P. Deligne. The compactification is still defined for any number of points $x_1, \dots, x_n \in \mathbb{P}^1$.

Bundles with connections of this kind can be thought of as modules over a TDO ring D_λ on the projective line with doubled points $x_1, \dots, x_4$ (D_λ depends on the conjugacy classes of residues). Thus, we get an $\mathcal{M}$-family of D_λ-modules. We claim this family has properties similar to those of the Poincaré bundle: we prove it is orthogonal as a family of $O_{\mathcal{M}}$-modules, and, by the results of S. Lysenko ([15]), it is also orthogonal as a family of D_λ-modules. Combining the statements, we see the family defines an equivalence between the derived category of D_λ-modules and the full subcategory of the derived category of quasicoherent sheaves on $\mathcal{M}$ formed by objects on which $-1 \in \mu_2$ acts as -1 (since $\mathcal{M}$ is a μ_2-gerbe over an algebraic space, μ_2 acts on any sheaf on $\mathcal{M}$).

Notation. In this paper, the ground field is $\mathbb{C}$, in other words, "space" means "$\mathbb{C}$-space", $\mathbb{P}^1$ means $\mathbb{P}^1_{\mathbb{C}}$, and so on.

For any schemes (or stacks) $X_1, \dots, X_k$, p_i stands for the natural projection $X_1 \times \dots \times X_k \to X_i$, p_{ij} is the projection $X_1 \times \dots \times X_k \to X_i \times X_j$, and so on.

We denote by $(F^0 \xrightarrow{d^0} F^1 \xrightarrow{d^1} \dots)$ the complex (or the corresponding object of the derived category) $F^\bullet$ with $F^i = 0$ for $i < 0$ (here F^i are objects of some abelian category).

1. Formulation of main results

Let us fix $x_1, \dots, x_n \in \mathbb{P}^1$ and $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ such that $x_i \neq x_j$ for $i \neq j$, $n \geq 4$, $2\lambda_i \notin \mathbb{Z}$, and

$$\sum_{i=1}^{n} \epsilon_i \lambda_i \notin \mathbb{Z} \tag{1}$$

for any $\epsilon_i \in \mu_2 := \{1, -1\}$.

Definition 1. A $(\lambda_1, \dots, \lambda_n)$-bundle is a triple (L, ∇, φ) such that L is a rank 2 vector bundle on $\mathbb{P}^1$, $\nabla : L \to L \otimes \Omega_{\mathbb{P}^1}(x_1 + \dots + x_n)$ is a connection, $\varphi : \bigwedge^2 L \widetilde{\to} O_{\mathbb{P}^1}$ is a horizontal isomorphism, and the residue $R_i := \operatorname{res}_{x_i} \nabla$ of ∇ at x_i has eigenvalues $\{\lambda_i, -\lambda_i\}$.

The following definition is a generalization of Definition 1.

Suppose E is a one-dimensional vector space, $\epsilon \in E$, L is a rank 2 vector bundle on $\mathbb{P}^1$, $\nabla : L \to L \otimes \Omega_{\mathbb{P}^1}(x_1 + \cdots + x_n) \otimes E$ is a $\mathbb{C}$-linear map, $\varphi : \bigwedge^2 L \widetilde{\to} O_{\mathbb{P}^1}$. Let $l_i \subset L_{x_i}$ be a one-dimensional subspace for each $i = 1, \dots, n$.

Definition 2. A collection $(L, \nabla, \varphi; E, \epsilon; l_1, \dots, l_n)$ is called an ϵ-bundle if the following conditions hold:

- (i) $\nabla(fs) = f\nabla s + s \otimes df \otimes \epsilon$ for $f \in O_{\mathbb{P}^1}, s \in L$;
- (ii) $\varphi(\nabla s_1 \wedge s_2) + \varphi(s_1 \wedge \nabla s_2) = d(\varphi(s_1 \wedge s_2)) \otimes \epsilon$ for $s_1, s_2 \in L$;
- (iii) The map $R_i : L_{x_i} \to (L \otimes \Omega(x_1 + \cdots + x_n) \otimes E)_{x_i} = L_{x_i} \otimes E$ induced by ∇ satisfies $R_i|_{l_i} = \epsilon\lambda_i$;
- (iv) (L, ∇) is irreducible, that is, there is no rank 1 subbundle $L_0 \subset L$ such that $\nabla(L_0) \subset L_0 \otimes \Omega(x_1 + \cdots + x_n) \otimes E$.

Remark 1. From a certain point of view, it is natural to consider collections $(L, \nabla, \varphi; E, \epsilon)$ that satisfy (i), (ii), (iv), and the following condition

- (iii') The map $R_i : L_{x_i} \to L_{x_i} \otimes E$ has eigenvalues $\pm\lambda_i\epsilon$.

For $n = 4$, these two definitions are equivalent (in other words, l_i is uniquely determined by $(L, \nabla, \varphi; E, \epsilon)$). So, for $n = 4$, we also use the term ϵ-bundle for $(L, \nabla, \varphi; E, \epsilon)$ that satisfies (i), (ii), (iii'), and (iv). However, the advantage of Definition 2 is that the moduli stack of ϵ-bundles is smooth for any n (Theorem 1). One can check that the moduli stack of collections $(L, \nabla, \varphi; E, \epsilon)$ that satisfy (i), (ii), (iii'), and (iv) is no longer smooth if $n > 4$ (although the stack is still complete).

Example. Let (L, ∇, φ) be a $(\lambda_1, \dots, \lambda_n)$-bundle. Set $l_i := \mathrm{Ker}(R_i - \lambda_i) \subset L_{x_i}$. Then $(L, \nabla, \varphi; \mathbb{C}, 1; l_1, \dots, l_n)$ satisfies conditions (i)–(iii) of Definition 2. Let us check (iv). Assume $L_0 \subset L$ is a ∇-invariant subbundle of rank 1. Then ∇ induces a connection $\nabla_0 : L_0 \to L_0 \otimes \Omega_{\mathbb{P}^1}(x_1 + \cdots + x_n)$ such that $\mathrm{res}_{x_i} \nabla_0 = \pm\lambda_i$. This contradicts (1), so $(L, \nabla, \varphi; \mathbb{C}, 1; l_1, \dots, l_n)$ is an ϵ-bundle.

Example. Let L be a rank 2 bundle on $\mathbb{P}^1$, $\nabla \in \mathrm{Hom}_{O_{\mathbb{P}^1}}(L, L \otimes \Omega(x_1 + \cdots + x_n))$, and $\varphi : \bigwedge^2 L \widetilde{\to} O_{\mathbb{P}^1}$. Suppose $\det(\nabla) \in H^0(\mathbb{P}^1, \Omega^{\otimes 2}(x_1 + \cdots + x_n))$ and $\mathrm{tr}(\nabla) = 0$. In this case, $R_i \in \mathrm{End}(L_{x_i})$ is nilpotent, so one can choose $l_i \subset L_{x_i}$ such that $R_i|_{l_i} = 0$. Clearly $(L, \nabla, \varphi; \mathbb{C}, 0; l_1, \dots, l_n)$ satisfies conditions (i)–(iii) of Definition 2. If $\det(\nabla) = 0$, then any rank 1 subbundle $L_0 \subset \mathrm{Ker}\,\nabla$ is ∇-invariant; hence (L, ∇) is reducible. Conversely, assume (L, ∇) is reducible. Then ∇ has eigenvalues $\omega_\pm \in H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}(x_1 + \cdots + x_n))$. We have $\omega_+ + \omega_- = \mathrm{tr}(\nabla) = 0$ and $\omega_+\omega_- = \det(\nabla) \in H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^{\otimes 2}(x_1 + \cdots + x_n))$. Hence $\omega_\pm \in H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}) = 0$, this implies $\det(\nabla) = \omega_+\omega_- = 0$. So $(L, \nabla, \varphi; \mathbb{C}, 0; l_1, \dots, l_n)$ is an ϵ-bundle if and only if $\det(\nabla) \neq 0$.

Remark 2. For an ϵ-bundle $(L, \nabla, \varphi; E, \epsilon; l_1, \dots, l_n)$, one can pick an isomorphism $E \widetilde{\to} \mathbb{C}$ such that ϵ maps either to $1 \in \mathbb{C}$ or to $0 \in \mathbb{C}$. So the above two examples describe all ϵ-bundles. Hence we can replace condition (iv) in Definition 2 with the condition

(iv') If $\epsilon = 0$, then $\det(\nabla) \neq 0$.

Let $\overline{\mathcal{M}}$ be the moduli stack of ϵ-bundles (so $\overline{\mathcal{M}}_U$ is the groupoid of U-families of ϵ-bundles). Vector spaces E for ϵ-bundles $(L, \nabla, \varphi; E, \epsilon; l_1, \dots, l_n)$ form an invertible sheaf $\mathcal{E}$ on $\overline{\mathcal{M}}$ together with a natural section $\epsilon \in H^0(\overline{\mathcal{M}}, \mathcal{E})$. Denote by $\mathcal{M}_H \subset \overline{\mathcal{M}}$ the closed substack defined by the equation $\epsilon = 0$. We identify the stack of $(\lambda_1, \dots, \lambda_n)$-bundles with $\mathcal{M} := \overline{\mathcal{M}} \setminus \mathcal{M}_H$.

Theorem 1.

(i) *$\overline{\mathcal{M}}$ is a complete Deligne–Mumford stack;*
(ii) *$\mathcal{M}$, $\overline{\mathcal{M}}$, and $\mathcal{M}_H$ are smooth algebraic stacks.*

For $x \in \mathbb{P}^1$, let ξ_x be the bundle on $\overline{\mathcal{M}}$ whose fiber at $(L, \nabla, \varphi; E, \epsilon; l_1, \dots, l_n)$ is L_x.

Theorem 2. *Suppose $x, y \in \mathbb{P}^1$, $n = 4$. Then*

(i) $H^i(\mathcal{M}, \xi_x \otimes \xi_y) = 0$ *for* $x \neq y$, $i \geq 0$.
(ii) $H^i(\mathcal{M}, O_{\mathcal{M}}) = \begin{cases} \mathbb{C}, & i = 0 \\ 0, & i > 0 \end{cases}$.
(iii) $H^i(\mathcal{M}, \mathrm{Sym}^2(\xi_x)) = 0$ *for* $i \geq 0$, $x \notin \{x_1, \dots, x_4\}$.

Remark. This result is announced in [1]. (ii) is proved in [1].

Remark. Clearly $\bigwedge^2 \xi_x = O_{\overline{\mathcal{M}}}$. So (ii) and (iii) of Theorem 2 imply

$$H^i(\mathcal{M}, \xi_x^{\otimes 2}) = \begin{cases} \mathbb{C}, & i = 0 \\ 0, & i > 0 \end{cases} \qquad \text{for } x \in \mathbb{P}^1 \setminus \{x_1, \dots, x_4\}.$$

Theorem 2 describes $H^i(\mathcal{M}, \xi_x \otimes \xi_y)$ for some $(x, y) \in (\mathbb{P}^1)^2$. It is natural to consider $\xi_x \otimes \xi_y$ as a family of bundles on $\mathcal{M}$ parametrized by $(x, y) \in (\mathbb{P}^1)^2$. Then the problem is to compute the push-forward of the bundle with respect to $\mathcal{M} \times (\mathbb{P}^1)^2 \to (\mathbb{P}^1)^2$ (actually, $\mathbb{P}^1$ should be replaced by another curve: see Theorem 3 for the precise statement).

Denote by $p : P \to \mathbb{P}^1$ the projective line with doubled points $x_1, \dots, x_4$. In other words, P is obtained by gluing two copies of $\mathbb{P}^1$ outside $x_1, \dots, x_4$. Let $x_i^\pm \in P$ be the preimages of $x_i \in \mathbb{P}^1$, $[\lambda] := \sum_{i=1}^4 \lambda_i (x_i^+ - x_i^-) \in \operatorname{div} P \otimes_{\mathbb{Z}} \mathbb{C}$, where $\operatorname{div} P$ is the group of divisors on P. Denote by D_λ the TDO ring corresponding to $[\lambda]$ (see [3] for the definition of TDO rings).

For a $(\lambda_1, \dots, \lambda_4)$-bundle L, we denote by L_λ the D_λ-module generated by $p^* L$. More precisely, $L_\lambda := j_{!*}(L|_U)$, where $j : U := \mathbb{P}^1 \setminus \{x_1, \dots, x_4\} \hookrightarrow P$ is the natural embedding. Since $L|_U$ is a D_U-module and $[\lambda]$ is supported outside of U, L_λ is well-defined. This construction still makes sense for families of $(\lambda_1,\dots,\lambda_4)$-bundles. Hence we can apply it to the universal family of $(\lambda_1,\dots,\lambda_4)$-bundles, getting an $\mathcal{M}$-family ξ_λ of D_λ-modules.

Consider $p_{12}: P\times P\times\mathcal{M}\to P\times P$ and $p_{13},p_{23}: P\times P\times\mathcal{M}\to P\times\mathcal{M}$. Set $\mathcal{F}_P := (p_{13}^*\xi_\lambda)\otimes(p_{23}^*\xi_\lambda)$ (since ξ_λ is a flat $O_{P\times\mathcal{M}}$-module, $(p_{13}^*\xi_\lambda)\otimes(p_{23}^*\xi_\lambda)=(p_{13}^*\xi_\lambda)\otimes^L(p_{23}^*\xi_\lambda)$). Note that p_{13}^* and p_{23}^* stand for the O-module pull-back (from the viewpoint of D-modules, these pull-back functors should include a cohomological shift).

$Rp_{12,*}\mathcal{F}_P$ is an object of the derived category of $p_1^\bullet D_\lambda \circledast p_2^\bullet D_\lambda$-modules, where $p_1,p_2: P\times P\to P$ are the projections. Here $p_i^\bullet$ (resp. $\circledast$) stands for the inverse image (resp. the Baer sum) of TDO rings (the corresponding functors on Picard Lie algebroids are described in [3]).

Theorem 3. *$Rp_{12,*}\mathcal{F}_P=\delta_{\Delta'}[-1]$, where $\Delta'\subset P\times P$ is the graph of the involution $\sigma: P\to P$ such that $\sigma(x_i^\pm)=x_i^\mp$, and $\delta_{\Delta'}$ is the direct image of $O_{\Delta'}$ as a $D_{\Delta'}$-module.*

Remark. In general, for a map $f: X\to Y$ and a TDO ring D_1 on Y, there is a functor $f_+:\mathcal{D}^b(f^\bullet D_1)\to\mathcal{D}^b(D_1)$, where $\mathcal{D}^b(D_1)$ is the derived category of D_1-modules. For the embedding $i:\Delta'\hookrightarrow P\times P$, one easily checks $i^\bullet(p_1^\bullet D_\lambda\circledast p_2^\bullet D_\lambda)$ is the (non-twisted) differential operator ring $D_{\Delta'}$, so $\delta_{\Delta'}:=i_+(O_{\Delta'})$ is well-defined as a $p_1^\bullet D_\lambda\circledast p_2^\bullet D_\lambda$-module.

By Theorem 3, ξ_λ is an orthogonal P-family of $O_\mathcal{M}$-bundles. To construct an equivalence of categories, one should also show that ξ_λ is orthogonal as an $\mathcal{M}$-family of D_λ-modules. Let us give the precise statement. We follow closely S. Lysenko's unpublished notes [15].

Consider $\mathcal{F}_\mathcal{M}:=p_{13}^*\xi_\lambda\otimes p_{23}^*(id_\mathcal{M}\times\sigma)^*\xi_\lambda$ (here $p_{13},p_{23}:\mathcal{M}\times\mathcal{M}\times P\to\mathcal{M}\times P$ are the projections and $\sigma: P\widetilde{\to}P$ is the involution introduced in Theorem 3).

$\mathcal{F}_\mathcal{M}$ can be viewed as a family of D_P-modules parametrized by $\mathcal{M}\times\mathcal{M}$. Consider the de Rham complex of $\mathcal{F}_\mathcal{M}$ in the direction of P:

$$\mathbb{DR}(\mathcal{F}_\mathcal{M})=\mathbb{DR}_P(\mathcal{F}_\mathcal{M}):=(\mathcal{F}_\mathcal{M}\to\mathcal{F}_\mathcal{M}\otimes\Omega_{\mathcal{M}\times\mathcal{M}\times P/\mathcal{M}\times\mathcal{M}}).$$

Our aim is to compute $Rp_{12,*}\,\mathbb{DR}(\mathcal{F}_\mathcal{M})$.

$\mathcal{M}\times\mathcal{M}$ is a $\mu_2\times\mu_2$-gerbe over some algebraic space (actually, a scheme), so $\mu_2\times\mu_2$ acts on any quasicoherent sheaf $\mathcal{F}$ on $\mathcal{M}$. Therefore, $\mathcal{F}$ can be decomposed with respect to the characters of $\mu_2\times\mu_2$. Denote by $\mathcal{F}^\psi$ the component of $\mathcal{F}$ corresponding to the character $\psi:\mu_2\times\mu_2\to\mathbf{G}_m$ defined by $(a,b)\mapsto ab$.

Let $\mathrm{diag}:\mathcal{M}\to\mathcal{M}\times\mathcal{M}$ be the diagonal morphism.

Theorem 4 (S. Lysenko)**.** $Rp_{12,*}\,\mathbb{DR}(\mathcal{F}_\mathcal{M})=(\mathrm{diag}_*\,O_\mathcal{M})^\psi[-2]$.

Remark. The definition of $(\lambda_1,\dots,\lambda_n)$-bundles can be carried out using l-adic sheaves instead of bundles with connections. In this situation, the moduli space of

"$(\lambda_1, \ldots, \lambda_n)$-$l$-adic sheaves" exists only infinitesimally: there is no analogue of $\mathcal{M}$ (so the analogue of Theorem 3 cannot be formulated), but the formal neighborhood of a point on $\mathcal{M}$ can still be defined. So Theorem 4 admits an l-adic version, which is also proved in [15].

Since $\mathcal{M}$ is a μ_2-gerbe, the derived category $\mathcal{D}_{qc}(\mathcal{M})$ of quasicoherent sheaves on $\mathcal{M}$ naturally decomposes as $\mathcal{D}_{qc}(\mathcal{M}) = \mathcal{D}_{qc}(\mathcal{M})^+ \times \mathcal{D}_{qc}(\mathcal{M})^-$, where $\mathcal{F} \in \mathcal{D}_{qc}(\mathcal{M})^\pm$ if and only if $-1 \in \mu_2$ acts as ± 1 on $H^i(\mathcal{F})$ for any i.

Using base change, one easily derives from Theorems 3 and 4 the following equivalence of categories:

Theorem 5. *The functor*

$$\Phi_{\mathcal{M}\to P} : \mathcal{F} \mapsto Rp_{2,*}(\xi_\lambda \otimes_{O_{\mathcal{M}\times P}} p_1^*\mathcal{F})[1]$$

is an equivalence between $\mathcal{D}_{qc}(\mathcal{M})^-$ and the derived category of D_λ-modules. The inverse functor is given by

$$\Phi_{P\to\mathcal{M}} : \mathcal{F} \mapsto Rp_{1,*}\, \mathbb{DR}_P((id_{\mathcal{M}} \times \sigma)^*\xi_\lambda \otimes_{O_{\mathcal{M}\times P}} p_2^*\mathcal{F})[1].$$

□

Remark. The functors $\Phi_{\mathcal{M}\to P}$ and $\Phi_{P\to\mathcal{M}}$ provide an equivalence between the derived category of coherent D_λ-modules and the full subcategory of $\mathcal{D}_{qc}(\mathcal{M})^-$ consisting of objects with coherent cohomologies. Note that both categories are equipped with natural anti-equivalences. Namely, for coherent D_λ-modules, we consider the composition of the Verdier duality with the pull-back functor σ^*, while on the derived category of coherent $O_{\mathcal{M}}$-modules, the anti-equivalence is given by $\mathcal{H}om(\bullet, \Omega^2_{\mathcal{M}})$ (Serre's duality). $\Phi_{\mathcal{M}\to P}$ and $\Phi_{P\to\mathcal{M}}$ agree with the anti-equivalences. The proof of these statements will be given elsewhere.

This paper has the following structure:

In Section 2, we explain the place of our results in the geometric Langlands philosophy. The proof of Theorem 1 occupies Sections 3 (the first statement of the theorem) and 4 (its second statement). In Sections 5–8, we prove Theorem 2 by first studying the behavior of $\xi_x \otimes \xi_y$ on $\mathcal{M}_H$ (Section 5), its infinitesimal neighborhood (Section 6), and $\overline{\mathcal{M}}$ (Section 7). These results are used in Section 8 to prove Theorem 2. Theorem 3 is derived from Theorem 2 in Section 9. In Section 10, we sketch S. Lysenko's proof of Theorem 4.

2. Relation to the geometric Langlands program

In this section, we explain the meaning of Theorem 5 from the viewpoint of the geometric Langlands conjecture. Detailed proofs of the statements will be given elsewhere. This section is independent from the rest of the text.

Set $G := PGL(2) = GL(2)/\mathbf{G}_m$. Fix a Borel subgroup $B \subset G$, a Cartan subgroup $T \subset G$, and an isomorphism $T \tilde{\to} \mathbf{G}_m$.

Denote by $\mathcal{B}un_{\mathrm{qp}}(\mathrm{G})$ the moduli stack of principal G bundles $\mathcal{F}$ on $\mathbb{P}^1$ together with $l_i \in (\mathcal{F}_{x_i})/B$ for $i = 1, \dots, n$ (a quasiparabolic structure). Let $\xi_i^{(B)}$ be the principal B-bundle on $\mathcal{B}un_{\mathrm{qp}}(\mathrm{G})$ whose fiber at $(\mathcal{F}, l_1, \dots, l_n)$ is l_i (viewed as a B-orbit in $\mathcal{F}_{x_i}$). The map $B \to T \tilde{\to} \mathbf{G}_m$ transforms $\xi_i^{(B)}$ into a $\mathbf{G}_m$-torsor (in other words, a line bundle) ξ_i. The fiber of the line bundle ξ_i at $(\mathcal{F}, l_1, \dots, l_n)$ is the cotangent space to the projective line $(\mathcal{F}_{x_i})/B$ at l_i.

Consider the map $\mathcal{B}un_{\mathrm{qp}}(\mathrm{G}) \to \mu_2$ that sends $(\mathcal{F}, l_1, \dots, l_n)$ to $\delta([\mathcal{F}])$, where $[\mathcal{F}] \in H^1(\mathbb{P}^1, PGL(2))$ is the isomorphism class of $\mathcal{F}$, and $\delta : H^1(\mathbb{P}^1, PGL(2)) \to \mu_2 = H^2(\mathbb{P}^1, \mu_2)$ is the coboundary map corresponding to

$$1 \to \mu_2 \to SL(2) \to PGL(2) \to 1.$$

Clearly, the map $\mathcal{B}un_{\mathrm{qp}}(\mathrm{G}) \to \mu_2$ is locally constant. Denote by $\mathcal{B}un_{\mathrm{qp}}^{\mathrm{odd}}(\mathrm{G}) \subset \mathcal{B}un_{\mathrm{qp}}(\mathrm{G})$ the preimage of $-1 \in \mu_2$.

Consider $\sum_i \lambda_i[\xi_i] \in (\operatorname{Pic} \mathcal{B}un_{\mathrm{qp}}(\mathrm{G})) \otimes_{\mathbb{Z}} \mathbb{C}$, where $[\xi_i] \in \operatorname{Pic} \mathcal{B}un_{\mathrm{qp}}(\mathrm{G})$ is the isomorphism class of ξ_i. Let $D(\mathcal{B}un_{\mathrm{qp}}(\mathrm{G}))_\lambda$ be the corresponding TDO ring. A. Beilinson and V. Drinfeld explained that the geometric Langlands philosophy predicts a canonical equivalence between the derived category of $O_{\mathcal{M}}$-modules and the derived category of $D(\mathcal{B}un_{\mathrm{qp}}(\mathrm{G}))_\lambda$-modules such that $\mathcal{D}_{qc}(\mathcal{M})^-$ is mapped onto the derived category of $D(\mathcal{B}un_{\mathrm{qp}}^{\mathrm{odd}}(\mathrm{G}))_\lambda$-modules. Here $D(\mathcal{B}un_{\mathrm{qp}}^{\mathrm{odd}}(\mathrm{G}))_\lambda$ is the restriction of $D(\mathcal{B}un_{\mathrm{qp}}(\mathrm{G}))_\lambda$ to $\mathcal{B}un_{\mathrm{qp}}^{\mathrm{odd}}(\mathrm{G})$. Under assumption (1), one can replace $\mathcal{B}un_{\mathrm{qp}}(\mathrm{G})$ by a smaller stack:

A quasiparabolic bundle $(\mathcal{F}, l_1, \dots, l_n)$ is *decomposable* if it admits a T-structure $\mathcal{F}_T \subset \mathcal{F}$ that agrees with l_i for $i = 1, \dots, n$. Any decomposable quasiparabolic bundle clearly possesses a nontrivial automorphism. Actually, for quasiparabolic bundles on $\mathbb{P}^1$ the converse is also true (cf. [1, Proposition 3]).

Let $\mathcal{B}un'_{\mathrm{qp}}(\mathrm{G}) \subset \mathcal{B}un_{\mathrm{qp}}(\mathrm{G})$ be the open substack formed by undecomposable quasiparabolic bundles $(\mathcal{F}, l_1, \dots, l_n)$. One can check that, provided (1) holds, there are no nonzero $D(\mathcal{B}un_{\mathrm{qp}}(\mathrm{G}))_\lambda$-modules M such that $M|_{\mathcal{B}un'_{\mathrm{qp}}(\mathrm{G})} = 0$. In other words, the embedding $\mathcal{B}un'_{\mathrm{qp}}(\mathrm{G}) \hookrightarrow \mathcal{B}un_{\mathrm{qp}}(\mathrm{G})$ induces an equivalence between the derived category of $D(\mathcal{B}un'_{\mathrm{qp}}(\mathrm{G}))_\lambda$-modules and the derived category of $D(\mathcal{B}un_{\mathrm{qp}}(\mathrm{G}))_\lambda$-modules.

From now on, we assume $n = 4$. In this case, one can easily construct an isomorphism between $\mathcal{B}un'^{\mathrm{odd}}_{\mathrm{qp}}(\mathrm{G}) := \mathcal{B}un'_{\mathrm{qp}}(\mathrm{G}) \cap \mathcal{B}un^{\mathrm{odd}}_{\mathrm{qp}}(\mathrm{G})$ and P. The image of $\sum_i \lambda_i[\xi_i] \in \operatorname{Pic} \mathcal{B}un_{\mathrm{qp}}(\mathrm{G})_0 \otimes_{\mathbb{Z}} \mathbb{C}$ in $\operatorname{Pic} P \otimes_{\mathbb{Z}} \mathbb{C}$ via this isomorphism equals

$$\sum_{i=1}^{4} \lambda'_i([x_i^+] - [x_i^-]) \in \operatorname{Pic} P \otimes_{\mathbb{Z}} \mathbb{C}.$$

Here $[x_i^\pm] \in \operatorname{Pic} P$ is the image of $x_i^\pm \in \operatorname{div} P$, and λ_i' are given by

$$\lambda_i' = \lambda_i - \frac{1}{2}\sum_{j=1}^{4} \lambda_j.$$

Note that if λ_i satisfy (1), then so do λ_i', so Theorem 5 yields a canonical equivalence between the derived category of $D(\mathcal{B}un_{\mathrm{qp}}^{\mathrm{odd}}(\mathrm{G}))_\lambda$-modules and $\mathcal{D}_{qc}(\mathcal{M}')^-$ (here $\mathcal{M}'$ is the moduli stack of $(\lambda_1', \dots, \lambda_4')$-bundles). Let M (resp. M') be the coarse moduli space corresponding to $\mathcal{M}$ (resp. $\mathcal{M}'$). Tensor multiplication by ξ_1 gives an equivalence between $\mathcal{D}_{qc}(M)$ and $\mathcal{D}_{qc}(\mathcal{M})^-$. Here ξ_1 is the line bundle on $\mathcal{M}$ whose fiber at (L, ∇, φ) is $l_1 := \operatorname{Ker}(R_1 - \lambda_1) \subset L_{x_1}$. Similarly, we get an equivalence between $\mathcal{D}_{qc}(M')$ and $\mathcal{D}_{qc}(\mathcal{M}')^-$. By [2, Theorem 1], there is a natural isomorphism $M \tilde{\to} M'$; hence $\mathcal{D}_{qc}(\mathcal{M})^-$ is indeed equivalent to the derived category of $D(\mathcal{B}un_{\mathrm{qp}}^{\mathrm{odd}}(\mathrm{G}))_\lambda$-modules in our case.

Remark. It is also possible to construct the equivalence by first using Theorem 5 to get from $\mathcal{D}_{qc}(\mathcal{M})^-$ to the derived category of D_λ-modules on P, and then using a version of the Radon transform to prove the derived categories of D_λ-modules and $D(\mathcal{B}un_{\mathrm{qp}}^{\mathrm{odd}}(\mathrm{G}))_\lambda$-modules are equivalent. This approach gives a canonical equivalence between the categories, while our construction, rigorously speaking, depends on the choice of $x_1 \in \{x_1, \dots, x_4\}$. Another way of dealing with this problem is to prove that the isomorphism $M \tilde{\to} M'$ lifts canonically to the line bundles $\xi_i \otimes \xi_j$ (the existence of such lifting is guaranteed by [2, Theorem 1]).

3. Completeness of $\overline{\mathcal{M}}$

Let $\mathcal{F}ib_2$ be the moduli stack of rank 2 vector bundles on $\mathbb{P}^1$, $\mathcal{F}ib_2^{\mathrm{k}} \subset \mathcal{F}ib_2$ the open substack formed by bundles L such that $H^1(\mathbb{P}^1, L(k)) = 0$ (k is an integer). It is well-known that $\mathcal{F}ib_2$ is an algebraic stack and $\mathcal{F}ib_2^{\mathrm{k}}$ is an algebraic stack of finite type (cf. [14, Theorem 4.6.2.1]). Using the morphism $\mathcal{M} \to \mathcal{F}ib_2$ which sends $(L, \nabla, \varphi; E, \epsilon; l_1, \dots, l_n)$ to L, it is easy to see $\mathcal{M}$ is algebraic. Moreover, condition (iv) of Definition 2 guarantees the image of $\mathcal{M}$ is contained in $\mathcal{F}ib_2^{\mathrm{k}}$ for $k \gg 0$, so $\mathcal{M}$ is of finite type.

It is easy to see the diagonal $\overline{\mathcal{M}} \to \overline{\mathcal{M}} \times \overline{\mathcal{M}}$ is unramified, so [14, Theorem 8.1] implies that $\overline{\mathcal{M}}$ is a Deligne–Mumford stack. Using the valuative criterion for Deligne–Mumford stacks (see [6, Theorems 2.2,2.3]) we derive the first statement of Theorem 1 from the following statement:

Proposition 1. *Suppose A is a complete discrete valuation ring, K is the fraction field of A, $\eta := \operatorname{Spec}(K)$, $y^0 = (L^0, \nabla^0, \varphi^0; E^0, \epsilon^0; l_1^0, \dots, l_n^0) \in \overline{\mathcal{M}}_\eta$.*

(i) *If an extension of y^0 to $y \in \overline{\mathcal{M}}_U$ exists, it is unique. Here $U := \operatorname{Spec} A$;*

(ii) *There is a finite extension $K' \supset K$ such that the inverse image of y^0 to $\eta' := \operatorname{Spec} K'$ can be extended to $y' \in \overline{\mathcal{M}}_{U'}$. Here $U' := \operatorname{Spec}(A')$, A' is the integral closure of A in K'.*

Denote by $v : K \to \mathbb{R} \cup \{\infty\}$ the valuation, by $m \subset A$ the maximal ideal, by $k := A/m$ the residue field of A. So $\operatorname{Spec} k \in U$ is the special point. Let $\operatorname{Spec} \mathbb{C}(z) \in \mathbb{P}^1$ be the generic point, $\Omega_{\mathbb{C}(z)}$ the generic fiber of $\Omega_{\mathbb{P}^1}$. Denote by $\tilde{\eta} := \operatorname{Spec} K(z) \in U \times \mathbb{P}^1$ the generic point, by $\tilde{p} := \operatorname{Spec} k(z) \in \mathbb{P}^1_k \subset U \times \mathbb{P}^1$ the generic point of the special fiber, by $\tilde{A} \subset K(z)$ the local ring of $\tilde{p}$, and by $\tilde{m} \subset \tilde{A}$ the maximal ideal. $\tilde{A}$ is a valuation ring. Let $\tilde{v} : K(z) \to \mathbb{R} \cup \{\infty\}$ be the corresponding valuation such that $\tilde{v}|_K = v$.

Lemma 1. *Let X be a smooth scheme of dimension 2, $S \subset X$ a finite subscheme, F a locally free sheaf on $X_0 := X \setminus S$. Then j_*F is a locally free sheaf on X, where $j : X_0 \hookrightarrow X$ is the natural embedding.*

Proof (communicated by B. Conrad). By [7, Proposition VIII.3.2], $\overline{F} := j_*F$ is a coherent O_X-module. Clearly $\overline{F}$ is torsion-free. Besides, for any open subscheme $j_U : U \hookrightarrow X$ such that $\dim X \setminus U = 0$ we have $\overline{F} = (j_U)_*(\overline{F}|_U)$, so $\overline{F}$ is reflexive by [10, Proposition 1.6]. Now [10, Corollary 1.4] implies $\overline{F}$ is locally free. □

For a point $r \in U \times \mathbb{P}^1$ and an $O_{U\times\mathbb{P}^1}$-module L we denote by L_r (resp. $L_{(r)}$) the fiber (resp. the stalk) of L over r as a module over the residue field (resp. the local ring) of r.

Lemma 2. *Let $\mathcal{C}$ be the category of pairs (L, Q), where L is a vector bundle on $\eta \times \mathbb{P}^1 = \mathbb{P}^1_K$, $Q \subset L_{\tilde{\eta}}$ is an $\tilde{A}$-lattice (i.e., Q is a finitely generated $\tilde{A}$-submodule such that $L_{\tilde{\eta}} = K(z) \otimes_{\tilde{A}} Q$). Denote by $\operatorname{Fib}(U\times\mathbb{P}^1)$ the category of vector bundles on $U \times \mathbb{P}^1$. The correspondence $L \mapsto (L|_{\eta\times\mathbb{P}^1}, L_{(\tilde{p})})$ defines an equivalence of categories $F : \operatorname{Fib}(U \times \mathbb{P}^1) \to \mathcal{C}$.*

Proof. Clearly F is fully faithful. If $S \subset \mathbb{P}^1_k$ is a finite subscheme, any vector bundle on $(U \times \mathbb{P}^1) \setminus S$ has a unique extension to $U \times \mathbb{P}^1$ (Lemma 1). Hence F is essentially surjective. □

Let us describe all extensions of $y^0 = (L^0, \nabla^0, \varphi^0; E^0, \epsilon^0; l^0_1, \dots, l^0_n) \in \overline{\mathcal{M}}_\eta$ to $y = (L, \nabla, \varphi; E, \epsilon; l_1, \dots, l_n) \in \overline{\mathcal{M}}_U$. Fix y^0. Denote by $\mathcal{E}x$ the category of all extensions $y \in \overline{\mathcal{M}}_U$. Clearly $\mathcal{E}x$ is a discrete category. Denote by Ex the set of isomorphism classes of extensions.

Consider pairs $(Q \subset (L^0)_{\tilde{\eta}}, E \subset E^0)$, where Q is an $\tilde{A}$-lattice, E is a free A-submodule of rank 1 (i.e., $E \subset E^0$ is an A-lattice), $\epsilon \in E$. Let us introduce the following conditions:

(a) $\nabla^0(Q) \subset Q \otimes_A E \otimes_{\mathbb{C}(z)} \Omega_{\mathbb{C}(z)}$;
(b) $\varphi^0(\bigwedge^2_{\tilde{A}} Q) = \tilde{A}$;

(c) The map $(\nabla^0 \bmod \tilde{m}) : (Q/\tilde{m}Q) \to (Q/\tilde{m}Q) \otimes_k (E/mE) \otimes_{\mathbb{C}(z)} \Omega_{\mathbb{C}(z)}$ is irreducible (this map is well-defined if (a) holds), that is, there is no $(\nabla^0 \bmod \tilde{m})$-invariant subspace $V \subset Q/\tilde{m}Q$, $\dim_{k(z)} V = 1$.

(c') If $\epsilon^0 \in mE$, then $(\nabla^0 \bmod \tilde{m})$ is not nilpotent (in this case $(\nabla^0 \bmod \tilde{m})$ is $k(z)$-linear).

Denote by Ex_1 (resp. Ex_1') the set of all $(Q \subset (L^0)_{\tilde{\eta}}, E \subset E^0)$ that satisfy (a)–(c) (resp. (a), (b), and (c')). If (a) holds, then (c) implies (c'), so $Ex_1 \subset Ex_1'$.

Lemma 3. *The map $F_{Ex} : [y] \mapsto (L_{(\tilde{p})} \subset (L^0)_{\tilde{\eta}}, E \subset E^0)$ gives an isomorphism $Ex \widetilde{\to} Ex_1' = Ex_1$. Here $y = (L, \nabla, \varphi; E, \epsilon; l_1, \ldots, l_n) \in \overline{\mathcal{M}}_U$, and $[y] \in Ex$ is the isomorphism class of y.*

Proof. For $[y] \in Ex$ its image $F_{Ex}[y]$ clearly satisfies (a) and (b). Besides y satisfies the condition (iv) of Definition 2, so $F_{Ex}[y]$ satisfies (c). Hence $F_{Ex}(Ex) \subset Ex_1 \subset Ex_1'$. Lemma 2 implies F_{Ex} is injective.

Let us prove that $F_{Ex} : Ex \to Ex_1'$ is surjective. Fix $(Q \subset (L^0)_{\tilde{\eta}}, E \subset E^0) \in Ex_1'$. Using Lemma 2, we can extend L^0 to L. Then ∇^0 and φ^0 have unique extensions ∇ and φ. (a) (resp. (b)) implies that ∇ (resp. φ) has no poles. l_i^0 uniquely extends to free rank 1 submodule $l_i \subset L|_{U \times x_i}$. (i)–(iii) of Definition 2 are automatically satisfied. (c') implies condition (iv') of Remark 2. So $[(L, \nabla, \varphi; E, \epsilon = \epsilon^0; l_1, \ldots, l_n)] \in Ex$. □

Remark. In this proof we used the equivalence (iv) $\iff$ (iv'), which holds only for $\mathbb{P}^1$, not for the curves of higher genus. Actually Proposition 1 holds only for $\mathbb{P}^1$.

Proof of Proposition 1. (i) It is enough to prove the set Ex_1 has at most one element.

Let $(Q \subset (L^0)_{\tilde{\eta}}, E \subset E^0) \in Ex_1$. Denote by $v_Q : (L^0)_{\tilde{\eta}} \to \mathbb{R} \cup \{\infty\}$ (resp. $v_E : E^0 \to \mathbb{R} \cup \{\infty\}$) the valuation induced by $Q \subset (L^0)_{\tilde{\eta}}$ and $\tilde{v}$ (resp. $E \subset E^0$ and v).

Fix $s_1 \in (L^0)_{\tilde{\eta}}$, $e \in E^0$, $s_1 \neq 0$, $e \neq 0$. Let us prove that Q and E are uniquely determined by $v_s := v_Q(s_1)$ and $v_e := v_E(e)$. Choose $f_s, f_e \in K$ such that $v(f_s) = -v_s$ and $v(f_e) = -v_e$. Then $E = Af_e e$ is determined by v_e. Fix $\omega \in \Omega_{\mathbb{C}(z)}$, $\omega \neq 0$. Then $\tilde{\nabla} := (\omega f_e e)^{-1} \nabla^0 : (L^0)_{\tilde{\eta}} \to (L^0)_{\tilde{\eta}}$ preserves Q and its reduction modulo $\tilde{m}$ is irreducible. Since $v_Q(f_s s_1) = 0$, $f_s s_1$ and $\tilde{\nabla}(f_s s_1) = f_s f_e^{-1} s_2$ generate Q. Here $s_2 := f_e \tilde{\nabla} s_1 = (\omega e)^{-1} \nabla^0 s_1$ does not depend on v_s and v_e, so Q is uniquely determined.

Let us find v_s and v_e. Set $s_3 := ((\omega e)^{-1} \nabla^0)^2 s_1$. Since s_1 and s_2 are linearly independent over $K(z)$, $s_3 = f_1 s_1 + f_2 s_2$ for some $f_1, f_2 \in K(z)$. Since $(\tilde{\nabla} \bmod \tilde{m})$ is irreducible, $\tilde{\nabla}\tilde{\nabla}(f_s s_1) = f_s f_e^{-2} s_3 \in Q$ does not vanish modulo $\tilde{m}$. Since $f_s f_e^{-2} s_3 = f_e^{-2} f_1 (f_s s_1) + f_e^{-1} f_2 (f_s f_e^{-1} s_2)$ and $\{f_s s_1, f_s f_e^{-1} s_2\}$ is a basis in Q, we get $\min(\tilde{v}(f_e^{-2} f_1), \tilde{v}(f_e^{-1} f_2)) = 0$ and

$$v_e = -\min\left(\frac{1}{2}\tilde{v}(f_1), \tilde{v}(f_2)\right). \tag{2}$$

Besides, (b) implies $\tilde{v}(\varphi(f_s s_1 \wedge f_s f_e^{-1} s_2)) = 0$. This gives $\tilde{v}(f_s^2 f_e^{-1}\varphi(s_1 \wedge s_2)) = 0$ and finally

$$v_s = \frac{1}{2}\big(\tilde{v}(\varphi(s_1 \wedge s_2)) + v_e\big). \tag{3}$$

So v_s and v_e (and hence Q and E) are uniquely determined.

(ii) Let e, ω, $\tilde{\nabla}$, s_1, s_2, and s_3 be the same as above. Since (L^0, ∇^0) is irreducible over $\tilde{\eta}$, $s_3 = f_1 s_1 + f_2 s_2$ for some $f_1, f_2 \in K(z)$, and equations (2) and (3) make sense. We take a finite extension $K' \supset K$ with a valuation $v' : K' \to \mathbb{R} \cup \{\infty\}$ such that $v'|_K = v$ and the equations (2) and (3) have a solution (v_s, v_e) in $v'(K')$. For the sake of simplicity let us assume $K' = K$.

Set $Q := \tilde{A} f_s s_1 \oplus \tilde{A} f_s f_e^{-1} s_1$, $E' := A f_e e$, where $f_s, f_e \in K$, $v(f_s) = -v_s$, $v(f_e) = -v_e$. If $\epsilon \notin E'$, we set $E := \tilde{A}\epsilon$, otherwise $E := E'$. Let us prove that $(Q \subset (L^0)_{\tilde{\eta}}, E \subset E^0) \in Ex'_1$. (2) implies that Q is $\tilde{\nabla}$-invariant. Hence

$$\nabla^0(Q) \subset Q \otimes_A E \otimes_{\mathbb{C}(z)} \Omega_{\mathbb{C}(z)} \subset Q \otimes_A E' \otimes_{\mathbb{C}(z)} \Omega_{\mathbb{C}z}$$

and (a) holds. (3) implies (b). Let us prove (c').

If $\epsilon \in mE$, then $E = E'$. Besides $\tilde{\nabla}\tilde{\nabla}(f_s s_1) = f_e^{-2} s_3 \notin \tilde{m} Q$, so $(\tilde{\nabla} \bmod \tilde{m})$ (and hence $(\nabla^0 \bmod \tilde{m})$) is not nilpotent. □

Remark. Let us sketch another way to prove $\overline{\mathcal{M}}$ is separated.

Suppose there are two U-families of ϵ-bundles $(L^i, \nabla^i, \varphi^i; E^i, \epsilon^i; l_1^i, \dots, l_n^i)$ $(i = 1, 2)$ which coincide on $\eta \times \mathbb{P}^1$. Without loss of generality we may assume $E^1 \subset E^2$. Since L^i are $SL(2)$-sheaves, either $L^1 = L^2$ or $L^1 \not\subset L^2$ and $L^1 \not\supset L^2$. Hence $L^1 \cap L^2 \subset L^2$ is a ∇-invariant subsheaf and it does not vanish over the special fiber. So $L^1 \cap L^2 = L^2$ and $L^1 = L^2$ because L^2 is irreducible over any fiber. Then it is clear that $E^1 = E^2$.

4. Smoothness of $\overline{\mathcal{M}}$

Consider the quotient stack $\mathbf{G}_m \setminus \mathbb{A}^1$. Recall that $\mathbf{G}_m \setminus \mathbb{A}^1$ is the moduli stack of pairs (E, ϵ), where E is a dimension 1 vector space, $\epsilon \in E$. Define $r : \overline{\mathcal{M}} \to \mathbf{G}_m \setminus \mathbb{A}^1$ by $(L, \nabla, \varphi; E, \epsilon; l_1, \dots, l_n) \mapsto (E, \epsilon)$.

To complete the proof of Theorem 1, it is enough to show r is smooth. By [14, Proposition 4.15], we should check that r is formally smooth. Since $\mathbf{G}_m \setminus \mathbb{A}^1$ is Noetherian, it suffices to prove the following lemma (cf. [8, Proposition 17.4.2]):

Lemma 4. *Suppose A is a local Artinian ring with a maximal ideal $m \subset A$, $A/m = \mathbb{C}$, $I \subset A$ is an ideal, $mI = 0$. Set $A_0 := A/I$, $U_0 := Spec(A_0) \subset U := Spec(A)$. For any $\epsilon \in A$ and $y^0 = (L^0, \nabla^0, \varphi^0; A_0, \epsilon + I; l_1^0, \dots, l_n^0) \in \overline{\mathcal{M}}_{U_0}$, there exists an extension $y = (L, \nabla, \varphi; A, \epsilon; l_1, \dots, l_n) \in \overline{\mathcal{M}}_U$.*

Proof. Denote by $(L^1, \nabla^1, \varphi^1; A/m, \epsilon + m; l_1^1, \dots, l_n^1) \in \overline{\mathcal{M}}_{\operatorname{Spec}\mathbb{C}}$ the reduction of y_0 modulo m.

Clearly, one can always extend y^0 to y locally on $\mathbb{P}^1$. Obstructions to a global extension lie in $H^2 := \mathbb{H}^2(\mathbb{P}^1, \mathcal{F}^\bullet \otimes_{\mathbb{C}} I)$. Here $\mathcal{F}^\bullet$ is the complex of sheaves defined by $\mathcal{F}^i := 0$ for $i \neq 0, 1$,

$$\mathcal{F}^0 := \left\{ s \in \mathcal{E}nd(L^1) | \operatorname{tr}(s) = 0; s(x_i)(l_i^1) \subset l_i^1 \right\},$$

$$\mathcal{F}^1 := \{ s \in \mathcal{E}nd(L^1) \otimes \Omega_{\mathbb{P}^1}(x_1 + \cdots + x_n) | \operatorname{tr}(s) = 0; s(x_i)|_{l_i^1} = 0 \},$$

and $d : \mathcal{F}^0 \to \mathcal{F}^1$ maps s to $s\nabla^1 - \nabla^1 s$. In other words, d is the ϵ-connection on $\mathcal{F}^0$ induced by ∇^1.

Consider the dual map $d^* : (\mathcal{F}^1)^* \otimes \Omega_{\mathbb{P}^1} \to (\mathcal{F}^0)^* \otimes \Omega_{\mathbb{P}^1}$. The natural pairing $\mathcal{E}nd(L_1) \times \mathcal{E}nd(L_1) \to O_{\mathbb{P}^1}$ induces an isomorphism between complexes $(\mathcal{F}^1)^* \otimes \Omega_{\mathbb{P}^1} \to (\mathcal{F}^0)^* \otimes \Omega_{\mathbb{P}^1}$ and $\mathcal{F}^\bullet$. By Serre's duality, $H^2 = \operatorname{Coker}(H^1(\mathbb{P}^1, \mathcal{F}^0) \to H^1(\mathbb{P}^1, \mathcal{F}^1)) \otimes_{\mathbb{C}} I = \operatorname{Ker}(H^1(\mathbb{P}^1, \mathcal{F}^1)^* \to H^1(\mathbb{P}^1, \mathcal{F}^0)^*) \otimes_{\mathbb{C}} I = \operatorname{Ker}(H^0(\mathbb{P}^1, \mathcal{F}^0) \to H^0(\mathbb{P}^1, \mathcal{F}^1))^* \otimes_{\mathbb{C}} I$. So it suffices to prove that any $B \in \mathcal{E}nd\, L^1$ such that $\nabla^1 B = B\nabla^1$ is scalar.

For any $\lambda \in \mathbb{C}$, $\operatorname{Ker}(B - \lambda) \subset L^1$ is a ∇^1-invariant subbundle. So either $\operatorname{Ker}(B - \lambda) = 0$ or $\operatorname{Ker}(B - \lambda) = L^1$. Since B has an eigenvalue $\lambda \in \mathbb{C}$, the statement easily follows. □

5. Geometric description of $\mathcal{M}_H$ and $\xi_x|_{\mathcal{M}_H}$

For the rest of the paper, we assume $n = 4$.

Recall that $\mathcal{M}_H$ is the moduli stack of $(L, \nabla, \varphi; E, 0)$, where L is a rank 2 vector bundle on $\mathbb{P}^1$, E is a dimension 1 vector space, $\varphi : \bigwedge^2 L \tilde{\to} O_{\mathbb{P}^1}$, $\nabla : L \to L \otimes \Omega_{\mathbb{P}^1}(x_1 + \cdots + x_4) \otimes E$ is an $O_{\mathbb{P}^1}$-linear homomorphism, $\operatorname{tr} \nabla = 0$, $\det \nabla \neq 0$, and $\det \nabla \in H^0(\mathbb{P}^1, \Omega^{\otimes 2}(x_1 + \cdots + x_4)) \otimes E^{\otimes 2}$.

Remark. $\det \nabla$ defines a section of $(\mathcal{E}|_{\mathcal{M}_H})^{\otimes 2} \otimes H^0(\mathbb{P}^1, \Omega^{\otimes 2}(x_1 + \cdots + x_4))$ with no zeros. In particular, $(\mathcal{E}|_{\mathcal{M}_H})^{\otimes 2} \simeq O_{\mathcal{M}_H}$.

Let us fix $\mu \in H^0(\mathbb{P}^1, \Omega^{\otimes 2}(x_1 + \cdots + x_4))$, $\mu \neq 0$. One can choose an isomorphism $E \simeq \mathbb{C}$ such that $\det \nabla = \mu$ (there are two choices for such $E \simeq \mathbb{C}$).

Denote by $\mathcal{Y}$ the moduli stack of triples (L, ∇, φ), where (L, φ) is an $SL(2)$-bundle on $\mathbb{P}^1$, $\nabla \in H^0(\mathbb{P}^1, \operatorname{End}(L) \otimes \Omega(x_1 + \cdots + x_4))$, $\operatorname{tr} \nabla = 0$, $\det \nabla = \mu$. We have proved the following statement:

Proposition 2. *The correspondence $(L, \nabla, \varphi) \mapsto (L, \nabla, \varphi; \mathbb{C}, 0)$ yields a double cover $\pi_{(1)} : \mathcal{Y} \to \mathcal{M}_H$. Besides, $\mathcal{M}_H$ is identified with the quotient stack $\mu_2 \setminus \mathcal{Y}$, where $\pm 1 \in \mu_2$ acts on $\mathcal{Y}$ by $(L, \nabla, \varphi) \mapsto (L, \pm\nabla, \varphi)$.* □

It follows directly from the definition of $\mathcal{Y}$ that $\pi_{(1)}^*(\mathcal{E}) = O_{\mathcal{Y}}$.

Set $\mathcal{A} := O_{\mathbb{P}^1} \oplus (\Omega_{\mathbb{P}^1}(x_1 + \cdots + x_4))^{-1}$. $\mathcal{A}$ is a sheaf of $O_{\mathbb{P}^1}$-algebras with respect to the multiplication $(f_1, \tau_1) \times (f_2, \tau_2) := (f_1 f_2 - \mu \otimes \tau_1 \otimes \tau_2, f_1 \tau_2 + f_2 \tau_1)$. Set

$\pi : Y := \mathcal{S}pec(\mathcal{A}) \to \mathbb{P}^1$. Denote by $y_i \in Y$ the preimage of $x_i \in \mathbb{P}^1$, and by $\sigma : Y \to Y$ the involution induced by $\sigma^* : \mathcal{A} \to \mathcal{A} : (f, \tau) \mapsto (f, -\tau)$.

For an invertible sheaf l on Y, there is a natural action of σ on the sheaf $l \otimes \sigma^* l$. So there is a natural invertible sheaf $\operatorname{norm}(l)$ on $\mathbb{P}^1$ such that $l \otimes \sigma^* l = \pi^* \operatorname{norm}(l)$. Moreover, $\bigwedge^2(\pi_* l) = \bigwedge^2(\pi_* O_Y) \otimes \operatorname{norm}(l) = \bigwedge^2 \mathcal{A} \otimes \operatorname{norm}(l) = (\Omega_{\mathbb{P}^1}(x_1 + \cdots + x_4))^{-1} \otimes \operatorname{norm}(l)$.

Proposition 3. *$\mathcal{Y}$ is the moduli stack of (l, ψ), where l is a line bundle on Y, $\psi : \operatorname{norm}(l) \widetilde{\to} \Omega_{\mathbb{P}^1}(x_1 + \cdots + x_4)$.*

Proof. Let (L, ∇, φ) be a point of $\mathcal{Y}$. Then L is an $\mathcal{A}$-module with respect to the multiplication $(f, \tau)s := fs + \tau \otimes \nabla s$. Since L is a torsion-free $\mathcal{A}$-module, L defines an invertible sheaf l on Y. φ induces $\psi : \operatorname{norm}(l) \widetilde{\to} \Omega_{\mathbb{P}^1}(x_1 + \cdots + x_4)$. The inverse construction is given by $l \mapsto L := \pi_* l$. □

If (l, ψ) is a point of $\mathcal{Y}$, then $\deg l = 2$. Conversely, if l is a degree 2 invertible sheaf on Y, then $\deg(\operatorname{norm}(l)) = 2$, so there is $\psi : \operatorname{norm}(l) \widetilde{\to} \Omega_{\mathbb{P}^1}(x_1 + \cdots + x_4)$. Hence the natural morphism $\mathcal{Y} \to \mathbf{P}\mathrm{ic}^2(Y) := \{\gamma \in \mathbf{P}\mathrm{ic}\, Y \mid \deg \gamma = 2\}$ that sends (l, ψ) to the class of l is a μ_2-gerbe, which is actually neutral. Here $\mathbf{P}\mathrm{ic}\, Y$ denotes the coarse moduli space of invertible sheaves on Y.

Denote by $\sigma_{(1)} : \mathcal{Y} \widetilde{\to} \mathcal{Y}$ the involution defined by $(l, \psi) \mapsto (\sigma^* l, \psi')$, where ψ' is the composition

$$\operatorname{norm}(\sigma^* l) = \operatorname{norm}(l) \xrightarrow{\psi} \Omega_{\mathbb{P}^1}(x_1 + \cdots + x_4) \xrightarrow{-1} \Omega_{\mathbb{P}^1}(x_1 + \cdots + x_4).$$

Note that $\sigma_{(1)}$ coincides with the action of $-1 \in \mu_2$. Clearly, the corresponding involution of $\mathbf{P}\mathrm{ic}^2\, Y$ sends the class of l to the class of $\sigma^* l$.

Let ζ_y be the line bundle on $\mathcal{Y}$ whose fiber over (l, ψ) is l_y, $y \in Y$.

Suppose $x \in \mathbb{P}^1 \setminus \{x_1, \ldots, x_4\}$, $\pi^{-1}(x) = \{y_+, y_-\}$. Then $\pi^*_{(1)}(\xi_x) = \zeta_{y_+} \oplus \zeta_{y_-}$. For $x = x_i$, $y = y_i$ we have a natural injection $\zeta_y \to \pi^*_{(1)}(\xi_x)$ and its cokernel is isomorphic to ζ_y.

Define $\deg : \operatorname{Pic} \mathcal{Y} \to \frac{1}{2}\mathbb{Z}$ by $\gamma \mapsto \frac{1}{2}\deg(\gamma^{\otimes 2})$ ($\gamma^{\otimes 2}$ is a class of sheaves on $\mathbf{P}\mathrm{ic}^2\, Y$, so the right hand side is well-defined). Actually $\deg : \operatorname{Pic} \mathcal{Y} \to \mathbb{Z}$.

Lemma 5.

(i) $\zeta_y^* = \zeta_{\sigma(y)} = \sigma^*_{(1)} \zeta_y$;
(ii) $\deg \zeta_y = 0$ *for* $y \in Y$;
(iii) $\zeta_{y_1} \not\simeq \zeta_{y_2}$ *for* $y_1 \neq y_2$, $y_1, y_2 \in Y$.

Proof. (i) Since $\bigwedge^2 \pi^*_{(1)}(\xi_x) = O_{\mathcal{Y}}$ (for $x = \pi(y)$), this statement is obvious.

(ii) The bundles ζ_y form a Y-family, so $\deg \zeta_y$ does not depend on y. Now (ii) follows from (i).

(iii) Fix $y_0 \in Y$. Consider the invertible sheaf on $Y \times \mathbf{P}\mathrm{ic}^2 Y$ whose fiber over (y, l) is $l_y \otimes (l_{y_0})^{-1}$. This is a universal $\mathbf{P}\mathrm{ic}^2 Y$-family of invertible sheaves of degree 2 on Y. It is well-known that this invertible sheaf can be viewed as a universal Y-family of invertible sheaves of degree 0 on $\mathbf{P}\mathrm{ic}^2 Y$. In particular, two different sheaves in this Y-family are not isomorphic. Hence $\zeta_{y_1} \otimes (\zeta_{y_0})^{-1} \not\simeq \zeta_{y_2} \otimes (\zeta_{y_0})^{-1}$ for any $y_1, y_2 \in Y$. □

Let γ be a sheaf on $\mathcal{M}_H$. Then $\pi_{(1)}^* \gamma$ is a sheaf on $\mathcal{Y}$ with an action of μ_2. Clearly $H^i(\mathcal{M}_H, \gamma) = (H^i(\mathcal{Y}, \pi_{(1)}^* \gamma))^{\mu_2}$, where $V^{\mu_2} \subset V$ is the subspace of μ_2-invariants.

Corollary 1. *If $x, y \in \mathbb{P}^1$, $x \neq y$, then $H^i(\mathcal{M}_H, \xi_x \otimes \xi_y \otimes \mathcal{E}^{\otimes k}) = 0$ for any i, k.*

Proof. It is enough to prove that $H^i(\mathcal{Y}, \pi_{(1)}^*(\xi_x \otimes \xi_y \otimes \mathcal{E}^{\otimes k})) = 0$. Since $\pi_{(1)}^* \mathcal{E} = O_{\mathcal{Y}}$, we must prove $H^i(\mathcal{Y}, \pi_{(1)}^*(\xi_x) \otimes \pi_{(1)}^*(\xi_y)) = 0$. But $\pi_{(1)}^*(\xi_x) \otimes \pi_{(1)}^*(\xi_y)$ has a filtration with quotients $\zeta_{x_\pm} \otimes \zeta_{y_\pm}$, where $\pi(x_\pm) = x$, $\pi(y_\pm) = y$. Since $\mathcal{Y}$ is a μ_2-gerbe over an elliptic curve, Lemma 5 implies $H^i(\mathcal{Y}, \zeta_{x_\pm} \otimes \zeta_{y_\pm}) = 0$. □

Proposition 4.

(i) $H^i(\mathcal{M}_H, O_{\mathcal{M}_H}) = \begin{cases} \mathbb{C}, & i = 0 \\ 0, & i \neq 0; \end{cases}$

(ii) $\dim H^i(\mathcal{M}_H, \mathcal{E}) = \begin{cases} 1, & i = 1 \\ 0, & i \neq 1. \end{cases}$

Remark. If γ' is an invertible sheaf on $\mathcal{Y}$, $\deg \gamma' = 0$, $\gamma' \not\simeq O_{\mathcal{Y}}$, then $H^i(\mathcal{Y}, \gamma') = 0$ for all i. Hence if γ is an invertible sheaf on $\mathcal{M}_H$, $\deg \pi_{(1)}^* \gamma = 0$, $\gamma \not\simeq O_{\mathcal{M}_H}, \mathcal{E}|_{\mathcal{M}_H}$, then $H^i(\mathcal{M}_H, \gamma) = 0$ for all i.

Proof. (i) We have

$$H^i(\mathcal{M}_H, O_{\mathcal{M}_H}) = (H^i(\mathcal{Y}, O_{\mathcal{Y}}))^{\mu_2} = (H^i(\mathbf{P}\mathrm{ic}^2 Y, O_{\mathbf{P}\mathrm{ic}^2 Y}))^{\mu_2},$$

where the action of μ_2 on $O_{\mathcal{Y}}$ is trivial. Since $\mathbf{P}\mathrm{ic}^2 Y$ is an elliptic curve,

$$\dim H^i(\mathbf{P}\mathrm{ic}^2 Y, O_{\mathbf{P}\mathrm{ic}^2 Y}) = \begin{cases} 1, & i = 0, 1 \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that $-1 \in \mu_2$ acts on $H^1(\mathbf{P}\mathrm{ic}^2 Y, O_{\mathbf{P}\mathrm{ic}^2 Y})$ as -1. This completes the proof.

(ii) Clearly $\pi_{(1)}^* \mathcal{E} = O_{\mathcal{Y}}$, but $-1 \in \mu_2$ acts on $\pi_{(1)}^* \mathcal{E}$ as -1. So $H^i(\mathcal{Y}, O_{\mathcal{Y}}) = (H^i(\mathcal{Y}, \pi_{(1)}^* \mathcal{E}))^{\mu_2} \oplus (H^i(\mathcal{Y}, O_{\mathcal{Y}}))^{\mu_2}$. The statement follows immediately. □

6. Infinitesimal neighborhood of $\mathcal{M}_H$

Denote by $\mathcal{M}_{H(k)}$ the k-th infinitesimal neighborhood of $\mathcal{M}_H$. In other words, $\mathcal{M}_{H(k)} \subset \overline{\mathcal{M}}$ is the closed substack defined by the sheaf of ideals $O_{\overline{\mathcal{M}}}(-k\mathcal{M}_H) \subset O_{\overline{\mathcal{M}}}$. Let $\mathcal{M}_{H(\infty)} := \varinjlim \mathcal{M}_{H(k)}$ be the formal completion of $\overline{\mathcal{M}}$ along $\mathcal{M}_H$.

Since the étale topology does not depend on nilpotents in the structure sheaf, there is a unique extension of $\mathcal{Y} \to \mathcal{M}_H$ to a double cover $\pi_{(k)} : \mathcal{Y}_{(k)} \to \mathcal{M}_{H(k)}$. Besides, there is a unique extension of $\sigma : \mathcal{Y} \to \mathcal{Y}$ to $\sigma_{(k)} : \mathcal{Y}_{(k)} \to \mathcal{Y}_{(k)}$ such that $\pi_{(k)} = \pi_{(k)} \circ \sigma_{(k)}$. We identify $\mathcal{Y}_{(k)}$ with a closed substack of $\mathcal{Y}_{(l)}$ for $l > k$. Set $\mathcal{Y}_{(\infty)} := \varinjlim \mathcal{Y}_{(k)}$. Denote by $\sigma_{(\infty)} : \mathcal{Y}_{(\infty)} \to \mathcal{Y}_{(\infty)}$ the involution such that $\sigma_{(\infty)}|_{\mathcal{Y}_{(k)}} = \sigma_{(k)}$.

Set $C := Y \setminus \{y_1, \dots, y_4\}$, $\mathcal{Y}_{(\infty)} \hat{\times} C := \varinjlim(\mathcal{Y}_{(k)} \times C)$. Denote by $\xi_{(\infty)}$ the pull-back of the natural bundle on $\overline{\mathcal{M}} \times \mathbb{P}^1$ to $\mathcal{Y}_{(\infty)} \hat{\times} C$ and by $\mathcal{E}_{(\infty)}$ the pull-back of $\mathcal{E}$ to $\mathcal{Y}_{(\infty)}$.

Let $\epsilon_{(\infty)} \in H^0(\mathcal{Y}_{(\infty)}, \mathcal{E}_{(\infty)})$ be the pull-back of $\epsilon \in H^0(\overline{\mathcal{M}}, \mathcal{E})$, and $\nabla : \xi_{(\infty)} \to \xi_{(\infty)} \otimes \Omega_C \otimes \mathcal{E}_{(\infty)}$ the natural $\epsilon_{(\infty)}$-connection along C. There is a decomposition $\xi_{(\infty)}|_{\mathcal{Y} \times C} = \zeta \oplus \sigma^* \zeta$, where ζ is the family of bundles ζ_y, $y \in C$. Let us extend this decomposition to $\mathcal{Y}_{(\infty)} \hat{\times} C$.

It is enough to show that locally on $\mathcal{Y}_{(\infty)}$ this decomposition has a unique ∇-invariant extension. So we can use the following lemma:

Lemma 6. *Let $C = \operatorname{Spec} A$ be a smooth curve, $Y_0 := \operatorname{Spec} B$ an affine scheme, $Y := \operatorname{Spec} B[[\epsilon]]$, L a rank 2 bundle on $C \hat{\times} Y := \operatorname{Spec}(A \otimes B)[[\epsilon]]$, $\nabla : L \to L \otimes \Omega_C$ an ϵ-connection on L along C (i.e., ∇ is $B[[\epsilon]]$-linear and $\nabla(fs) = f\nabla s + \epsilon s d_C f$, where $d_C f$ is the differential of f along C). Set $L_0 := L|_{C \times Y_0}$ and let $\nabla_0 \in \operatorname{Hom}_{O_{C \times Y_0}}(L_0, L_0 \otimes \Omega_C)$ be the reduction of ∇ modulo ϵ. Suppose there exists a ∇_0-invariant decomposition $L_0 = L_0^+ \oplus L_0^-$ such that the eigenvalues $\omega_0^\pm := \nabla_0|_{L_0^\pm}$ differ at any point of $C \times Y_0$. Then there is a unique ∇-invariant decomposition $L = L^+ \oplus L^-$ such that $L^\pm|_{C \times Y_0} = L_0^\pm$.* □

This lemma is a bit generalized version of [5, Proposition 1.2] (see also [21, Theorem 25.2]) and can be proved by the same method.

Proposition 5. *For $y \in Y \setminus \{y_1, \dots, y_4\}$, there is a rank 1 subbundle $\zeta_y^{(\infty)} \subset \pi_{(\infty)}^* \xi_{\pi(y)}$ such that*

(i) $\zeta_y^{(\infty)}|_{\mathcal{Y}} = \zeta_y \subset \pi_{(1)}^* \xi_{\pi(y)}$;

(ii) *The natural isomorphism $\pi_{(\infty)}^* \xi_{\pi(y)} = \sigma_{(\infty)}^*(\pi_{(\infty)}^* \xi_{\pi(y)})$ identifies $\zeta_{\sigma(y)}^{(\infty)}$ with $\sigma_{(\infty)}^* \zeta_y^{(\infty)}$.*

(iii) $\pi_{(\infty)}^* \xi_{\pi(y)} = \zeta_y^{(\infty)} \oplus \zeta_{\sigma(y)}^{(\infty)}$.

Proof. By Lemma 6, there is a ∇-invariant decomposition $\xi_{(\infty)} = \zeta^+_{(\infty)} \oplus \zeta^-_{(\infty)}$ that extends $\zeta_{(\infty)}|_{\mathcal{Y}\times C} = \zeta \oplus \sigma^*\zeta$. Set $\zeta_y^{(\infty)} := \zeta^+_{(\infty)}|_{\mathcal{Y}_{(\infty)}\times\{y\}}$. Statement (i) is clear. Since this ∇-invariant decomposition is unique, (ii) immediately follows. (iii) follows from (ii). □

Consider $\mathcal{M}_{H(2)}$. Recall that $\mathcal{M}_H$ is the zero set of $\epsilon \in H^0(\overline{\mathcal{M}}, \mathcal{E})$, so $O_{\overline{\mathcal{M}}}(\mathcal{M}_H) = \mathcal{E}$.

Lemma 7. *Set* $V := \mathrm{Ker}(\mathrm{Pic}\,\mathcal{M}_{H(2)} \to \mathrm{Pic}\,\mathcal{M}_H)$. *Then*

(i) *V is torsion-free;*
(ii) *Let γ be an invertible sheaf on $\mathcal{M}_{H(2)}$ such that $\gamma|_{\mathcal{M}_H} \simeq O_{\mathcal{M}_H}$, $\gamma \not\simeq O_{\mathcal{M}_{H(2)}}$. Then $H^i(\mathcal{M}_{H(2)}, \gamma) = 0$ for all i.*

Proof. (i) Consider the exact sequence

$$0 \to \mathcal{E}|_{\mathcal{M}_H} \to O^*_{\mathcal{M}_{H(2)}} \to O^*_{\mathcal{M}_H} \to 1.$$

It yields an exact sequence

$$0 \to H^1(\mathcal{M}_H, \mathcal{E}) \to \mathrm{Pic}\,\mathcal{M}_{H(2)} \to \mathrm{Pic}\,\mathcal{M}_H \to 1.$$

So $V = H^1(\mathcal{M}_H, \mathcal{E}) \simeq \mathbb{C}$.

(ii) Consider the exact sequence

$$0 \to \gamma(-\mathcal{M}_H) \to \gamma \to \gamma/\gamma(-\mathcal{M}_H) \to 0.$$

Clearly, $\gamma/\gamma(-\mathcal{M}_H)$ is the direct image of $\gamma|_{\mathcal{M}_H} \simeq O_{\mathcal{M}_H}$, so Proposition 4 implies

$$\dim H^i(\mathcal{M}_{H(2)}, \gamma/\gamma(-\mathcal{M}_H)) = \begin{cases} 1, & i = 0 \\ 0, & i \neq 0. \end{cases}$$

Similarly, $\gamma(-\mathcal{M}_H)$ is equal to the direct image of $\gamma|_{\mathcal{M}_H} \otimes (O(-\mathcal{M}_H))|_{\mathcal{M}_H} \simeq \gamma|_{\mathcal{M}_H} \otimes \mathcal{E}|_{\mathcal{M}_H} \simeq \mathcal{E}|_{\mathcal{M}_H}$, so

$$\dim H^i(\mathcal{M}_{H(2)}, \gamma(-\mathcal{M}_H)) = \begin{cases} 1, & i = 1 \\ 0, & i \neq 1. \end{cases}$$

Since $\gamma \neq O_{\mathcal{M}_{H(2)}}$, a nonzero section of $\gamma/\gamma(-\mathcal{M}_H)$ cannot be lifted to a section of γ. So the coboundary map

$$H^0(\mathcal{M}_{H(2)}, \gamma/\gamma(-\mathcal{M}_H)) \to H^1(\mathcal{M}_{H(2)}, \gamma(-\mathcal{M}_H))$$

is bijective, and the statement follows immediately. □

Set $N_{\mathcal{M}_{H(2)}} := O(\mathcal{M}_{H(2)})|_{\mathcal{M}_{H(2)}} = \mathcal{E}^{\otimes 2}|_{\mathcal{M}_{H(2)}}$.

Proposition 6. *The sheaf $N_{\mathcal{M}_{H(2)}}$ is not trivial.*

Remark. Let $[N_{\mathcal{M}_{H(2)}}] \in \operatorname{Pic} \mathcal{M}_{H(2)}$ be the isomorphism class of $N_{\mathcal{M}_{H(2)}}$. Then $[N_{\mathcal{M}_{H(2)}}] \in V$.

Proof. Set $\mathcal{F}_k := O_{\overline{\mathcal{M}}}(k\mathcal{M}_H)$. Clearly $\mathcal{F}_k/\mathcal{F}_{k-1} = \iota_* N_{\mathcal{M}_H}^{\otimes k}$, where $N_{\mathcal{M}_H} \simeq \mathcal{E}|_{\mathcal{M}_H}$ is the normal bundle to $\mathcal{M}_H \subset \overline{\mathcal{M}}$, $\iota : \mathcal{M}_H \hookrightarrow \overline{\mathcal{M}}$ is the embedding. So Proposition 4 implies that

$$H^i(\overline{\mathcal{M}}, \mathcal{F}_k/\mathcal{F}_{k-1}) = \begin{cases} \mathbb{C}, i = 0 & \text{and k is even;} \\ \mathbb{C}, i = 1 & \text{and k is odd;} \\ 0, & \text{otherwise.} \end{cases}$$

Besides, $H^i(\mathcal{M}_{H(2)}, N_{\mathcal{M}_{H(2)}}) = H^i(\overline{\mathcal{M}}, \mathcal{F}_2/\mathcal{F}_0)$, so it is enough to prove that the coboundary map $H^0(\overline{\mathcal{M}}, \mathcal{F}_2/\mathcal{F}_1) \to H^1(\overline{\mathcal{M}}, \mathcal{F}_1/\mathcal{F}_0)$ does not vanish.

Let us construct a rational section $F \in \mathcal{E}^{\otimes 2} = \mathcal{F}_2$. Fix $x \in \mathbb{P}^1 \setminus \{x_1, \dots, x_4\}$ and $\omega \in \Omega_x$. Let $(L, \nabla, \varphi; E, \epsilon)$ be an ϵ-bundle. Consider the map $F_1 : H^0(\mathbb{P}^1, L) \to L_x \otimes E : s \mapsto \omega^{-1}(\nabla s)(x)$ and the map $F_2 : H^0(\mathbb{P}^1, L) \to L_x : s \mapsto s(x)$. Set $F := \det(F_1)(\det(F_2))^{-1}$.

Now denote by δ the invertible sheaf on $\overline{\mathcal{M}}$ whose fiber at $(L, \nabla, \varphi; E, \epsilon)$ is $\bigwedge^2 H^0(\mathbb{P}^1, L) = \det R\Gamma(\mathbb{P}^1, L)$. Then $\det(F_2)$ is naturally a section of δ^{-1}, and $\det(F_1)$ is a section of $\mathcal{E}^{\otimes 2} \otimes \delta^{-1}$. The zero divisor of $\det(F_2) \in H^0(\overline{\mathcal{M}}, \delta^{-1})$ is the closed reduced substack $\mathcal{M}_1 \subset \overline{\mathcal{M}}$ formed by ϵ-bundles $(L, \nabla, \varphi, E, \epsilon)$ with $L \simeq O_{\mathbb{P}^1}(-1) \oplus O_{\mathbb{P}^1}(1)$.

It is easy to see the restriction $\det(F_1)|_{\mathcal{M}_1} \in H^0(\mathcal{M}_1, \mathcal{E}^{\otimes 2} \otimes \delta^{-1})$ has a zero of order 1 at $\mathcal{M}_H \cap \mathcal{M}_1$. Clearly $F|_{\mathcal{M}_H} \in H^0(\mathcal{M}_H, \mathcal{E}^{\otimes 2})$ equals $\det(\nabla)(x)\omega^{-2}$ at $(L, \nabla, \varphi; E, \epsilon)$. So $F|_{\mathcal{M}_H}$ has no zero.

Hence any global section of $\mathcal{F}^2/\mathcal{F}^0$ has a form $aF + G$, where $a \in \mathbb{C}$, $G \in \mathcal{F}^1/\mathcal{F}^0 = \mathcal{E}|_{\mathcal{M}_H}$. More precisely, $G \in H^0(\mathcal{M}_H, \mathcal{E}(\mathcal{M}_1 \cap \mathcal{M}_H))$. But it follows from the explicit description of $\mathcal{M}_H$ (see Section 4) that $H^0(\mathcal{M}_H, \mathcal{E}(\mathcal{M}_1 \cap \mathcal{M}_H)) = 0$. Since F has a pole of order 1 along $\mathcal{M}_1$, $a = 0$ (otherwise aF is not regular). □

Remark. There is another way to prove this proposition. Let us sketch the proof. Assume the converse. Then

$$H^0(\overline{\mathcal{M}}, \mathcal{F}_2/\mathcal{F}_0) = H^0(\overline{\mathcal{M}}, \mathcal{F}_2/\mathcal{F}_1) = H^0(\mathcal{M}_H, O_{\mathcal{M}_H}).$$

Let $\tilde{f} \in H^0(\overline{\mathcal{M}}, \mathcal{F}_2/\mathcal{F}_0)$ correspond to $1 \in H^0(\mathcal{M}_H, O_{\mathcal{M}_H})$. Since $H^1(\overline{\mathcal{M}}, O_{\overline{\mathcal{M}}}) = 0$ (see Proposition 7) one can lift $\tilde{f}$ to $f \in H^0(\overline{\mathcal{M}}, O_{\overline{\mathcal{M}}}(2\mathcal{M}_H)) \setminus H^0(\overline{\mathcal{M}}, O_{\overline{\mathcal{M}}})$. Clearly f has no zero on $\mathcal{M}_H$. Since $f^{-1} \notin H^0(\overline{\mathcal{M}}, O_{\overline{\mathcal{M}}})$, the divisor $(f)_0$ of zeros of f is not empty. Then $(f)_0 \subset \mathcal{M}$ is a complete substack and its image in the coarse moduli space M corresponding to $\mathcal{M}$ is a projective curve. By the Riemann–Hilbert correspondence M is analytically isomorphic to an affine variety, so M contains no projective curve. □

The following statements are used in Section 8.

Corollary 2. $H^i(\mathcal{M}_{H(2)}, (N_{\mathcal{M}_{H(2)}})^{\otimes k}) = 0$ *for any* $i \geq 0$, $k \neq 0$.

Proof. Let $[N_{\mathcal{M}_{H(2)}}] \in V \subset \operatorname{Pic} \mathcal{M}_{H(2)}$ be the isomorphism class of $N_{\mathcal{M}_{H(2)}}$. By Proposition 6, we have $[N_{\mathcal{M}_{H(2)}}] \neq 0$. By Lemma 7(i), $[N^{\otimes k}_{\mathcal{M}_{H(2)}}] \neq 0$. Lemma 7(ii) completes the proof. □

Corollary 3. *Set* $\mathcal{F} := \operatorname{Sym}^2 \xi_x \otimes \mathcal{E}|_{\mathcal{M}_{H(2)}}$. *Then* $H^i(\mathcal{M}_{H(2)}, \mathcal{F} \otimes N^{\otimes k}_{\mathcal{M}_{H(2)}}) = 0$ *for any* i, k, *and* $x \in \mathbb{P}^1 \setminus \{x_1, \ldots, x_4\}$.

Proof. For the sake of simplicity, we write ζ_y for $\zeta_y^{(\infty)}|_{\mathcal{Y}_{(2)}}$, $y \in Y \setminus \{y_1, \ldots, y_4\}$.

Clearly, the inverse image of $\operatorname{Sym}^2 \xi_x$ to $\mathcal{Y}_{(2)}$ is $\operatorname{Sym}^2(\zeta_{y_+} \oplus \zeta_{y_-}) = O_{\mathcal{Y}_{(2)}} \oplus (\zeta_{y_+}^{\otimes 2} \oplus \zeta_{y_-}^{\otimes 2})$. Here $\{y_+, y_-\} = \pi^{-1}(x)$. $O_{\mathcal{Y}_{(2)}} \subset \operatorname{Sym}^2(\zeta_{y_+} \oplus \zeta_{y_-})$ is μ_2-invariant and $-1 \in \mu_2$ acts on this sheaf as -1. Denote by $O^-_{\mathcal{M}_{H(2)}} \subset \operatorname{Sym}^2 \xi_x|_{\mathcal{M}_{H(2)}}$ the corresponding $O_{\mathcal{M}_{H(2)}}$-submodule. The same arguments as in the proof of Corollary 1 show that

$$H^i(\mathcal{Y}_{(2)}, (\zeta_{y_+}^{\otimes 2} \oplus \zeta_{y_-}^{\otimes 2}) \otimes \mathcal{E} \otimes N^{\otimes k}_{\mathcal{M}_{H(2)}}) = 0,$$

so

$$H^i(\mathcal{M}_{H(2)}, \mathcal{F} \otimes N^{\otimes k}_{\mathcal{M}_{H(2)}}) = H^i(\mathcal{M}_{H(2)}, O^-_{\mathcal{M}_{H(2)}} \otimes \mathcal{E} \otimes N^{\otimes k}_{\mathcal{M}_{H(2)}}).$$

But $(O^-_{\mathcal{M}_{H(2)}} \otimes \mathcal{E})^{\otimes 2} = \mathcal{E}^{\otimes 2}$, so the class of $O^-_{\mathcal{M}_{H(2)}} \otimes \mathcal{E} \otimes N^{\otimes k}_{\mathcal{M}_{H(2)}}$ in $\operatorname{Pic} \mathcal{M}_{H(2)}$ is $[N_{\mathcal{M}_{H(2)}}](k + \frac{1}{2}) \neq 0$. Lemma 7 completes the proof. □

7. Bundles ξ_x on $\overline{\mathcal{M}}$

Proposition 7. $H^i(\overline{\mathcal{M}}, O_{\overline{\mathcal{M}}}) = \begin{cases} \mathbb{C}, & i = 0 \\ 0, & i \neq 0 \end{cases}$.

Proof. Consider the four points on $\mathcal{M}_H$ that correspond to the classes $[\gamma] \in \mathbf{P}\mathrm{ic}^2 Y$ such that $\gamma \simeq \sigma^* \gamma$. Denote by $\overline{\mathcal{M}}^? \to \overline{\mathcal{M}}$ the blow-up in these four points. Then the coarse moduli space $\overline{M}^?$ corresponding to $\overline{\mathcal{M}}^?$ is a smooth rational projective scheme (note that the coarse moduli space corresponding to $\overline{\mathcal{M}}$ is not smooth). So $H^i(\overline{\mathcal{M}}, O_{\overline{\mathcal{M}}}) = H^i(\overline{\mathcal{M}}^?, O_{\overline{\mathcal{M}}^?}) = H^i(\overline{M}^?, O_{\overline{M}^?}) = \begin{cases} \mathbb{C}, & i = 0 \\ 0, & i \neq 0 \end{cases}$. □

Remark. Denote by M the coarse moduli space corresponding to $\mathcal{M}$. The variety M has the least smooth compactification $\overline{M} \supset M$ (see [1]). Hence there is a natural map $\overline{M}^? \to \overline{M}$. Actually $\overline{M}^? = \overline{M}$.

Suppose $(L, \nabla, \varphi; E, \epsilon)$ is an ϵ-bundle. Set $L' := \{s \in L | s(x_1) \in l_1\} \subset L$, where $l_1 := \operatorname{Ker}(R_1 - \lambda_1 \otimes \epsilon) \subset L_{x_1}$ is the eigenspace of the residue of ∇. Condition (iv) of Definition 2 implies $(L', \nabla|_{L'})$ is irreducible. Hence $L' \simeq O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-1)$, so $\dim H^0(\mathbb{P}^1, L') = 1$. Denote by ξ_+ the line bundle on $\overline{\mathcal{M}}$ whose fiber over $(L, \nabla, \varphi; E, \epsilon)$ equals $H^0(\mathbb{P}^1, L')$.

Lemma 8. *For $i \geq 0$,*

(i) $H^i(\overline{\mathcal{M}}, O_{\overline{\mathcal{M}}}(-\mathcal{M}_H)) = 0$;
(ii) $H^i(\overline{\mathcal{M}}, (\xi_+)^{\otimes 2}(-\mathcal{M}_H)) = 0$;
(iii) $H^i(\overline{\mathcal{M}}, ((\xi_+)^*)^{\otimes 2}(-\mathcal{M}_H)) = 0$.

Proof. (i) Consider the exact sequence

$$0 \to O_{\overline{\mathcal{M}}}(-\mathcal{M}_H) \to O_{\overline{\mathcal{M}}} \to O_{\overline{\mathcal{M}}}/O_{\overline{\mathcal{M}}}(-\mathcal{M}_H) \to 0.$$

By Propositions 7 and 4 the natural map

$$H^i(\overline{\mathcal{M}}, O_{\overline{\mathcal{M}}}) \to H^i(\overline{\mathcal{M}}, O_{\overline{\mathcal{M}}}/O_{\overline{\mathcal{M}}}(-\mathcal{M}_H)) = H^i(\mathcal{M}_H, O_{\mathcal{M}_H})$$

is bijective. So the first statement is obvious.

(ii) Let $(L, \nabla, \varphi; E, \epsilon)$ be an ϵ-bundle, $L' := \{s \in L | s(x_1) \in l_1\} \subset L$. Take $s \in H^0(\mathbb{P}^1, L')$, $s \neq 0$. Then $\nabla s \in H^0(\mathbb{P}^1, L' \otimes \Omega_{\mathbb{P}^1}(x_1 + \cdots + x_4) \otimes E)$, so $\nabla(s) \wedge s \in H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}((x_1 + \cdots + x_4) - x_1)) \otimes E$, $\nabla(s) \wedge s \neq 0$. Let $z \in \mathbb{P}^1$ be the unique zero of $\nabla(s) \wedge s$. Clearly z does not depend on the choice of s. Define $q : \overline{\mathcal{M}} \to \mathbb{P}^1$ by $(L, \nabla, \varphi; E, \epsilon) \mapsto z$.

Fix $x_0 \in \mathbb{P}^1 \setminus \{x_1\}$, $\omega \in \Omega(x_1 + \cdots + x_4)_{x_0}$, $\omega \neq 0$. The correspondence $s \mapsto (\nabla(s) \wedge s)(x_0)\omega^{-1} \in \bigwedge^2 L_{x_0} \otimes E$ defines a map $(\xi_+)^{\otimes 2} \to \mathcal{E}$. This map induces an isomorphism $(\xi_+)^{\otimes 2} \widetilde{\to} q^*(O_{\mathbb{P}^1}(-1)) \otimes \mathcal{E}$. So $(\xi_+)^{\otimes 2}(-\mathcal{M}_H) \simeq q^*(O_{\mathbb{P}^1}(-1))$.

Set $\overline{\mathcal{M}}_x := q^{-1}(x)$ for $x \in \mathbb{P}^1$. Clearly $\overline{\mathcal{M}}_x$ is a μ_2-gerbe over some algebraic space for $x \neq x_1, \ldots, x_4$. [1, Theorem 3] implies that $\overline{\mathcal{M}}_x \cap \mathcal{M}$ is a μ_2-gerbe over $\mathbb{A}^1$ for $x \neq x_1, \ldots, x_4$. So, there is $x \in \mathbb{P}^1$ such that $\overline{\mathcal{M}}_x$ is a μ_2-gerbe over $\mathbb{P}^1$ and $x' := \overline{\mathcal{M}}_x \cap \mathcal{M}_H$ is a μ_2-gerbe over a point.

Since $(\xi_+)^{\otimes 2}(-\mathcal{M}_H) \simeq q^*(O_{\mathbb{P}^1}(-1)) \simeq O_{\overline{\mathcal{M}}}(-\overline{\mathcal{M}}_x)$, it is enough to prove $H^i(\overline{\mathcal{M}}, O_{\overline{\mathcal{M}}}(-\overline{\mathcal{M}}_x)) = 0$. Consider the exact sequence

$$0 \to O_{\overline{\mathcal{M}}}(-\overline{\mathcal{M}}_x) \to O_{\overline{\mathcal{M}}} \to O_{\overline{\mathcal{M}}}/O_{\overline{\mathcal{M}}}(-\overline{\mathcal{M}}_x) \to 0.$$

Clearly

$$H^i(\overline{\mathcal{M}}, O_{\overline{\mathcal{M}}}/O_{\overline{\mathcal{M}}}(-\overline{\mathcal{M}}_x)) = H^i(\overline{\mathcal{M}}_x, O_{\overline{\mathcal{M}}_x}) = \begin{cases} \mathbb{C}, & i = 0 \\ 0, & i \neq 0. \end{cases}$$

Now Proposition 7 completes the proof.

(iii) Let x, $\overline{\mathcal{M}}_x$, and x' have the same meaning as above. Clearly

$$(\xi_+^*)^{\otimes 2}(-\mathcal{M}_H) \simeq (q^* O_{\mathbb{P}^1}(1))(-2\mathcal{M}_H) \simeq O_{\overline{\mathcal{M}}}(-2\mathcal{M}_H + \overline{\mathcal{M}}_x).$$

Let $\iota_1 : \overline{\mathcal{M}}_x \hookrightarrow \overline{\mathcal{M}}$ and $\iota_2 : \mathcal{M}_H \hookrightarrow \overline{\mathcal{M}}$ be the natural embeddings, $\mathcal{F}_1 := \iota_1^* O_{\overline{\mathcal{M}}}(\overline{\mathcal{M}}_x - \mathcal{M}_H)$, $\mathcal{F}_2 := \iota_2^* O_{\overline{\mathcal{M}}}(\overline{\mathcal{M}}_x - \mathcal{M}_H)$. Consider the exact sequences

$$0 \to O_{\overline{\mathcal{M}}}(-\mathcal{M}_H) \to O_{\overline{\mathcal{M}}}(\overline{\mathcal{M}}_x - \mathcal{M}_H) \to (\iota_1)_* \mathcal{F}_1 \to 0$$

$$0 \to O_{\overline{\mathcal{M}}}(\overline{\mathcal{M}}_x - 2\mathcal{M}_H) \to O_{\overline{\mathcal{M}}}(\overline{\mathcal{M}}_x - \mathcal{M}_H) \to (\iota_2)_* \mathcal{F}_2 \to 0.$$

By (i), $H^i(\overline{\mathcal{M}}, O_{\overline{\mathcal{M}}}(-\mathcal{M}_H)) = 0$, so it is enough to prove that $H^i(\overline{\mathcal{M}}_x, \mathcal{F}_1) = H^i(\mathcal{M}_H, \mathcal{F}_2) = 0$. Note that $\overline{\mathcal{M}}_x$ is a fiber of q, so $\iota_1^* O_{\overline{\mathcal{M}}}(\overline{\mathcal{M}}_x) \simeq O_{\overline{\mathcal{M}}_x}$ and $\mathcal{F}_1 \simeq O_{\overline{\mathcal{M}}_x}(-x')$. Since $\overline{\mathcal{M}}_x$ is a μ_2-gerbe over a projective line,

$$H^i(\overline{\mathcal{M}}_x, \mathcal{F}_1) \simeq H^i(\overline{\mathcal{M}}_x, O_{\overline{\mathcal{M}}_x}(-x')) = 0.$$

Finally, we have $\mathcal{F}_2 = (\mathcal{E}|_{\mathcal{M}_H})(x')$. The pull-back of $(\mathcal{E}|_{\mathcal{M}_H})(x')$ to $\mathcal{Y}$ is of degree 2, so

$$H^1(\mathcal{Y}, \pi_{(1)}^*((\mathcal{E}|_{\mathcal{M}_H})(x'))) = 0.$$

Hence $H^1(\mathcal{M}_H, (\mathcal{E}|_{\mathcal{M}_H}))(x') = 0$. Proposition 4 implies $\chi((\mathcal{E}|_{\mathcal{M}_H})(x')) = 1 + \chi(\mathcal{E}|_{\mathcal{M}_H}) = 0$, so $H^0(\mathcal{M}_H, (\mathcal{E}|_{\mathcal{M}_H}))(x') = 0$. ☐

Proposition 8. *Suppose $x, y \in \mathbb{P}^1$. For any i,*

(i) $H^i(\overline{\mathcal{M}}, \xi_x \otimes \xi_y(-\mathcal{M}_H)) = 0$;
(ii) $H^i(\overline{\mathcal{M}}, \operatorname{Sym}^2 \xi_x(-\mathcal{M}_H)) = 0$.

Proof. Without loss of generality we may assume that $x \neq x_1$, $y \neq y_1$. Then the natural maps $\xi_+ \to \xi_x$, $\xi_+ \to \xi_y$ are injective and their cokernels are isomorphic to $(\xi_+)^*$. We use these maps to identify ξ_+ with subbundles of ξ_x, ξ_y.

Consider the filtration $\mathcal{F}_0 := 0 \subset \mathcal{F}_1 := \xi_+ \otimes \xi_+ \subset \mathcal{F}_2 := (\xi_+ \otimes \xi_y) + (\xi_x \otimes \xi_+) \subset \mathcal{F}_4 := \xi_x \otimes \xi_y$. It follows from Lemma 8 that $H^i(\overline{\mathcal{M}}, (\mathcal{F}_k / \mathcal{F}_{k-1})(-\mathcal{M}_H)) = 0$. This implies (i). Since $\xi_x^{\otimes 2} = \operatorname{Sym}^2 \xi_x \oplus O_{\overline{\mathcal{M}}}$, (ii) follows from (i). ☐

8. Proof of Theorem 2

Denote by $j : \mathcal{M} \hookrightarrow \overline{\mathcal{M}}$ and $i_{(2)} : \mathcal{M}_{H(2)} \hookrightarrow \overline{\mathcal{M}}$ the natural embeddings. For a vector bundle $\mathcal{F}$ on $\overline{\mathcal{M}}$ we consider the filtration

$$\mathcal{F}_0 := \mathcal{F} \subset \cdots \subset \mathcal{F}_k := \mathcal{F}(k\mathcal{M}_{H(2)}) \subset \cdots \subset \mathcal{F}_\infty := j^* j_* \mathcal{F}.$$

This yields $H^\bullet(\mathcal{M}, \mathcal{F}|_{\mathcal{M}}) = H^\bullet(\overline{\mathcal{M}}, \mathcal{F}_\infty) = \varinjlim H^\bullet(\overline{\mathcal{M}}, \mathcal{F}_k)$. Besides, $\mathcal{F}_k / \mathcal{F}_{k-1} = (i_{(2)})_*(\mathcal{F}_k|_{\mathcal{M}_{H(2)}}) = (i_{(2)})_*(\mathcal{F}|_{\mathcal{M}_{H(2)}} \otimes (N_{\mathcal{M}_{H(2)}})^{\otimes k})$. The following lemma is clear.

Lemma 9. *Suppose $\mathcal{F}$ is a vector bundle on $\overline{\mathcal{M}}$ such that*

$$H^\bullet(\mathcal{M}_{H(2)}, \mathcal{F}|_{\mathcal{M}_{H(2)}} \otimes (N_{\mathcal{M}_{H(2)}})^{\otimes k}) = 0$$

for any $k > 0$. Then the natural maps $H^\bullet(\overline{\mathcal{M}}, \mathcal{F}) \to H^\bullet(\mathcal{M}, \mathcal{F}|_{\mathcal{M}})$ are isomorphisms. □

Proof of Theorem 2.

(i) Set $\mathcal{F} := \xi_x \otimes \xi_y(-\mathcal{M}_H)$. Using Corollary 1 and Lemma 9, we get $H^\bullet(\overline{\mathcal{M}}, \mathcal{F}) = H^\bullet(\mathcal{M}, \mathcal{F})$. Now Proposition 8 (i) completes the proof.
(ii) Set $\mathcal{F} := O_{\overline{\mathcal{M}}}$. Combining Lemma 9 with Corollary 2 and Proposition 7, we obtain the required formula.
(iii) Set $\mathcal{F} := \operatorname{Sym}^2 \xi_x(-\mathcal{M}_H)$. Corollary 3, Proposition 8 (ii), and Lemma 9 imply $H^\bullet(\mathcal{M}, \mathcal{F}) = H^\bullet(\overline{\mathcal{M}}, \mathcal{F}) = 0$. □

Remark. There is another way to prove Theorem 2 without Lemma 8. Let us sketch the proof.

Set $\mathcal{F}_{xy} := \xi_x \otimes \xi_y(-\mathcal{M}_H)$. Suppose $x \neq y$. Corollary 1 implies

$$H^i(\overline{\mathcal{M}}, \mathcal{F}_{xy}(k\mathcal{M}_{H(2)})/\mathcal{F}_{xy}((k-1)\mathcal{M}_{H(2)})) = 0.$$

So $H^i(\overline{\mathcal{M}}, \mathcal{F}_{xy}(k\mathcal{M}_{H(2)})) = H^i(\mathcal{M}, \mathcal{F}_{xy})$. But $H^0(\overline{\mathcal{M}}, \mathcal{F}_{xy}(k\mathcal{M}_{H(2)})) = 0$ for $k \ll 0$. Besides, $H^2(\mathcal{M}, \mathcal{F}_{xy}) = 0$ (see [1, Theorem 1]). Hence $H^i(\overline{\mathcal{M}}, \mathcal{F}_{xy}) = 0$ for $i \neq 1$.

In the same way, $H^i(\overline{\mathcal{M}}, \operatorname{Sym}^2 \xi_x(-\mathcal{M}_H)) = 0$ for $i \neq 1$, $x \neq x_1, \dots, x_4$. Since $H^i(\overline{\mathcal{M}}, O_{\overline{\mathcal{M}}}(-\mathcal{M}_H)) = 0$, we have $H^i(\overline{\mathcal{M}}, \mathcal{F}_{xy}) = H^i(\mathcal{M}, \mathcal{F}_{xy}) = 0$ for $i \neq 1$, $(x, y) \neq (x_j, x_j)$. But [1, Theorem 2] implies $H^i(\mathcal{M}, \mathcal{F}_{x_1x_2}) = 0$ for all i. Since $\mathcal{F}_{xy}$ form a $\mathbb{P}^1 \times \mathbb{P}^1$-family of coherent $O_{\overline{\mathcal{M}}}$-modules, we have $\chi(\mathcal{F}_{xy}) = 0$ for any (x, y). Hence $H^i(\overline{\mathcal{M}}, \mathcal{F}_{xy}) = 0$ for $(x, y) \neq (x_j, x_j)$ $(j = 1, \dots, 4)$. This part of Proposition 8 is enough for Theorem 2.

9. Orthogonality for families: Theorem 3

Recall that $p : P \to \mathbb{P}^1$ is the projective line with doubled points $x_1, \dots, x_4$, $p^{-1}(x_i) = \{x_i^+, x_i^-\}$, $[\lambda] := \sum_{i=1}^4 \lambda_i(x_i^+ - x_i^-)$ is a $\mathbb{C}$-divisor on P, D_λ is the corresponding TDO. For a $(\lambda_1, \dots, \lambda_4)$-bundle L, the D_λ-module L_λ is defined by $L_\lambda := j_{*!}(L|_U)$, where $U := \mathbb{P}^1 \setminus \{x_1, \dots, x_4\}$, $j : U \hookrightarrow P$ is the natural embedding. We identify U with $j(U)$. Clearly, $(L_\lambda)_x = L_x$ for $x \in U$, and $(L_\lambda)_x = l_i^\mp := \operatorname{Ker}(R_i \pm \lambda_i) \subset L_{x_i}$ for $x = x_i^\pm$.

Theorem 2 and [1, Theorem 2] imply the following proposition.

Proposition 9. *Let $y \in P \times P$ be a point. Then*

(i) $H^i(\{y\} \times \mathcal{M}, \mathcal{F}_P) = 0$ *for* $i \neq 0$;

(ii) $H^0(\{y\} \times \mathcal{M}, \mathcal{F}_P) = \begin{cases} 0, & y \notin \Delta' \\ \mathbb{C}, & y \in \Delta'; \end{cases}$

(iii) *Let* $y = (y_1, y_2) \in \Delta'$, $x = p(y_1) = p(y_2) \in \mathbb{P}^1$. *Then the map* $\mathbb{C} \to \bigwedge^2 L_x \to (L_x)^{\otimes 2} = (p^*L)_{y_1} \otimes (p^*L)_{y_2} \to (L_\lambda)_{y_1} \otimes (L_\lambda)_{y_2}$ *for an* ϵ*-bundle* L *defines a morphism* $O_\mathcal{M} \to \mathcal{F}_P|_{\{y\}\times\mathcal{M}}$ *such that the corresponding map* $\mathbb{C} = H^0(\mathcal{M}, O_\mathcal{M}) \to H^0(\{y\} \times \mathcal{M}, \mathcal{F}_P)$ *is bijective.* □

It is easy to see that this proposition is valid for all (not necessarily closed) points $y \in P \times P$.

Let Z be a scheme. Consider the derived category of quasicoherent O_Z-modules $\mathcal{D}^-_{qc}(Z)$.

We need the following lemma:

Lemma 10.

(i) *Let $V \subset Z$ be a closed subscheme that locally can be defined by one equation, $\mathcal{F} \in \mathcal{D}^-_{qc}(Z)$. Suppose $Li^*\mathcal{F} = 0$, where $i : Z \hookrightarrow X$ is the natural embedding. Then the natural mapping $\mathcal{F} \to j_*j^*\mathcal{F}$ is an isomorphism. Here $j : Z \backslash V \hookrightarrow Z$ (note that both j_* and j^* are exact).*

(ii) *Suppose Z is Noetherian. Let $\mathcal{F} \in \mathcal{D}^-_{qc}(Z)$ satisfy $(Li_y^*)\mathcal{F} = 0$ for all (not necessarily closed) points $i_y : y \hookrightarrow Z$. Then $\mathcal{F} = 0$.*

Remark. Statement (i) still holds in the case of a closed subscheme $V \subset Z$ that locally can be defined by a finite number of equations. In this situation, $Li^*\mathcal{F} = 0$ implies that $\mathcal{F} \to (Rj_*)j^*\mathcal{F}$ is an isomorphism.

Proof. (i) The statement is local, so we assume $Z = \operatorname{Spec} A$, $V = \operatorname{Spec} A/(f)$, $Z \setminus V = \operatorname{Spec} A_f$. Set $\operatorname{Ann}(f) := \operatorname{Ker}(A \xrightarrow{f} A)$. Since $\mathcal{F} \otimes^L_A (A/(f)) = 0$, we have

$$\mathcal{F} \otimes^L_A \operatorname{Ann}(f) = (\mathcal{F} \otimes^L_A (A/(f))) \otimes^L_{A/(f)} \operatorname{Ann}(f) = 0.$$

Consider the exact sequence

$$0 \to \operatorname{Ann}(f) \to A \to A \to A/(f) \to 0,$$

where the map $A \to A$ is multiplication by f. Multiplying it by $\mathcal{F}$, we see that $\mathcal{F} \xrightarrow{f} \mathcal{F}$ is a quasi-isomorphism. Hence $H^i(\mathcal{F}) \xrightarrow{f} H^i(\mathcal{F})$ is an isomorphism and $H^i(\mathcal{F}) = H^i(\mathcal{F}) \otimes A_f = H^i(\mathcal{F} \otimes A_f)$.

(ii) The statement is local, so we suppose $Z = \operatorname{Spec} A$. Assume $\mathcal{F} \neq 0$. Consider all closed subschemes $i_Y : Y \hookrightarrow Z$ such that $(Li_Y^*)\mathcal{F} \neq 0$. Since Z is Noetherian, there is a minimal subscheme Y with this property. Without loss of generality, we assume $Y = Z$.

Statement (i) implies that multiplication by any $f \in A$, $f \neq 0$ induces an isomorphism $f : H^i(\mathcal{F}) \to H^i(\mathcal{F})$. If A is not integral, we take $f, g \in A$ such that $f \neq 0$, $g \neq 0$, $fg = 0$. The composition $H^i(\mathcal{F}) \xrightarrow{f} H^i(\mathcal{F}) \xrightarrow{g} H^i(\mathcal{F})$ is zero, so $H^i(\mathcal{F}) = 0$ for all i.

If A is integral, we get $H^i(\mathcal{F}) = H^i(\mathcal{F}) \otimes_A K$, where K is the fraction field of A. Since K is a flat A-module, the assumptions of the lemma imply $H^i(\mathcal{F}) = H^i(\mathcal{F}) \otimes_A K = H^i(\mathcal{F} \otimes_A K) = 0$. $\square$

Proof of Theorem 3. Set $\mathcal{F} := Rp_{12,*}\mathcal{F}_P$ (as before, $p_{12} : P \times P \times \mathcal{M} \to P \times P$ is the projection).

Step 1. Set $V := P \times P \setminus \overline{\Delta}$, where $\overline{\Delta}$ is the closure of $\Delta' \subset P \times P$. Lemma 10 and Proposition 9 imply $\mathcal{F}|_V = 0$.

By Kashiwara's theorem ([4, Theorem 7.13]), $\mathcal{F} = \bar{i}_+(L\bar{i}^*\mathcal{F})[-1]$, where $\bar{i} : \overline{\Delta} \hookrightarrow P \times P$ is the embedding, $\bar{i}^*$ is the O-module pull-back. So it is enough to prove that $L\bar{i}^*\mathcal{F} = O_{\Delta'}$.

Step 2. Clearly $\overline{\Delta} \setminus \Delta'$ consists of eight points $(x_i^{\pm}, x_i^{\pm})$. By Proposition 9, the inverse image of $\mathcal{F}$ to this points is quasi-isomorphic to 0. So, by Lemma 10, it is enough to prove that $Li^*\mathcal{F} = O_{\Delta'}$, where $i : \Delta' \hookrightarrow P \times P$ is the natural embedding.

Step 3. There is a natural embedding $O_{\Delta' \times \mathcal{M}} \to \mathcal{F}_P|_{\Delta' \times \mathcal{M}}$. By Proposition 9 and Lemma 10, $R(p_{12})_*((\mathcal{F}_P|_{\Delta' \times \mathcal{M}})/O_{\Delta' \times \mathcal{M}}) = 0$. Hence

$$Li^*\mathcal{F} = R(p_{12})_*(\mathcal{F}_P|_{\Delta' \times \mathcal{M}}) = R(p_{12})_*(O_{\Delta' \times \mathcal{M}}) = O_{\Delta'} \otimes_{\mathbb{C}} \mathrm{R}\Gamma(\mathcal{M}, O_{\mathcal{M}}).$$

By Theorem 2 (ii), $\mathrm{R}\Gamma(\mathcal{M}, O_{\mathcal{M}}) = \mathbb{C}$, so $Li^*\mathcal{F} = O_{\Delta'}$. $\square$

10. Orthogonality for families: Theorem 4

We will need the following easy (and well-known) statement:

Lemma 11. *Let t be a local parameter at $x_i \in \mathbb{P}^1$, (L, ∇, φ) a λ-bundle. The restriction of (L, ∇) to the formal neighborhood of x is isomorphic to $\mathbb{C}[[t]] \oplus \mathbb{C}[[t]]$ with $\nabla : \mathbb{C}[[t]] \oplus \mathbb{C}[[t]] \to (\mathbb{C}[[t]] \oplus \mathbb{C}[[t]])dt$ given by $\nabla(f, g) = (df + f\lambda_i t^{-1}dt, dg - g\lambda_i t^{-1}dt)$.* $\square$

Let $(L_1, \nabla_1, \varphi_1)$ and $(L_2, \nabla_2, \varphi_2)$ be λ-bundles. Consider the D_P-module $\sigma^*((L_1)_\lambda) \otimes_{O_P} (L_2)_\lambda$. Set $L_{12} := \mathcal{H}om_{O_{\mathbb{P}^1}}(L_1, L_2)$. The natural connection $\nabla : L_{12} \to L_{12} \otimes \Omega_{\mathbb{P}^1}(x_1 + \cdots + x_4)$ gives a D_U-module structure on $L_{12}|_U$. Set $V_i := \mathrm{Im}(\mathrm{res}_{x_i} \nabla : (L_{12})_{x_i} \to (L_{12})_{x_i})$. Denote by $\widetilde{L}_{12}$ the modification of L_{12} whose sheaf of sections is $\{s \in L_{12} : s(x_i) \in V_i \subset (L_{12})_{x_i}; i = 1, \ldots, 4\}$.

Lemma 11 implies the following two statements.

Lemma 12. *The identification*

$$L_{12}|_U \widetilde{\to} j^*(\sigma^*((L_1)_\lambda) \otimes_{O_P} (L_2)_\lambda) = j^*(L_1) \otimes_{O_U} j^*(L_2)$$

extends to an isomorphism $j_{!*}(L_{12}|_U) \widetilde{\to} Rp_*(\sigma^*((L_1)_\lambda) \otimes_{O_P} (L_2)_\lambda)$. □

Lemma 13. *The map* $L_{12} \hookrightarrow j_{!*}(L_{12})$ *induces a quasi-isomorphism*

$$(L_{12} \xrightarrow{\nabla} \widetilde{L}_{12} \otimes \Omega_{\mathbb{P}^1}(x_1 + \cdots + x_4)) \to \mathbb{DR}(j_{!*}(L_{12})).$$

□

All the above constructions are still valid for families of λ-bundles. In particular, since the complex $(L_{12} \to \widetilde{L}_{12} \otimes \Omega_{\mathbb{P}^1}(x_1 + \cdots + x_4))$ is formed by coherent sheaves, Lemmas 12 and 13 imply $R^i p_{12,*}\, \mathbb{DR}(\mathcal{F}_\mathcal{M})$ is coherent for any i (here $p_{12} : \mathcal{M} \times \mathcal{M} \times P \to \mathcal{M} \times \mathcal{M}$).

Set $\mathbb{H}^i := \mathbb{H}^i(\mathbb{P}^1, (L_{12} \xrightarrow{\nabla} \widetilde{L}_{12} \otimes \Omega_{\mathbb{P}^1}(x_1 + \cdots + x_4)))$. The following proposition computes $\dim \mathbb{H}^i$.

Proposition 10.

(i) *If* L_1 *and* L_2 *are not isomorphic,* $\mathbb{H}^i = 0$ *for any* i*;*

(ii) *If* L_1 *and* L_2 *are isomorphic,*

$$\dim \mathbb{H}^i = \begin{cases} 1, & \text{if } i = 0, 2 \\ 2, & \text{if } i = 1 \\ 0, & \text{otherwise.} \end{cases}$$

(iii) *Suppose* $L_1 = L_2 = L$ *(so* $L_{12} = \mathcal{E}nd_{O_{\mathbb{P}^1}}(L)$*). Consider the map of complexes*

$$(O_{\mathbb{P}^1} \xrightarrow{d} \Omega_{\mathbb{P}^1}) \hookrightarrow (L_{12} \xrightarrow{\nabla} \widetilde{L}_{12} \otimes \Omega_{\mathbb{P}^1}(x_1 + \cdots + x_4))$$

induced by $O_{\mathbb{P}^1} \to L_{12} : f \mapsto f\,\mathrm{Id}_L$*. Then the induced map* $H^i_{DR}(\mathbb{P}^1, \mathbb{C}) := \mathbb{H}^i(\mathbb{P}^1, O \xrightarrow{d} \Omega_{\mathbb{P}^1}) \to \mathbb{H}^i$ *is an isomorphism for* $i = 0, 2$*.*

Proof. The Riemann-Roch Theorem implies $\dim \mathbb{H}^0 + \dim \mathbb{H}^2 - \dim \mathbb{H}^1 = \deg(L_{12}) - \deg(\widetilde{L}_{12} \otimes \Omega_{\mathbb{P}^1}(x_1 + \cdots + x_4)) = 0$. Since $\mathbb{H}^0 = \{A \in \mathrm{Hom}(L_1, L_2) : \nabla A = A\nabla\}$, $\mathbb{H}^0$ has the required dimension. On the other hand, by Serre's duality,

$$(\mathbb{H}^2)^* = \mathbb{H}^0(\mathbb{P}^1, (\widetilde{L}_{12}(x_1 + \cdots + x_4))^* \xrightarrow{\nabla^*} L_{12}^* \otimes \Omega_{\mathbb{P}^1}).$$

The $SL(2)$-structure on L_1 and L_2 yields a canonical paring $L_{12} \times L_{12} \to O_{\mathbb{P}^1}$. Since the pairing agrees with ∇, $((\widetilde{L}_{12}(x_1 + \cdots + x_4))^* \xrightarrow{\nabla^*} L_{12}^* \otimes \Omega_{\mathbb{P}^1})$ is naturally a subcomplex of $(L_{12}\xrightarrow{\nabla}\widetilde{L}_{12}(x_1+\dots+x_4)\otimes\Omega_{\mathbb{P}^1})$, and $(\mathbb{H}^2)^*$ is naturally a subspace of $\mathbb{H}^0$. In particular, $\dim\mathbb{H}^2\leq\dim\mathbb{H}^0$. This proves (i).

Note that $(O_{\mathbb{P}^1}\xrightarrow{d}\Omega_{\mathbb{P}^1})$ is a direct summand in $(L_{12}\xrightarrow{\nabla}\widetilde{L}_{12}\otimes\Omega_{\mathbb{P}^1}(x_1+\dots+x_4))$ if $L_1=L_2$. This implies (iii). (ii) follows from (iii). □

The map $(\mathbb{H}^2)^*\to\mathbb{H}^0$ constructed above still makes sense for S-points over an arbitrary scheme S. Since we are going to need the result, let us give a precise statement.

Lemma 14. *Let S be a locally Noetherian scheme, $i: S\to\mathcal{M}\times\mathcal{M}$. Set $\mathcal{F}_{(S)}:=Rp_{1,*}(\mathbb{DR}((i\times id_P)^*\mathcal{F}_{\mathcal{M}}))$ (here $p_1: S\times P\to S$). Then $\mathcal{H}om(H^2(\mathcal{F}_{(S)}),O_S)$ is isomorphic to a subsheaf of $H^0(\mathcal{F}_{(S)})$.*

Proof. The proof repeats that of Proposition 10, the only difference is that Serre's duality should be replaced by an appropriate "relative" result, e.g., [11, Theorem 21] or [9, Theorem III.5.1] (or [9, Corollary VII.3.4(c)] for a general statement). □

Remark. Actually, it is easy to see that $\mathcal{H}om(H^2(\mathcal{F}_{(S)}),O_S)=H^0(\mathcal{F}_{(S)})$.

Clearly, diag : $\mathcal{M}\to\mathcal{M}\times\mathcal{M}$ is a μ_2-torsor over diag$(\mathcal{M})$. Denote by Hom the line bundle (i.e., a $\mathbf{G}_m$-torsor) on diag$(\mathcal{M})$ obtained by applying $\mu_2\hookrightarrow\mathbf{G}_m$ to this torsor. Note that the fiber of Hom over $((L_1,\nabla_1,\varphi_1),(L_2,\nabla_2,\varphi_2))$ equals $\{A\in\mathrm{Hom}_{O_{\mathbb{P}^1}}(L_1,L_2): A\nabla_1=\nabla_2A\}$.

Proposition 10 still holds for all (not necessarily closed) points of $\mathcal{M}\times\mathcal{M}$ (that is, L_1 and L_2 can be Spec K-families of λ-bundles for a field K). The following corollary is obvious (p_{12} stands for the projection $\mathcal{M}\times\mathcal{M}\times P\to\mathcal{M}\times\mathcal{M}$).

Corollary 4.

(i) *$Rp_{12,*}\,\mathbb{DR}(\mathcal{F}_{\mathcal{M}})$ vanishes if restricted to $\mathcal{M}\times\mathcal{M}\setminus\mathrm{diag}(\mathcal{M})$.*
(ii) *The map $(p_{12}^*\,\mathrm{Hom})\to\mathcal{F}_{\mathcal{M}}|_{\mathrm{diag}(\mathcal{M})\times P}$ induces an isomorphism* $\mathrm{Hom}=\mathrm{Hom}\otimes\mathbb{H}^2(\mathbb{P}^1,(O_{\mathbb{P}^1}\to\Omega_{\mathbb{P}^1}))\to R^2p_{12,*}(\mathbb{DR}(\mathcal{F}_{\mathcal{M}})|_{\mathrm{diag}(\mathcal{M})})$. □

Now let us prove Theorem 4. The proof is based on the following observation (cf. [16, Lemma in §13])

Lemma 15. *Let Z be a locally Noetherian scheme, $V\subset Z$ a closed subscheme that is locally a complete intersection of pure codimension n. Denote by $i: V\hookrightarrow Z$ and $j: Z\setminus V\hookrightarrow Z$ the natural embeddings.*

(i) *Let F be a quasicoherent sheaf on Z such that $F|_{Z\setminus V}=0$, $L_ni^*F=0$. Then $F=0$.*
(ii) *Let $F^\bullet=(F^0\to F^1\to\dots)$ be a complex of flat O_Z-modules such that $H^i(F^\bullet)|_{Z\setminus V}=0$ for all $i<n$. Then $H^i(F^\bullet)=0$ for $i<n$.*

Proof. (i) Using the Koszul resolution, one easily sees that ([9, Corollary III.7.3])

$$L_n i^* F = \mathcal{H}om_{O_Z}(\omega_{V/Z}^{-1}, F).$$

Here $\omega_{V/Z}^{-1}$ is the determinant of the normal bundle of $V \subset Z$. On the other hand, the kernel of the natural map $F \to (R^0 j_*)j^*(F)$ is

$$F^{(V_\infty)} := \bigcup_{k=1}^{\infty} \{f \in F : O_Z(-kV)f = 0\}.$$

Since $\{f \in F : O_Z(-V)f = 0\} = 0$, we have $F^{(V_\infty)} = 0$, so $F \to (R^0 j_*)j^*(F)$ is an injection.

(ii) Consider the spectral sequence

$$E_2^{pq} = L_{-p} i^* H^q(F^\bullet) \Rightarrow H^{p+q}(i^* F^\bullet)$$

(note that $\mathcal{F}^\bullet$ is a complex of flat modules, so $Li^* F^\bullet = i^* F^\bullet$). Clearly $H^k(i^* F^\bullet) = 0$ for $k < 0$. The spectral sequence implies $L_n i^* H^0(F^\bullet) = 0$, so by (i), $H^0(F^\bullet) = 0$. In a similar way, now we see $L_n i^* H^1(F^\bullet) = 0$, and so on. □

Lemma 15 (ii) and Corollary 4 (i) imply $R^i p_{12,*} \mathbb{DR}(\mathcal{F}_\mathcal{M}) = 0$ for $i \neq 2$. Set $\mathcal{F}^{(2)} := R^2 p_{12,*} \mathbb{DR}(\mathcal{F}_\mathcal{M})$. Corollary 4 (ii) implies Hom $= \mathcal{F}^{(2)}|_{\text{diag}(\mathcal{M})}$. To complete the proof, it is enough to check $\mathcal{F}^{(2)}$ is concentrated on diag$(\mathcal{M})$, which follows from Lemma 14.

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