MODULI SPACES OF D-CONNECTIONS AND DIFFERENCE PAINLEVÉ EQUATIONS

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Abstract

We show that difference Painlevé equations can be interpreted as isomorphisms of moduli spaces of difference connections (d-connections) *on* $\mathbb{P}^1$ *with given singularity structure. In particular, we derive a difference equation that lifts to an isomorphism between* $A_2^{(1)*}$*-surfaces in Sakai's classification (see [29]); it degenerates to both difference Painlevé V and classical (differential) Painlevé VI equations. This difference equation has been known before under the name of* asymmetric discrete Painlevé IV equation.

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1. Introduction

This article is about difference Painlevé equations and their geometric properties. The term *discrete* (*difference*, *q-difference*, or *elliptic*) *Painlevé equation* is rather vague; there exist different ways of discretizing the classical (second-order differential) Painlevé equations (see, e.g., [13], [19], [22], [23], [29]). We consider the equations that fit into Sakai's classification described in [29].

By definition, any equation of Sakai's hierarchy originates from a birational automorphism of $\mathbb{C}^2$ which lifts to a regular isomorphism between two blowups of $\mathbb{P}^2$

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at nine points. This geometric property allows us to classify the equations according to the type of the resulting surface. The hierarchy also includes the classical Painlevé equations for which the surfaces are viewed as spaces of initial conditions (see [24], [25]).

From 2001 to 2004, several researchers have computed the so-called gap probabilities in various discrete probabilistic models of random-matrix type (see [1], [4], [5], [7], [8] – [12]). Surprisingly, these quantities were often expressible in terms of certain specific solutions of equations from Sakai's hierarchy. Later, it was demonstrated that the equations arising in probabilistic models can be viewed as reductions of isomonodromy transformations of systems of linear difference equations with rational coefficients (see [6]). Further discussion of monodromy for difference equations can be found in [20].

The goal of this article is twofold. First, we show how the geometric approach to isomonodromy transformations implies that the transformations lift to isomorphisms between suitable surfaces. This provides a conceptual explanation of the above coincidence. The surfaces are geometrically interpreted as suitable moduli spaces of *d-connections* (short for difference connections) on the Riemann sphere. Second, we derive an equation of Sakai's hierarchy which lifts to an isomorphism between $A_2^{(1)*}$-surfaces in Sakai's classification (see [29]). We call this equation the *difference Painlevé VI*, or dPVI.

Let us briefly describe our results.

Consider a matrix linear difference equation

$$y(z+1) = A(z)y(z), \quad A(z) = A_0 z^n + \cdots + A_{n-1} z + A_n, \; A_i \in \operatorname{Mat}(m, \mathbb{C}). \tag{1.1}$$

We always assume that A_0 is invertible. According to [6], isomonodromy transformations of this equation consist of maps of the form

$$A(z) \mapsto A'(z) = R(z+1)A(z)R(z)^{-1} \tag{1.2}$$

for suitable rational matrix-valued functions $R(z)$. For generic $A(z)$, these transformations are parameterized by integral shifts of the zeros of $A(z)$ and of certain exponents at $z = \infty$ with total sum of shifts equal to zero (see [6, Theorem 2.1]). We can then express the matrix elements of $A'(z)$ as functions of the matrix elements of $A(z)$; in special cases, the expressions give rise to the difference Painlevé equations.

However, the isomonodromy transformation is defined only when $A(z)$ is generic enough. Therefore, the resulting maps are rational rather than regular; that is, the formulas for matrix elements of $A'(z)$ have singularities. In order to resolve these singularities, it is convenient to use the geometric approach.

Let $\mathscr{L}$ be a vector bundle on $\mathbb{P}^1$ of rank m. Assume that we are given a d-connection on $\mathscr{L}$ which is, by definition, a linear operator $\mathscr{A}(z) : \mathscr{L}_z \to \mathscr{L}_{z+1}$ which depends on z polynomially. (Here $\mathscr{L}_z$ is the fiber of $\mathscr{L}$ over z.) If $\mathscr{L}$ is the trivial vector bundle, $\mathscr{A}(z)$ is a matrix difference equation (see (1.1)).

There is a natural operation on vector bundles with d-connection called *modification*; it is induced by a rational isomorphism $\mathscr{R} : \mathscr{L} \dashrightarrow \mathscr{L}'$ between two vector bundles. A d-connection $\mathscr{A}$ on $\mathscr{L}$ then induces a d-connection $\mathscr{A}'$ on $\mathscr{L}'$, and vice versa. Isomonodromy transformations can be viewed as special cases of such modifications.

Let us consider the example that leads to the difference Painlevé V equation (dPV). Take $m =$ (rank of $\mathscr{L}$) $= 2$; assume that $\mathscr{A}(z)$ has four simple zeros $a_1, a_2, a_3, a_4 \in \mathbb{C}$; and assume that there exists a trivialization of $\mathscr{L}$ in a neighborhood of $z = \infty$ with respect to which the matrix of $\mathscr{A}(z)$ has the form

$$A(z) = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} z^2 + \begin{bmatrix} \rho_1 d_1 & 0 \\ 0 & \rho_2 d_2 \end{bmatrix} z + O(1), \quad z \to \infty.$$

PROPOSITION (Isomonodromy transformation)
Under certain nondegeneracy conditions on the parameters $(a_1, \ldots, a_4, \rho_1, \rho_2, d_1, d_2)$, for any vector bundle $\mathscr{L}$ with d-connection $\mathscr{A}$ and any integral shifts of the parameters $a_1, \ldots, a_4, d_1, d_2$, there exists a unique vector bundle $\mathscr{L}'$ with d-connection $\mathscr{A}'$ related to $(\mathscr{L}, \mathscr{A})$ by a modification and such that it satisfies the above assumptions with shifted values of parameters.

Note that we do not need to assume that $(\mathscr{L}, \mathscr{A})$ is generic. This means that the modifications of this proposition give (regular, not birational) isomorphisms of the moduli spaces of vector bundles with d-connections with given singularity structure, provided that the parameters are generic.

From now on, let us also assume that

$$\deg(\mathscr{L}) = -(a_1 + \cdots + a_4 + d_1 + d_2) = -1.$$

This condition implies that $\mathscr{L}$ is always isomorphic to $\mathscr{O} \oplus \mathscr{O}(-1)$. (Notice that an isomonodromy transformation fixes $\deg(\mathscr{L})$ if and only if the corresponding shifts of the parameters $a_1, \ldots, a_4, d_1, d_2$ add up to zero.) By a choice of basis in $\mathscr{L}$, the moduli space of d-connections can be identified with equivalence classes of (2×2)-matrices A with polynomial entries satisfying

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \deg a_{11} \leq 2,\ \deg a_{22} \leq 2,\ \deg a_{21} \leq 1,\ \deg a_{12} \leq 3,$$

$$\det A(z) = \rho_1 \rho_2 (z - a_1)(z - a_2)(z - a_3)(z - a_4),$$

$$a_{11} + a_{22}(1 + z^{-1}) = (\rho_1 + \rho_2) z^2 + (d_1 \rho_1 + d_2 \rho_2) z + O(1),$$

modulo the gauge transformations of the form (1.2) with polynomial

$$R = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}, \quad r_{11} = \text{const}, \ r_{22} = \text{const}, \ \deg r_{12} \le 1.$$

It is not hard to see that this moduli space is two-dimensional. We show that its smallest compactification is a surface of the Sakai-type $D_4^{(1)}$; in particular, it is a blowup of $\mathbb{P}^2$ at nine points. (We use a different description as a blowup of $\mathbb{P}^1 \times \mathbb{P}^1$.) The moduli space itself is the complement of five curves (the support of the unique effective anticanonical divisor) inside this surface.

In order to connect this picture to dPV, we introduce coordinates on the moduli spaces.

THEOREM (dPV)
Take the zero of the linear polynomial a_{21} as the first coordinate, and denote it by q; take the value of the matrix element a_{11} at q divided by $(q-a_3)(q-a_4)$ as the second coordinate, and denote it by p. Consider the modification of $(\mathscr{L}, \mathscr{A})$ to $(\mathscr{L}', \mathscr{A}')$ which shifts

$$a_1 \mapsto a_1 - 1, \qquad a_2 \mapsto a_2 - 1, \qquad d_1 \mapsto d_1 + 1, \qquad d_2 \mapsto d_2 + 1.$$

Then the coordinates (p', q') on the moduli space of $(\mathscr{L}', \mathscr{A}')$ are related to (p, q) by

$$\begin{cases} q' + q = a_3 + a_4 + \dfrac{\rho_1(d_1 + a_3 + a_4)}{p - \rho_1} + \dfrac{\rho_2(d_2 + a_3 + a_4)}{p - \rho_2}, \\ p' p = \dfrac{(q' - a_1 + 1)(q' - a_2 + 1)}{(q' - a_3)(q' - a_4)} \cdot \rho_1 \rho_2. \end{cases}$$

This is exactly the dPV equation of [14] and [29].*

Remark. The idea of using (q, p) as coordinates on the moduli space is by no means new. For Painlevé equations, it has been used, for example, in [18], in the continuous situation, and in [19], in the discrete situation.

Another example that we consider in detail deals with rank 2 vector bundles $\mathscr{L}$ with d-connection $\mathscr{A}(z)$ which has six simple zeros $a_1, \dots, a_6 \in \mathbb{C}$ and whose behavior near $z = \infty$ in a suitable trivialization is given by

$$A(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} z^3 + \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} z^2 + O(z).$$

*A different reduction of an isomonodromy transformation to dPV can be found in [5].

Quite similarly to the case of dPV, there is an action of $\mathbb{Z}^8$ which is parametrized by integral shifts of a_i's and d_j's. The group acts by isomorphisms of moduli spaces. Let us again assume that $\deg(\mathscr{L}) = -(a_1 + \cdots + a_6 + d_1 + d_2) = -1$. Then the corresponding moduli spaces can be identified with equivalence classes of (2×2)-polynomial matrices A satisfying

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \deg a_{11} \leq 3,\ \deg a_{22} \leq 3,\ \deg a_{21} \leq 1,\ \deg a_{12} \leq 3,$$

$$\det A(z) = (z - a_1)(z - a_2)(z - a_3)(z - a_4)(z - a_5)(z - a_6),$$

$$(a_{11} - z^3)\bigl(a_{22}(1 + z^{-1}) - z^3\bigr) - a_{12}a_{21} = d_1 d_2 z^4 + O(z^3),$$

modulo the same gauge transformations as in the case of dPV.

Once again, we show that such a moduli space is two-dimensional and that its smallest compactification can be identified with $\mathbb{P}^2$ blown up at nine points. The corresponding surface has type $A_2^{(1)*}$, in Sakai's notation, and the moduli space is the complement of three curves (the support of the effective anticanonical divisor) in this surface.

Similarly to the case of dPV, in order to get explicit equations, we need to introduce coordinates on the moduli spaces.

THEOREM (dPVI)

Take the zero of the matrix element a_{21} as the first coordinate, and denote it by q; take the value of the matrix element a_{11} at q divided by $(q - a_4)(q - a_5)(q - a_6)$ as the second coordinate, and denote it by p. Consider the modification of $\mathscr{L}$ to $\mathscr{L}'$ which shifts

$$a_1 \mapsto a_1 - 1, \qquad a_2 \mapsto a_2 - 1, \qquad d_1 \mapsto d_1 + 1, \qquad d_2 \mapsto d_2 + 1. \tag{1.3}$$

Then the coordinates (p', q') on the moduli space of $\mathscr{L}'$ are related to (p, q) by

$$\begin{cases} q' = (p-1)(q+1-a_1-a_2) + pa_3 + \displaystyle\sum_{j=1,2} \frac{c_j p}{q - ((p(1-a_1-a_2-d_j) - a_3)/(p-1))}, \\ p' \cdot p = \dfrac{(q'-a_1+1)(q'-a_2+1)}{(q'-a_4)(q'-a_5)(q'-a_6)} \cdot \bigl((p-1)(q'-q) + q' - a_3\bigr), \end{cases}$$

where

$$c_j = \frac{(d_j + a_1 + a_2 + a_4 - 1)(d_j + a_1 + a_2 + a_5 - 1)(d_j + a_1 + a_2 + a_6 - 1)}{(d_j - d_{3-j})}.$$

We call the relations above the *difference Painlevé VI equation*.

Remark. The difference Painlevé VI equation is equivalent to the asymmetric dPIV equation of [15] (see also earlier references therein). Indeed, introducing the new variable r instead of p via $p = (q - a_3)/(q + r)$, we can rewrite our relations as

$$\begin{cases} (q+r)(q'+r) = \dfrac{(r+a_3)(r+a_4)(r+a_5)(r+a_6)}{(r+1-a_1-a_2-d_1)(r+1-a_1-a_2-d_2)}, \\ (q'+r)(q'+r') = \dfrac{(q'-a_3)(q'-a_4)(q'-a_5)(q'-a_6)}{(q'-(a_1-1))(q'-(a_2-1))}, \end{cases}$$

which, up to a change of notation, coincides with [15, (1.3)]. We are very grateful to the referee for pointing this out.

The reason we prefer seemingly more complicated expressions in the theorem is that the coordinates have a clear geometric meaning. This also simplifies various degenerations to other Painlevé equations.

It should be noted that formulas for all isomorphisms of Sakai surfaces in principle can be written using coordinates of [29]. The computation, however, can be rather tedious.

There are simple degenerations that turn dPVI into dPV and the classical PVI equations. In a sense, this can be done simultaneously. Let us consider, in addition to the flow given by the shift (1.3), the flow generated by the shift

$$a_3 \mapsto a_3 - 1, \qquad a_4 \mapsto a_4 - 1, \qquad d_1 \mapsto d_1 + 1, \qquad d_2 \mapsto d_2 + 1. \tag{1.4}$$

Clearly, the flow given by the shift (1.4) is also described by dPVI with a slightly different p-coordinate. Now let a_1, a_2, d_1, and d_2 go to infinity at speeds $-\rho_1$, $-\rho_2$, ρ_1, and ρ_2, respectively. In the limit, the dPVI equation corresponding to (1.3) converges to a continuous vector field that is equivalent to the classical PVI.* At the same time, the flow corresponding to (1.4) converges to a discrete flow described by dPV. As the result, we get two commuting flows on the same surface (of the Sakai-type $D_4^{(1)}$), a vector field given by dPVI and discrete dynamics given by dPV.

This limiting picture can be seen from two points of view. First, the classical PVI possesses the so-called Bäcklund transformations, which can be described via dPV; see [11]. Second, there is a natural continuous isomonodromy deformation that moves the parameters ρ_1, ρ_2 in the dPV setting; it can be reduced to the classical PVI. Finally, the geometric Mellin transform (a version of the Fourier transform) relates the two approaches. These interrelations (except for the Bäcklund transformations) are discussed in detail in the body of the article.

The article is organized as follows. In Section 1 we state our main results. In Section 2 we study general properties of d-connections and discuss various operations

*The classical PVI was also obtained as a limit of other discrete Painlevé equations in [19] and [26].

on them. Section 3 is dedicated to dPV and the corresponding moduli space. In Section 4 we describe the relations between dPV and PVI. Finally, in Section 5 we deal with dPVI, the associated moduli space, and degenerations of dPVI to dPV and PVI.

1.1. Notation

In this article the ground field is $\mathbb{C}$, so "variety" means "variety over $\mathbb{C}$," "$\mathbb{P}^1$" means "$\mathbb{P}^1_{\mathbb{C}}$," and so on. The coordinate on the projective line $\mathbb{P}^1$ is denoted by z. For a vector bundle $\mathscr{L}$ on $\mathbb{P}^1$, the fiber of $\mathscr{L}$ over $z \in \mathbb{P}^1$ is denoted by $\mathscr{L}_z$ and the space of global sections of $\mathscr{L}$ is denoted by $\Gamma(\mathbb{P}^1, \mathscr{L})$. $\mathscr{O}(k)$ stands for the line bundle (vector bundle of rank 1) on $\mathbb{P}^1$ whose sections are functions on $\mathbb{P}^1$ with a pole of order at most k (or zero of order at least $-k$, if $k < 0$) at $\infty \in \mathbb{P}^1$.

The diagonal $(m \times m)$-matrix with entries $\alpha_1, \dots, \alpha_m$ is denoted by $\operatorname{diag}(\alpha_1, \dots, \alpha_m)$.

2. Main results

2.1. d-connections and their moduli

Let $\mathscr{L}$ be a vector bundle on $\mathbb{P}^1$ of rank m.

Definition 2.1

A (rational) d-connection on $\mathscr{L}$ is a linear operator

$$\mathscr{A}(z) : \mathscr{L}_z \to \mathscr{L}_{z+1}$$

which depends on a point $z \in \mathbb{P}^1 - \{\infty\}$ in a rational way (in particular, $\mathscr{A}(z)$ is defined for all $z \in \mathbb{C}$ outside of a finite set); here $\mathscr{L}_z$ is the fiber of $\mathscr{L}$ over $z \in \mathbb{P}^1$. In other words, $\mathscr{A}$ is a rational map between the vector bundle $\mathscr{L}$ and its pullback $s^*(\mathscr{L})$ via the automorphism $s : \mathbb{P}^1 \to \mathbb{P}^1 : z \mapsto z + 1$.

Remark 2.2

Essentially, a d-connection is a system of (rational) linear difference equations $y(z+1) = \mathscr{A}(z)y(z)$ on a section $y(z)$ of the vector bundle $\mathscr{L}$. Notice that any vector bundle $\mathscr{L}$ is trivial when restricted to $\mathbb{A}^1 = \mathbb{P}^1 - \{\infty\}$. If we pick a trivialization $\mathscr{S}(z) : \mathbb{C}^m \xrightarrow{\sim} \mathscr{L}_z$, $z \in \mathbb{A}^1$, of this restriction (a *basis of* $\mathscr{L}$), $\mathscr{A}$ can be written in coordinates as the matrix-valued function $A(z) = \mathscr{S}(z+1)^{-1}\mathscr{A}(z)\mathscr{S}(z)$ (the *matrix of the d-connection*). For two trivializations $\mathscr{S}_i(z) : \mathbb{C}^m \xrightarrow{\sim} \mathscr{L}_z$ $(i = 1, 2)$, the corresponding matrices $A_i = \mathscr{S}_i(z+1)^{-1}\mathscr{A}(z)\mathscr{S}_i(z)$ differ by a *d-gauge transformation,*

$$A_2(z) = R(z+1)^{-1} A_1(z) R(z),$$

for the *d-gauge matrix* $R := \mathscr{S}_1^{-1}\mathscr{S}_2$. Thus, classification of d-connections is equivalent to the classification of their matrices up to the d-gauge transformation.

We work with d-connections that have simple zeros on $\mathbb{A}^1$ and whose behavior at infinity is simple in the sense of the following definition.

Definition 2.3
Let $\mathscr{L}$ be a rank 2 vector bundle on $\mathbb{P}^1$, and let $\mathscr{A}(z)$ be a d-connection on $\mathscr{L}$. Suppose that $\mathscr{A}(z)$ satisfies the following conditions.

(1) The only zeros and poles of $\mathscr{A}(z)$ are a pole of order n at infinity and simple zeros at k distinct points $a_1, \dots, a_k \in \mathbb{A}^1$. Here we say that a_i is a simple zero of $\mathscr{A}(z)$ if, at a_i, $\mathscr{A}(z)$ is regular and $\det(\mathscr{A}(z))$ has zero of order 1.

(2) On the formal neighborhood of $\infty \in \mathbb{P}^1$, there exists a trivialization $\mathscr{R}(z) : \mathbb{C}^2 \to \mathscr{L}_z$ ($\mathscr{R}(z)$ is essentially a matrix-valued Taylor series in z^{-1}) such that the matrix of $\mathscr{A}$ with respect to $\mathscr{R}$ satisfies

$$\mathscr{R}(z+1)^{-1}\mathscr{A}(z)\mathscr{R}(z) = \begin{bmatrix} \rho_1(z^n + d_1 z^{n-1}) & 0 \\ 0 & \rho_2(z^n + d_2 z^{n-1}) \end{bmatrix} \tag{2.1}$$

for some numbers $\rho_1, \rho_2, d_1, d_2 \in \mathbb{C}$.

We call such a d-connection $\mathscr{A}(z)$ (or, more precisely, we call the pair $(\mathscr{L}, \mathscr{A})$) a *d-connection of type* $\theta = (a_1, \dots, a_k; \rho_1, \rho_2, d_1, d_2; n)$.

Remark 2.4
One can also consider d-connections that have simple poles besides simple zeros. As it turns out, addition of poles does not lead to a significantly different object; in Section 3.2, we discuss an operation (*multiplication by a scalar*) that turns a pole of a d-connection into a zero, and vice versa.

Remark 2.5
The second condition of Definition 2.3 might seem unnatural; however, Corollary 3.4 shows that a generic d-connection satisfies it. See also Remark 3.2 for a reformulation of this condition in terms of formal solutions to a difference equation.

Denote by M_θ the moduli space of d-connections of type θ. One can think of M_θ in several different ways: as a set (the set of isomorphism classes of connections of given type), a category (the category of such connections), a scheme (the corresponding coarse moduli space), or an algebraic stack (the fine moduli stack). In this article we work with the coarse moduli space, although some results also hold for other incarnations of M_θ. (Note that we need to impose some conditions on θ to make sure that the coarse moduli space of d-connections of type θ is a scheme.)

It is easy to see (see Corollary 3.11) that M_θ is empty unless

$$k = 2n, \tag{2.2}$$

$$\deg(\theta) := -d_1 - d_2 - \sum_{i=1}^{k} a_i \text{ is an integer.} \tag{2.3}$$

Let us also consider the following nondegeneracy assumptions on θ:

$$-d_j - \sum_{i\in I} a_i \notin \mathbb{Z} \quad \text{for any } I \subset \{1, \dots, k\},\ j = 1, 2, \tag{2.4}$$

$$a_i - a_j \notin \mathbb{Z} \quad \text{for any } i \neq j, \tag{2.5}$$

$$\rho_1, \rho_2 \neq 0, \qquad \rho_1 \neq \rho_2. \tag{2.6}$$

Let Θ_{2n} be the set of all collections $\theta = (a_1, \dots, a_{2n}; \rho_1, \rho_2, d_1, d_2; n)$, and let $\Theta_{2n}^\sharp \subset \Theta_{2n}$ be the set of θ's which satisfy (2.2)–(2.6). Set $\Theta = \bigsqcup_n \Theta_{2n}$, $\Theta^\sharp = \bigsqcup_n \Theta_{2n}^\sharp$.

Remark 2.6
Informally speaking, we impose conditions (2.4)–(2.6) for the following reasons: (2.5) and (2.6) simplify modifications of d-connections (see Section 3.2), while (2.4) implies that d-connections of type θ are irreducible (see Lemma 3.12). Irreducibility can be used to prove that the moduli space M_θ is nice; for example, one can show (using the same ideas as in [3]) that M_θ is a smooth variety of dimension $2n - 2$ for any $\theta \in \Theta_{2n}^\sharp$.

2.2. *Difference PV*

We want to study the moduli space M_θ for $\theta \in \Theta_{2n}^\sharp$. As Remark 2.6 shows, the first interesting case is when $2n = 4$; then M_θ is a smooth algebraic surface. We also assume that $\deg(\theta) = -1$. (The degree is defined in (2.3).)

Remark 2.7
The assumption on degree is not too restrictive; using modifications of d-connections (described in Section 3.2), we can construct for any θ an isomorphism $M_\theta \xrightarrow{\sim} M_{\theta'}$, where $\deg(\theta') = -1$.

We describe the surface M_θ by introducing coordinates $(q, p) \in (\mathbb{P}^1)^2$; more precisely, M_θ is described as an open subset in a blowup of $(\mathbb{P}^1)^2$. The construction imitates the description of the moduli space of connections (see [3], [17]), which goes back to Okamoto [24], [25].

THEOREM A
Suppose that

$$\theta = (a_1, a_2, a_3, a_4; \rho_1, \rho_2, d_1, d_2; 2) \in \Theta_4^\sharp$$

has $\deg(\theta) = -1$. *Let* $\sigma_1 : K_1 \to (\mathbb{P}^1)^2$ *be the blowup of* $(\mathbb{P}^1)^2$ *at the following six points:* $(q, p) = (a_1, 0)$, $(a_2, 0)$, (a_3, ∞), (a_4, ∞), (∞, ρ_1), *and* (∞, ρ_2). *(Here* q *and* p *are the projections* $(\mathbb{P}^1)^2 \to \mathbb{P}^1$.*) Consider the two exceptional curves* $E_j = \sigma_1^{-1}(\infty, \rho_j) \subset K_1$, $j = 1, 2$*; homogeneous coordinates on* E_j *are given by* $(1/q : p - \rho_j)$. *Let* $\sigma_2 : K_2 \to K_1$ *be the blowup of* K_1 *at the two points* $(1/q : p - \rho_j) = (1 : \rho_j(d_j + a_3 + a_4))$, $j = 1, 2$ *(one point on each exceptional curve).*

(1) *There exists an open embedding* $P_2 : M_\theta \hookrightarrow K_2$.
(2) *The complement to* $P_2(M_\theta)$ *in* K_2 *is the union of the proper preimages of the curves* $\mathbb{P}^1 \times \{0\}$, $\mathbb{P}^1 \times \{\infty\}$, $\{\infty\} \times \mathbb{P}^1 \subset (\mathbb{P}^1)^2$ *and the two exceptional curves* $E_j \subset K_1$, $j = 1, 2$.

Remark 2.8
K_2 is the smallest smooth compactification of M_θ (see [3, Corollary 5]); any open embedding $M_\theta \hookrightarrow \overline{M}$ with smooth projective $\overline{M}$ induces a regular morphism $\overline{M} \to K_2$. Note also that $(K_2, K_2 - M_\theta)$ is an Okamoto-Painlevé pair (of type $\tilde{D}_4$), in the sense of [27] and [28]; in particular, K_2 is a surface of the Sakai-type $D_4^{(1)}$.

In particular, the composition $P : M_\theta \hookrightarrow K_2 \to (\mathbb{P}^1)^2$ is birational. Therefore, one can view the components of P as a kind of rational coordinates on M_θ. We denote the components by q and p, so that $P = (q, p)$.

The natural operations on d-connections (modifications and multiplications by scalars) define isomorphisms between the spaces M_θ for different θ (see Section 3.2). Our next result describes such an isomorphism for one of the simplest modifications of d-connections. The description can be viewed as a nonlinear difference equation in coordinates (p, q) (the difference PV).

As before, suppose that

$$\theta = (a_1, a_2, a_3, a_4; \rho_1, \rho_2, d_1, d_2; 2) \in \Theta_4^\sharp$$

has $\deg(\theta) = -1$. Set

$$\theta' = (a_1 - 1, a_2 - 1, a_3, a_4; \rho_1, \rho_2, d_1 + 1, d_2 + 1; 2) \in \Theta_4^\sharp.$$

Modification of d-connections defines an isomorphism $\mathrm{dPV} : M_\theta \to M_{\theta'}$. Explicitly, for every $(\mathscr{L}, \mathscr{A}) \in M_\theta$, the image $\mathrm{dPV}(\mathscr{L}, \mathscr{A}) = (\mathscr{L}', \mathscr{A}')$ is the only d-connection of type θ' which admits a rational isomorphism $\mathscr{R} : \mathscr{L}' \dashrightarrow \mathscr{L}$ which agrees with the d-connections: $\mathscr{R}(z+1)\mathscr{A}'(z) = \mathscr{A}(z)\mathscr{R}(z)$.

THEOREM B

Set $p' := p \circ \mathrm{dPV}$, $q' := q \circ \mathrm{dPV} : M_\theta \to \mathbb{P}^1$. *Then*

$$\begin{cases} q' + q = a_3 + a_4 + \dfrac{\rho_1(d_1 + a_3 + a_4)}{p - \rho_1} + \dfrac{\rho_2(d_2 + a_3 + a_4)}{p - \rho_2}, \\ p' \cdot p = \dfrac{(q' - a_1 + 1)(q' - a_2 + 1)}{(q' - a_3)(q' - a_4)} \cdot \rho_1 \rho_2. \end{cases} \tag{2.7}$$

2.3. *Difference* PV *and classical* PVI

As we mentioned above, d-connections and ordinary connections have many common properties. Let us consider the following class of (ordinary) connections.

Denote by $\Lambda \subset \mathbb{C}^8$ the set of all collections $\lambda = (\lambda_1^-, \lambda_1^+, \dots, \lambda_4^-, \lambda_4^+)$ such that

$$\sum_{i=1}^{4} (\lambda_i^- + \lambda_i^+) \in \mathbb{Z}, \qquad \lambda_i^+ - \lambda_i^- \notin \mathbb{Z}, \qquad \sum_{i=1}^{4} \lambda_i^{\epsilon_i} \notin \mathbb{Z}$$

for any choice of upper indexes $\epsilon_i \in \{+, -\}$. Let $X \subset (\mathbb{P}^1)^4$ be the set of all collections $x = (x_1, \dots, x_4)$ of four distinct points of $\mathbb{P}^1$:

$$X := \{(x_1, \dots, x_4) \,|\, x_i \neq x_j \text{ for } i \neq j\} \subset (\mathbb{P}^1)^4.$$

Definition 2.9

Suppose that $(x, \lambda) \in X \times \Lambda$. A *connection of type* (x, λ) is a pair $(\mathscr{L}, \nabla)$ such that $\mathscr{L}$ is a rank 2 vector bundle on $\mathbb{P}^1$, $\nabla : \mathscr{L} \to \mathscr{L} \otimes \Omega_{\mathbb{P}^1}(x_1 + \cdots + x_4)$ is a connection with simple poles at x_i's, and the residue of ∇ at x_i has eigenvalues $\{\lambda_i^-, \lambda_i^+\}$.

For $(x, \lambda) \in X \times \Lambda$, we denote the coarse moduli space of connections of type (x, λ) by $M_{(x,\lambda)}$. It can be thought of as the space of initial conditions of the Painlevé equation PVI. The space $M_{(x,\lambda)}$ has a geometric description that goes back to K. Okamoto [24], [25]; we recall the description in Proposition 5.1. It is easy to see from the description that M_θ and $M_{(x,\lambda)}$ are isomorphic for a suitable choice of parameters. (They are both surfaces of type $D_4^{(1)}$.)

THEOREM C

Suppose that

$$\theta = (a_1, a_2, a_3, a_4; \rho_1, \rho_2, d_1, d_2; 2) \in \Theta_4^\sharp$$

has $\deg(\theta) = -1$. *Set*

$$x = (x_1, x_2, x_3, x_4) := (0, \rho_1, \rho_2, \infty) \in X,$$

$$\lambda = (\lambda_1^-, \lambda_1^+, \dots, \lambda_4^-, \lambda_4^+) := (a_1, a_2, 0, d_1+a_3+a_4, 0, d_2+a_3+a_4, -a_3, -a_4) \in \Lambda.$$

Then $M_\theta \simeq M_{(x,\lambda)}$.

Remark 2.10
Theorem C can be proved by direct calculations, but it can also be explained in terms of moduli spaces. In Section 5.6 we describe a one-to-one correspondence between d-connections of type θ and connections of type (x, λ). Up to small twists, the correspondence is the geometric Mellin transform of [21]; it is constructed using de Rham cohomology and equivariant cohomology groups. The Mellin transform is a particular case of the duality for generalized one-motives (also defined in [21]).

Now let us fix $\lambda \in \Lambda$ and consider surfaces $M_{(x,\lambda)}$ for all $x \in X$. They can be viewed as fibers of an algebraic family $M_\lambda \to X$. The sixth Painlevé equation PVI is an algebraic connection on this family; the (analytic) integral curves of PVI correspond to isomonodromy deformation of connections.

By Theorem C, the sixth Painlevé equation PVI induces a connection on a family of moduli spaces of d-connections. It turns out that this connection can be defined for arbitrary $\theta \in \Theta_{2n}^\sharp$ (not necessarily when $2n = 4$). More precisely, we have the following.

THEOREM D
Let n be a positive integer. Fix $a_1, \dots, a_{2n}, d_1, d_2 \in \mathbb{C}$ which satisfy (2.3)–(2.5), and set $P = \{(\rho_1, \rho_2) \in \mathbb{C}^2 : \rho_1, \rho_2 \neq 0, \rho_1 \neq \rho_2\}$. For all $\rho := (\rho_1, \rho_2) \in P$, set $\theta(\rho) = (a_1, \dots, a_{2n}; \rho_1, \rho_2, d_1, d_2; n) \in \Theta_{2n}^\sharp$, and consider the coarse moduli spaces $M_{\theta(\rho)}$. Clearly, they form a family $M \to P$.
(1) *The family $M \to P$ carries a natural algebraic connection (defined in Section 5.3).*
(2) *In the case of $2n = 4$, this connection coincides with the* PVI *connection under the isomorphism of Theorem C.*

Remark 2.11
The connection of Theorem D can be thought of as a continuous isomonodromy deformation of d-connections.

2.4. Difference PVI

So far, we have worked with d-connections of type θ, where θ is nondegenerate, in the sense of (2.4)–(2.6). It turns out that a different class of d-connections enjoys similar properties. Namely, let us replace (2.6) with the conditions

$$\rho_1 = \rho_2 \neq 0, \qquad d_1 \neq d_2. \tag{2.8}$$

Let $\Theta_{2n}^{\flat} \subset \Theta_{2n}$ be the set of all θ's which satisfy (2.2)–(2.5) and (2.8), and set $\Theta^{\flat} = \bigsqcup_n \Theta_{2n}^{\flat}$. It can be shown that for $\theta \in \Theta_{2n}^{\flat}$, the coarse moduli space M_θ is a smooth variety of dimension $2n-4$. (Recall that for $\theta \in \Theta_{2n}^{\sharp}$, we have $\dim(M_\theta) = 2n-2$.) Therefore, the first interesting case is $\theta \in \Theta_6^{\flat}$; then M_θ is an algebraic surface. As before, we assume that $\deg(\theta) = -1$.

Similarly to Theorem A, we can describe the moduli space M_θ using coordinates $(q, p) \in (\mathbb{P}^1)^2$.

THEOREM E

Suppose that

$$\theta = (a_1, a_2, a_3, a_4, a_5, a_6; \rho, \rho, d_1, d_2; 3) \in \Theta_6^{\flat}$$

has $\deg(\theta) = -1$. *Let* $\sigma_1 : K_1 \to (\mathbb{P}^1)^2$ *be the blowup of* $(\mathbb{P}^1)^2$ *at the following seven points:* $(q, p) = (a_1, 0), (a_2, 0), (a_3, 0), (a_4, \infty), (a_5, \infty), (a_6, \infty)$, *and* (∞, ρ). *(Here q and p are the projections* $(\mathbb{P}^1)^2 \to \mathbb{P}^1$.*) Consider the exceptional curve* $E = \sigma_1^{-1}(\infty, \rho) \subset K_1$*; a homogeneous coordinate on E is given by* $(1/q : p - \rho)$. *Let* $\sigma_2 : K_2 \to K_1$ *be the blowup of* K_1 *at the two points* $(1/q : p - \rho) = (1 : \rho(d_j + a_4 + a_5 + a_6))$, $j = 1, 2$.

(1) *There exists an open embedding* $P_2 : M_\theta \hookrightarrow K_2$.

(2) *The complement to* $P_2(M_\theta)$ *in* K_2 *is the union of the proper preimages of the curves* $\mathbb{P}^1 \times \{0\}, \mathbb{P}^1 \times \{\infty\}, \{\infty\} \times \mathbb{P}^1 \subset (\mathbb{P}^1)^2$ *and the exceptional curve* $E \subset K_1$, $j = 1, 2$.

Remark 2.12

Using multiplication by scalar, it is easy to see that the moduli space M_θ for $\theta = (a_1, \dots, a_{2n}; \rho, \rho, d_1, d_2; n)$ does not depend on ρ. Therefore, we can assume that $\rho = 1$ without loss of generality.

Remark 2.13

K_2 is not the smallest smooth compactification of M_θ (unlike the case when $\theta \in \Theta_4^{\sharp}$; see Remark 2.8). Indeed, the proper preimage of $\{\infty\} \times \mathbb{P}^1 \subset (\mathbb{P}^1)^2$ is an exceptional curve in $K_2 - M_\theta$. Contracting the exceptional curve, we obtain the smallest smooth compactification of M_θ, which is a surface of the Sakai-type $A_2^{(1)*}$.

Modifications of d-connections define natural isomorphisms between spaces M_θ. Similarly to Theorem B, we describe a simple isomorphism of this kind explicitly. We call the resulting difference equation the *difference* PVI. As we see, it degenerates into both the difference PV (see Section 6.3) and the usual PVI (see Section 6.4).

Suppose that

$$\theta = (a_1, a_2, a_3, a_4, a_5, a_6; 1, 1, d_1, d_2; 3) \in \Theta_6^\flat$$

has $\deg(\theta) = -1$. Set

$$\theta' = (a_1 - 1, a_2 - 1, a_3, a_4, a_5, a_6; 1, 1, d_1 + 1, d_2 + 1; 3) \in \Theta_6^\flat.$$

Modification of d-connections induces an isomorphism $\mathrm{dPVI} : M_\theta \to M_{\theta'}$. Explicitly, for every $(\mathscr{L}, \mathscr{A}) \in M_\theta$, the image $\mathrm{dPVI}(\mathscr{L}, \mathscr{A}) = (\mathscr{L}', \mathscr{A}')$ is the only d-connection of type θ' which admits a rational isomorphism $\mathscr{R} : \mathscr{L}' \tilde{\dashrightarrow} \mathscr{L}$ which agrees with the d-connections $\mathscr{R}(z+1)\mathscr{A}'(z) = \mathscr{A}(z)\mathscr{R}(z)$.

THEOREM F

Set $p' := p \circ \mathrm{dPVI}$, $q' := q \circ \mathrm{dPVI} : M_\theta \to \mathbb{P}^1$. *For* $j = 1, 2$, *set*

$$c_j := \frac{(d_j + a_1 + a_2 + a_4 - 1)(d_j + a_1 + a_2 + a_5 - 1)(d_j + a_1 + a_2 + a_6 - 1)}{(d_j - d_{3-j})}.$$

(The denominator is $\pm(d_1 - d_2)$*.) Then*

$$\begin{cases} q' = (p-1)(q+1-a_1-a_2) + pa_3 + \displaystyle\sum_{j=1,2} \frac{c_j p}{q - (p(1-a_1-a_2-d_j) - a_3)/(p-1)}, \\ p' \cdot p = \dfrac{(q'-a_1+1)(q'-a_2+1)}{(q'-a_4)(q'-a_5)(q'-a_6)} \cdot \big((p-1)(q'-q) + q' - a_3\big). \end{cases} \tag{2.9}$$

Remark 2.14

Theorem C identifies M_θ for $\theta \in \Theta_4^\sharp$ with a moduli space of connections of certain kind. A similar statement holds for $\theta = (a_1, \dots, a_6; \rho, \rho, d_1, d_2; 3) \in \Theta_6^\flat$. In this case, M_θ is isomorphic to the moduli space of pairs $(\mathscr{L}, \nabla)$, where $\mathscr{L}$ is a rank 3 bundle on $\mathbb{P}^1$ and ∇ is a connection on $\mathscr{L}$ with first-order poles at ρ, 0, and ∞ (and no other poles); the residues at the poles have eigenvalues $\{0, d_1+a_4+a_5+a_6, d_2+a_4+a_5+a_6\}$, $\{a_1, a_2, a_3\}$, and $\{-a_4, -a_5, -a_6\}$, respectively. The isomorphism can be constructed using the Mellin transform (similarly to the construction in Section 5.6).

Notice that if we interpret M_θ as a moduli space of rank 3 bundles with connections on $\mathbb{P}^1$, then dPVI becomes an isomorphism between such moduli spaces (a Bäcklund transformation), which corresponds to a modification of such bundles.

3. General d-connections

3.1. Formal behavior at infinity

Let $\mathscr{L}$ be a vector bundle on $\mathbb{P}^1$, and let $\mathscr{A}(z) : \mathscr{L}_z \to \mathscr{L}_{z+1}$ be a rational d-connection on $\mathscr{L}$. Since $\infty \in \mathbb{P}^1$ is the only fixed point of the transformation $z \mapsto z+1$, it is natural to study the restriction of $\mathscr{A}$ to a neighborhood of infinity. Here the word "neighborhood" can be understood either analytically (a small disk) or formally (the formal disk). In this section we work with the formal neighborhood; the corresponding classification problem is significantly easier. The situation is somewhat similar to classification of irregular singularities for ordinary differential equations; the formal classification is much simpler than the analytic one (because of Stokes's phenomenon).

In the language of difference equations, the problem is to classify matrices $A(z)$ over the ring of formal Laurent series $\mathbb{C}((z^{-1}))$ modulo d-gauge transformations

$$A(z) \mapsto R(z+1)^{-1} A(z) R(z),$$

where the gauge matrix $R(z)$ is an invertible matrix over the ring of formal Taylor series $\mathbb{C}[[z^{-1}]]$.

If A is generic, the answer is given by the following easy statement (see, e.g., [6, Proposition 1.1]).

PROPOSITION 3.1
Suppose that the $(m \times m)$-matrix $A(z) = \sum_{i \le n} A_i z^i$ over $\mathbb{C}((z^{-1}))$ satisfies the following condition:

All eigenvalues of the leading term A_n are distinct and nonzero; in other words, A_n is invertible, regular, and semisimple. (3.1)

Then there exists a gauge matrix $R(z) = \sum_{i \le 0} R_i z^i$ with invertible R_0 such that

$$R(z+1)^{-1} A(z) R(z) = A'_n z^n + A'_{n-1} z^{n-1}, \tag{3.2}$$

where A'_n and A'_{n-1} are diagonal matrices. The matrix $R(z)$ is uniquely determined up to right multiplication by a permutation matrix and a constant diagonal matrix.

Denote the diagonal entries of A'_n by $\rho_1, \dots, \rho_m$; notice that ρ_i's are the eigenvalues of A_n; in particular, all ρ_i are distinct and nonzero. Denote the corresponding diagonal entries of A'_{n-1} by $c_1, \dots, c_m$. Set $d_i := c_i/\rho_i$; we work with d_i rather then c_i because it simplifies formulas (2.3) and (2.4). We call the collection $(\rho_1, \dots, \rho_m, d_1, \dots, d_m; n)$ the *formal type* of $A(z)$ at infinity. Proposition 3.1 implies that the formal type is determined by $A(z)$ up to a simultaneous permutation of ρ_i's and d_i's, that is, up to the action of the symmetric group S_m.

Remark 3.2
Proposition 3.1 is sometimes (e.g., in [6]) formulated in terms of formal solutions to the difference equation; the claim is that the equation $Y(z+1) = A(z)Y(z)$ has a formal solution of the form

$$Y(z) = \big(\Gamma(z)\big)^n \Big(\sum_{i\le 0} \hat{Y}_i z^i\Big) \operatorname{diag}(\rho_1^z z^{d_1}, \dots, \rho_m^z z^{d_m}),$$

where $\hat{Y}_i$ are $(m \times m)$-matrices, $\hat{Y}_0$ is invertible, and $\rho_1, \dots, \rho_m, d_1, \dots, d_m \in \mathbb{C}$.

Note that $\sum_{i\le 0} \hat{Y}_i z^i$ does not coincide with $R(z)$ of Proposition 3.1.

Remark 3.3
The formal type of $A(z)$ can be determined directly without diagonalizing $A(z)$. Indeed, denote by $\sigma_i(z)$ and $\sigma_i'(z)$ $(i = 1, \dots, m)$ the coefficients of the characteristic polynomials of $A(z)$ and $R(z+1)^{-1}A(z)R(z)$, respectively, so that $\sigma_1(z) = -\operatorname{tr} A(z)$ and $\sigma_m(z) = (-1)^m \det A(z)$. Clearly, $\sigma_i(z)$ and $\sigma_i'(z)$ have pole of order $i \cdot n$ at infinity. One can easily check that the order of pole of $\sigma_i(z) - \sigma_i'(z)$ is at most $i \cdot n - 2$. Thus, the two leading terms of $\sigma_i(z)$ and $\sigma_i'(z)$ coincide. It is now easy to see that the formal type of $A(z)$ is determined (up to the S_m-action) by the pairs of leading terms of $\sigma_i(z)$, $i = 1, \dots, m$.

In particular, if we assume that A_n is diagonal, then its diagonal entries are the ρ_i's, and the diagonal entries of A_{n-1} equal $\rho_i d_i$, even if A_{n-1} is not diagonal.

Let us now translate Proposition 3.1 into the language of d-connections. For simplicity, we consider vector bundles only of rank 2.

COROLLARY 3.4
Let $\mathscr{A}(z)$ be a d-connection on a rank 2 vector bundle $\mathscr{L}$. Denote by n the order of pole of $\mathscr{A}$ at infinity, and denote by $\mathscr{A}_n : \mathscr{L}_\infty \to \mathscr{L}_\infty$ the leading term of $\mathscr{A}$ (i.e., n is the smallest number such that the limit

$$\mathscr{A}_n := \lim_{z\to\infty} \mathscr{A}(z) z^{-n}$$

exists). Suppose that all eigenvalues of $\mathscr{A}_n$ are distinct and nonzero. Then $\mathscr{A}(z)$ satisfies Definition 2.3(2) (for some $\rho_1, \rho_2, d_1, d_2 \in \mathbb{C}$).

We call the collection $(\rho_1, \rho_2, d_1, d_2; n)$ the *formal type* of the d-connection $\mathscr{A}(z)$. It is determined by $\mathscr{A}(z)$ up to the action of S_2. Notice also that in the situation of Corollary 3.4, condition (2.6) holds automatically.

3.2. Operations on d-connections

Let us now discuss some natural operations on d-connections. The operations allow us to identify the moduli spaces (or moduli stacks, or sets of isomorphism classes, or categories) of d-connections of type θ for different θ. As a trivial example, notice that $M_{\theta'} = M_\theta$ if θ' is obtained from θ by a permutation of a_i's or a simultaneous permutation of ρ_i's and d_i's.

Multiplication by a scalar. Let $f(z) \neq 0$ be a rational function on $\mathbb{P}^1$, and let $\mathscr{A}(z)$ be a d-connection on a vector bundle $\mathscr{L}$. Clearly, the product $f(z)\mathscr{A}(z)$ is again a d-connection on $\mathscr{L}$.

In the language of difference equations, this operation corresponds to multiplication of solutions by Γ-functions. Indeed, let us write $f(z) = c\prod(z - z_i)^{k_i}$. Then $y(z)$ solves the difference equation $y(z+1) = \mathscr{A}(z)y(z)$ if and only if $\tilde{y}(z) = c^z \prod \Gamma(z - z_i)^{k_i} y(z)$ solves $\tilde{y}(z+1) = (f(z)\mathscr{A}(z))\tilde{y}(z)$.

On the other hand, multiplication by a scalar is also a special case of a tensor product of d-connections. We can view $f(z)$ as a d-connection on the trivial rank 1 bundle $\mathscr{O}_{\mathbb{P}^1}$; then $f(z)\mathscr{A}(z)$ becomes the natural d-connection on the tensor product $\mathscr{L} = \mathscr{L} \otimes \mathscr{O}_{\mathbb{P}^1}$ of two vector bundles with d-connections.

Remark 3.5

For any d-connection $\mathscr{A}(z)$, we can pick a function $f(z)$ such that the only pole of the product $f(z)\mathscr{A}(z)$ is at infinity. For instance, suppose that $\mathscr{L}$ has rank 2, and suppose that the d-connection $\mathscr{A}(z)$ has a simple pole at $z = z_0$; this means that all matrix elements of $A(z)$ (in some basis) have at most a simple pole and that $\det(A(z))$ has a simple pole at $z = z_0$. Then $(z - z_0)A(z)$ has a simple zero at z_0. In this way, classification of rank 2 d-connections with simple poles and simple zeros on $\mathbb{P}^1 - \{\infty\}$ is reduced to classification of d-connections with simple zeros only.

Now suppose that $(\mathscr{L}, \mathscr{A}(z)) \in M_\theta$ for $\theta \in \Theta$. Let $f(z)$ be a rational function; clearly, the product $(\mathscr{L}, f(z)\mathscr{A}(z))$ is a d-connection of type θ' (for some $\theta' \in \Theta$) if and only if the function $f(z) = c$ is a nonzero constant. If $f(z) = c \in \mathbb{C} - \{0\}$, then $(\mathscr{L}, c\mathscr{A}) \in M_{\theta'}$ for

$$\theta' = (a_1, \dots, a_k; c\rho_1, c\rho_2, d_1, d_2; n).$$

Clearly, the correspondence $(\mathscr{L}, \mathscr{A}) \mapsto (\mathscr{L}, c\mathscr{A})$ gives an isomorphism $\mu = \mu_c : M_\theta \xrightarrow{\sim} M_{\theta'}$; the inverse map is $\mu_{c^{-1}}$.

Modification. Suppose that $\mathscr{R} : \mathscr{L} \dashrightarrow \mathscr{L}'$ is a rational isomorphism between two vector bundles $\mathscr{L}$ and $\mathscr{L}'$ on $\mathbb{P}^1$. Then a d-connection $\mathscr{A}(z)$ on $\mathscr{L}$ induces a d-connection $\mathscr{A}'$ on $\mathscr{L}'$ (and vice versa).

In the language of difference equation, this operation is the d-gauge transformation

$$A'(z) = R(z+1)^{-1}A(z)R(z), \tag{3.3}$$

where R, A, and A' are the matrices of $\mathscr{R}$, $\mathscr{A}$, and $\mathscr{A}'$, respectively (corresponding to some choice of bases). We call $\mathscr{A}'$ a *modification* of $\mathscr{A}$. (Of course, $\mathscr{A}$ is also a modification of $\mathscr{A}'$.)

Remark 3.6
Modifications can also be viewed as isomonodromy deformations in the sense of [6]. Indeed, the monodromies of $\mathscr{A}$ and $\mathscr{A}'$ coincide. (For the monodromies to exist, $\mathscr{A}$ and $\mathscr{A}'$ have to satisfy the assumptions of Corollary 3.4.)

The simplest class of modifications is the so-called class of elementary modifications.

Definition 3.7
Suppose that the rational isomorphism $\mathscr{R} : \mathscr{L} \xrightarrow{\sim} \mathscr{L}'$ is regular and has exactly one simple zero. In this case, $\mathscr{A}'$ is an *elementary upper modification* of $\mathscr{A}$, and $\mathscr{A}$ is an *elementary lower modification* of $\mathscr{A}'$.

Note that an elementary upper modification $\mathscr{R} : \mathscr{L} \to \mathscr{L}'$ is uniquely determined by the pair (x, l), where $x \in \mathbb{P}^{\mathbf{1}}$ is the only zero of $\mathscr{R}$ and the one-dimensional subspace $l \subset \mathscr{L}_x$ is given by $l = \ker(\mathscr{R}(x) : \mathscr{L}_x \to \mathscr{L}'_x) \subset \mathscr{L}_x$. Conversely, any pair $(x \in \mathbb{P}^{\mathbf{1}}, l \subset \mathscr{L}_x)$ defines an elementary upper modification. Similarly, elementary lower modifications of $\mathscr{L}'$ are in one-to-one correspondence with pairs (x, l'), where $x \in \mathbb{P}^{\mathbf{1}}$, $l' \subset \mathscr{L}'_x$ is a subspace of codimension 1. (For $\mathscr{R} : \mathscr{L} \to \mathscr{L}'$, x is the only zero of $\mathscr{R}$, and $l' = \operatorname{im}(\mathscr{R}(x) : \mathscr{L}_x \to \mathscr{L}'_x)$.)

PROPOSITION 3.8
Suppose that $(\mathscr{L}, \mathscr{A}) \in M_\theta$ for $\theta = (a_1, \dots, a_k; \rho_1, \rho_2, d_1, d_2; n)$, and suppose that $\rho_1 \neq \rho_2$. Let $(\mathscr{L}', \mathscr{A}')$ be an elementary upper modification of $\mathscr{L}$ given by $(x \in \mathbb{P}^{\mathbf{1}}; l \subset \mathscr{L}_x)$. Then the only cases when $(\mathscr{L}', \mathscr{A}')$ belongs to $M_{\theta'}$ for some $\theta' \in \Theta$ are as follows.

(1) *If $x = \infty$, then l must be an eigenspace of $\mathscr{A}_n : \mathscr{L}_\infty \to \mathscr{L}_\infty$ (the leading term of $\mathscr{A} = \mathscr{A}_n z^n +$ lower-order terms). If, for instance, $l = \ker(\mathscr{A}_n - \rho_1) \subset L_\infty$, then $\theta' = (a_1, \dots, a_k; \rho_1, \rho_2, d_1 - 1, d_2; n)$, and an analogous formula holds when $l = \ker(\mathscr{A}_n - \rho_2)$.*

(2) *If $x = a_i$ is a zero of $\mathscr{A}$ and $x - 1 \neq a_j$ is not, then l must be the kernel of $\mathscr{A}(x) : \mathscr{L}_x \to \mathscr{L}_{x+1}$; in this case, $\theta' = (a_1, \dots, a_i - 1, \dots, a_k; \rho_1, \rho_2, d_1, d_2; n)$.*

In either case, the elementary modifications define an isomorphism $M_\theta \xrightarrow{\sim} M_{\theta'}$.

Remark 3.9
Sometimes an elementary modification of a d-connection of type θ has simple poles, which can be turned into simple zeros using multiplication by a scalar (e.g., this happens if neither x nor $x-1$ is a pole). However, this procedure does not lead to an isomorphism between the moduli spaces M_θ (at least assuming that (2.2)–(2.6) hold) because the corresponding spaces have different dimensions.

Thus, elementary modifications (upper or lower) allow us to identify $M_{\theta'}$ and M_θ if θ' is obtained from θ by adding or subtracting 1 to one of the a_i's or d_i's, provided certain conditions hold. Composing such identifications, we get other isomorphisms between M_θ for different $\theta \in \Theta_k$.

The situation is particularly simple if θ satisfies the conditions (2.5) and (2.6). Then M_θ and $M_{\theta'}$ are naturally isomorphic if θ' is obtained from θ by adding integers to a_i's and d_i's. In other words, we have a natural action of the group $G = (\mathbb{Z})^k \times (\mathbb{Z})^2$ on Θ_k, and for any $\theta \in \Theta_k$ satisfying (2.5) and (2.6) (in particular, for any $\theta \in \Theta_k^\sharp$), we get isomorphisms $M_\theta \to M_{g\theta}$ for all $g \in G$.

3.3. Irreducibility of d-connections

Let $\mathscr{A}(z)$ be a d-connection on a vector bundle $\mathscr{L}$ on $\mathbb{P}^1$. Assume that $\mathscr{A}(z)$ is nondegenerate at infinity in the sense that (3.1) holds. Denote by $(\rho_1, \dots, \rho_m, d_1, \dots, d_m; n)$ the formal type of $\mathscr{A}(z)$ at infinity.

For the morphism $\mathscr{A}(z) : \mathscr{L}_z \to \mathscr{L}_{z+1}$, its determinant is a map $\det \mathscr{A}(z) : \bigwedge^m \mathscr{L}_z \to \bigwedge^m \mathscr{L}_{z+1}$; in other words, $\det \mathscr{A}(z)$ is a d-connection on the line bundle $\det \mathscr{L} := \bigwedge^m \mathscr{L}$. It is easy to see that $\det \mathscr{L}$ has formal type $(\rho_1\rho_2 \cdots \rho_m, d_1 + \cdots + d_m; mn)$ at infinity. Let $a_1, \dots, a_k \in \mathbb{A}^1$ and $b_1, \dots, b_l \in \mathbb{A}^1$ be zeros and poles (counted with multiplicity), respectively, of $\det \mathscr{A}(z)$ on $\mathbb{A}^1$.

The following two statements are immediate.

LEMMA 3.10
The collection $(a_1, \dots, a_k; b_1, \dots, b_l; \rho_1, \dots, \rho_m, d_1, \dots, d_m; n)$ *satisfies the equalities*

$$mn = k - l,$$

$$\deg(\mathscr{L}) = -\sum_{i=1}^m d_i - \sum_{i=1}^k a_i + \sum_{i=1}^l b_i.$$

COROLLARY 3.11
Let $(\mathscr{L}, \mathscr{A})$ *be a d-connection of type*

$$\theta = (a_1, \dots, a_k; \rho_1, \rho_2, d_1, d_2; n) \in \Theta.$$

Then $k = 2n$ *and* $\deg(\theta) = \deg(\mathscr{L})$ *(see (2.3) for the definition of* $\deg(\theta)$*); in particular,* $\deg(\theta)$ *is an integer.*

LEMMA 3.12
Suppose that $\theta \in \Theta$ *satisfies (2.4). Then any* $(\mathscr{L}, \mathscr{A}) \in M_\theta$ *is irreducible; there is no rank* 1 *subbundle* $\ell \subset \mathscr{L}$ *such that* $\mathscr{A}(\ell_z) \subset \ell_{z+1}$ *for all* z*.*

Proof
(Both the statement and its proof are completely analogous to [3, Proposition 1].) Suppose that $\ell \subset \mathscr{L}$ is an invariant subbundle of rank 1, so that $\mathscr{A}$ induces a d-connection $\mathscr{A}|_\ell$ on ℓ. All zeros of $\mathscr{A}|_\ell$ belong to $\{a_1, \dots, a_k\}$; besides, the formal type of $\mathscr{A}|_\ell$ at infinity is either $(\rho_1, d_1; n)$ or $(\rho_2, d_2; n)$. Now Lemma 3.10 leads to a contradiction. □

COROLLARY 3.13
Suppose that $(\mathscr{L}, \mathscr{A}) \in M_\theta$*, and suppose that* $\theta \in \Theta_{2n}$ *satisfies (2.4). If* $\mathscr{L} \simeq \mathscr{O}(n_1) \oplus \mathscr{O}(n_2)$*, then* $|n_1 - n_2| \le n$*.*

Proof
Without loss of generality, we can assume that $n_1 \ge n_2$. Let $\ell \subset \mathscr{L}$ be a rank 1 subbundle of degree n_1. Since $(\mathscr{L}, \mathscr{A})$ is irreducible, ℓ is not $\mathscr{A}$-invariant, and so the rational map $\alpha : \ell \to \mathscr{L} \to s^*\mathscr{L} \to s^*(\mathscr{L}/\ell)$ is not identically zero. Notice that α can have at most a pole of order n at ∞ (and no other poles); thus, $n_1 = \deg(\ell) \le n + \deg(L/\ell) = n + n_2$. □

4. Difference PV

In this section we study M_θ for

$$\theta = (a_1, a_2, a_3, a_4; \rho_1, \rho_2, d_1, d_2; 2) \in \Theta_4^\sharp.$$

We assume that $\deg(\theta) = -1$ (i.e., $-d_1 - d_2 - \sum_{i=1}^4 a_i = -1$). Using modifications, we can make this assumption without loss of generality.

4.1. M_θ as a quotient

Let $(\mathscr{L}, \mathscr{A}) \in M_\theta$. By Corollary 3.13, $\mathscr{L}$ is isomorphic to $\mathscr{O} \oplus \mathscr{O}(-1)$. Let us choose an isomorphism $\mathscr{S} : \mathscr{O} \oplus \mathscr{O}(-1) \xrightarrow{\sim} \mathscr{L}$; then $\mathscr{A}$ induces the d-connection $\mathscr{S}(z+1)^{-1}\mathscr{A}(z)\mathscr{S}(z)$ of type θ on $\mathscr{O} \oplus \mathscr{O}(-1)$. Such a d-connection can be written as a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \begin{aligned} &a_{11}, a_{22} \in \Gamma\big(\mathbb{P}^1, \mathscr{O}(2)\big), \\ &a_{12} \in \Gamma\big(\mathbb{P}^1, \mathscr{O}(3)\big), \\ &a_{21} \in \Gamma\big(\mathbb{P}^1, \mathscr{O}(1)\big). \end{aligned} \tag{4.1}$$

Of course, $\mathscr{S}$ is not unique; it can be composed with an automorphism of $\mathscr{O} \oplus \mathscr{O}(-1)$. Such an automorphism can be written as a matrix

$$R = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}, \qquad \begin{array}{l} r_{11}, r_{22} \in \mathbb{C} - \{0\}, \\ r_{12} \in \Gamma\big(\mathbb{P}^1, \mathscr{O}(1)\big). \end{array} \tag{4.2}$$

If we replace $\mathscr{S}$ with $\mathscr{S} \circ R$, then A is replaced with its d-gauge transform

$$R(z+1)^{-1} A(z) R(z). \tag{4.3}$$

LEMMA 4.1
Let $\mathscr{A}$ be a d-connection on $\mathscr{O} \oplus \mathscr{O}(-1)$; its matrix A is of the form (4.1). We claim that $\mathscr{A}$ is of type θ if and only if A satisfies the conditions

$$\det(A) = (z-a_1)(z-a_2)(z-a_3)(z-a_4)\rho_1\rho_2, \tag{4.4}$$

$$a_{11} + a_{22}(1+z^{-1}) = (\rho_1+\rho_2)z^2 + (d_1\rho_1 + d_2\rho_2)z + t(z^{-1}), \tag{4.5}$$

where $t(z^{-1}) \in \mathbb{C}[[z^{-1}]]$ is a Taylor series in z^{-1}.

Proof
$\mathscr{A}$ is of type θ if and only if it satisfies the two conditions of Definition 2.3. Let us reformulate the conditions in terms of A.

Definition 2.3(1) is equivalent to the condition that

$$\det(A) = c(z-a_1)(z-a_2)(z-a_3)(z-a_4) \quad \text{for some } c \in \mathbb{C} - \{0\}. \tag{4.6}$$

(Here we use that $\det(A)$ is a polynomial of degree 4 in z.) Now set

$$S(z) := \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix}.$$

(S is essentially a basis of $\mathscr{O} \oplus \mathscr{O}(-1)$ in a neighborhood of $\infty \in \mathbb{P}^1$.) By Remark 3.3, Definition 2.3(2) is equivalent to the two conditions

$$\det\big(S(z+1)^{-1}A(z)S(z)\big) = \rho_1\rho_2 z^4 + \rho_1\rho_2(d_1+d_2)z^3 + t_1(z^{-1})z^2, \tag{4.7}$$

$$\operatorname{tr}\big(S(z+1)^{-1}A(z)S(z)\big) = (\rho_1+\rho_2)z^2 + (d_1\rho_1 + d_2\rho_2)z + t_2(z^{-1}). \tag{4.8}$$

Here t_1, t_2 are Taylor series in z^{-1}.

It is easy to see that (4.4) is equivalent to the combination of (4.6) and (4.7) (here we use the fact that $\deg(\theta) = -1$), and (4.5) is equivalent to (4.8). □

COROLLARY 4.2
Denote by X_θ the space of matrices A of the form (4.1) which satisfy (4.4) and (4.5); denote by G the group of matrices R of the form (4.2). Let G act on X_θ via d-gauge transformations (see (4.3)). Then the quotient X_θ/G is canonically isomorphic to M_θ.

4.2. Geometric description of M_θ

In this section we derive Theorem A from another geometric description of M_θ (see Theorem 4.4). Recall that Theorem A realizes M_θ as an open subset of a blowup of $(\mathbb{P}^1)^2$; in Theorem 4.4 we use a different rational surface in place of $(\mathbb{P}^1)^2$. Of the two descriptions, Theorem 4.4 uses somewhat more natural constructions (however, see Remark 4.5); for instance, all four points $a_1, \dots, a_4$ appear in a symmetric manner. On the other hand, the advantage of Theorem A is that $(\mathbb{P}^1)^2$ has natural coordinates (q, p), which can then be viewed as rational coordinates $q, p : M_\theta \to \mathbb{P}^1$. This makes Theorem A more suitable for writing formulas.

As before, $(\mathscr{L}, \mathscr{A}) \in M_\theta$, $\mathscr{S} : \mathscr{O} \oplus \mathscr{O}(-1) \xrightarrow{\sim} \mathscr{L}$, and A is the matrix of $\mathscr{A}$ relative to $\mathscr{S}$. Notice that the matrix element $a_{21} \in \Gamma(\mathbb{P}^1, \mathscr{O}(1))$ is not identically zero because $(\mathscr{L}, \mathscr{A})$ is irreducible. Therefore, a_{21} has a single zero on $\mathbb{P}^1$; let us denote it by $q \in \mathbb{P}^1$. Set $\tilde{p} := a_{11}(q) \in (\mathscr{O}(2))_q$.

PROPOSITION 4.3
The coordinates $\tilde{p}$ and q depend only on $(\mathscr{L}, \mathscr{A}) \in M_\theta$ and not on $\mathscr{S}$.

Proof
This statement can be easily checked directly by calculating the d-gauge transformation (4.3) with the gauge matrix (4.2). It is also possible to provide a geometric explanation in the spirit of [3, Section 4.1]. □

The pair $(q, \tilde{p})$ can be viewed as a map $\tilde{P} : M_\theta \to \widetilde{K}$, where $\widetilde{K} := \mathbb{V}(\mathscr{O}(2)^\vee)$ is the total space of the line bundle $\mathscr{O}(2)$. We prove in Theorem 4.4(1) that the map $\tilde{P} : M_\theta \to \widetilde{K}$ is a regular birational morphism. Since M_θ is a smooth algebraic surface, $\widetilde{P}$ identifies M_θ with an open subset of a blowup of $\widetilde{K}$. Let us describe the blowup.

Let us start with some general remarks about the geometry of $\widetilde{K}$. Clearly, $\widetilde{K}$ is fibered over $\mathbb{P}^1$ so that the fiber over $z \in \mathbb{P}^1$ is $\mathscr{O}(2)_z$. If f is a (rational) section of $O(2)$ which is regular at z, then its value $f(z) \in \mathscr{O}(2)_z$ can be viewed as a point of $\widetilde{K}$; we denote this point by $(z, f(z))$. For example, $(z, 0(z))$ is the zero element in the fiber of $\widetilde{K}$ over $z \in \mathbb{P}^1$.

Now let $\tilde{\sigma}_c : \widetilde{K}_c \to \widetilde{K}$ be the blowup of $\widetilde{K}$ at $c := (z, f(z))$. Then the exceptional divisor $\tilde{\sigma}_c^{-1}(c) \subset \widetilde{K}_c$ is isomorphic to the projective line $\mathbb{P}(T_c\widetilde{K})$; that is, points of $\tilde{\sigma}_c^{-1}(c)$ correspond to lines in the tangent space to $\widetilde{K}$ at c. Any smooth curve $C \subset \widetilde{K}$

which passes through c defines such a line (the tangent line to C at c). In particular, we can take C to be the graph $\{(x, f(x)) : x \in \mathbb{P}^1\}$ of f; denote the corresponding point of $\widetilde{K}_c$ by $(z, f'(z))$. Any other rational section g of $\mathscr{O}(2)$ defines a point $(z, g'(z)) \in \widetilde{K}_c$, provided that g is regular at z and $g(z) = f(z)$.

THEOREM 4.4

(1) *The map $\tilde{P} : M_\theta \to \widetilde{K}$ is a regular birational morphism of smooth algebraic surfaces.*

(2) *Let $\tilde{\sigma}_1 : \widetilde{K}_1 \to \widetilde{K}$ be the blowup of $\widetilde{K}$ at the following six points: $(a_i, 0(a_i))$ $(i = 1, \dots, 4)$ and $(\infty, (\rho_j z^2)(\infty))$ $(j = 1, 2)$. Let $\sigma_2 : \widetilde{K}_2 \to \widetilde{K}_1$ be the blowup of $\widetilde{K}_1$ at the two points $(\infty, (\rho_j z^2 + \rho_j d_j z)'(\infty))$, $j = 1, 2$. (These points belong to the preimages of $(\infty, (\rho_j z^2)(\infty))$, $j = 1, 2$.) Then the map $\widetilde{P}$ induces an open embedding $\widetilde{P}_2 : M_\theta \hookrightarrow \widetilde{K}_2$.*

(3) *The complement to $\widetilde{P}_2(M_\theta)$ in $\widetilde{K}_2$ is the union of the proper preimages of the following curves: the zero section $\{(z, 0(z)) : z \in \mathbb{P}^1\} \subset \widetilde{K}$, the fiber at infinity $\{(\infty, az^2(\infty)) : a \in \mathbb{C}\} \subset \widetilde{K}$, and two exceptional curves $\tilde{\sigma}_1^{-1}(\infty, (\rho_j z^2)(\infty)) \subset \widetilde{K}_1$.*

The proof of Theorem 4.4 is given in Section 4.3. Let us now derive Theorem A from Theorem 4.4.

Proof of Theorem A

For $(\mathscr{L}, \mathscr{A}) \in M_\theta$, consider the expression

$$p := \frac{\tilde{p}}{(q - a_3)(q - a_4)}. \tag{4.9}$$

Here the denominator is the value of the section $(z - a_3)(z - a_4) \in \Gamma(\mathbb{P}^1, \mathscr{O}(2))$ at $z = q \in \mathbb{P}^1$. Both the numerator and the denominator are elements of $\mathscr{O}(2)_q$; therefore, $p \in \mathbb{C}$, provided that the denominator does not vanish. We can view p as a rational mapping $p : M_\theta \to \mathbb{P}^1$. Actually, Theorem 4.4 implies that $p : M_\theta \to \mathbb{P}^1$ is regular; the corresponding rational mapping $\widetilde{K} \dashrightarrow \mathbb{P}^1$ has singularities at $(a_3, 0(a_3)), (a_4, 0(a_4)) \in \widetilde{K}$, but the blowup $\widetilde{K}_1 \to \widetilde{K}$ resolves the singularities. We therefore obtain a regular mapping $P := (q, p) : M_\theta \to (\mathbb{P}^1)^2$. We claim that P induces an embedding $P_2 : M_\theta \hookrightarrow K_2$, where K_2 is the blowup of $(\mathbb{P}^1)^2$ described in Theorem A.

Let us consider the birational mapping $\Phi : (q, \tilde{p}) \mapsto (q, p) : \widetilde{K} \dashrightarrow (\mathbb{P}^1)^2$. It is easy to see that Φ induces an open embedding $\Phi_1 : \widetilde{K}_1 \hookrightarrow K_1$, and the complement $K_1 - \Phi(\widetilde{K}_1)$ is the proper preimage of $\mathbb{P}^1 \times \{\infty\} \subset (\mathbb{P}^1)^2$ under the blowup $K_1 \to (\mathbb{P}^1)^2$. To complete the proof, we should now check that Φ_1 maps the centers of the blowup

$\widetilde{K}_2 \to \widetilde{K}_1$ to the centers of the blowup $K_2 \to K_1$. This also follows from the formulas. □

Remark 4.5
Geometrically, formula (4.9) can be explained as the multiplication of a d-connection by a scalar. For $(\mathscr{L}, \mathscr{A}) \in M_\theta$, consider the d-connection

$$\widetilde{\mathscr{A}} := \frac{1}{(z-a_3)(z-a_4)} \mathscr{A}$$

on $\mathscr{L}$. Then $\widetilde{\mathscr{A}}$ has simple zeros at a_1, a_2 and simple poles at a_3, a_4, and its formal type at infinity is $(\rho_1, \rho_2; d_1 + a_3 + a_4, d_2 + a_3 + a_4; 0)$. Moreover, we can then view M_θ as the moduli space of d-connections of this kind (as in Remark 3.5). For d-connections of this kind, p plays the role of $\tilde{p}$, and Theorem A plays the role of Theorem 4.4.

4.3. Proof of Theorem 4.4

The most direct way to prove Theorem 4.4 is by bringing matrices (4.1) to some normal form. We do not reproduce all calculations here; the idea of the proof is as follows.

Denote by $\widetilde{M}_\theta$ the open subset of $\widetilde{K}_2$ described in Theorem 4.4(3) (i.e., the complement of proper preimages of the zero section, the fiber at infinity, and two exceptional curves). We need to show that the map $\widetilde{P} : M_\theta \to \widetilde{K}$ lifts to an isomorphism $M_\theta \to \widetilde{M}_\theta$. Let us consider open sets

$$U_0 := q^{-1}(\mathbb{P}^1 - \{\infty\}) \subset M_\theta, \qquad U_\infty := q^{-1}(\mathbb{P}^1 - \{0\}) \subset M_\theta,$$
$$\widetilde{U}_0 := q^{-1}(\mathbb{P}^1 - \{\infty\}) \subset \widetilde{M}_\theta, \qquad \widetilde{U}_\infty := q^{-1}(\mathbb{P}^1 - \{\infty\}) \subset \widetilde{M}_\theta.$$

It suffices to show that $\widetilde{P}$ lifts to isomorphisms $U_0 \xrightarrow{\sim} \widetilde{U}_0$, $U_\infty \xrightarrow{\sim} \widetilde{U}_\infty$. We show this by writing U_0 and U_∞ explicitly as zero loci of polynomial equations.

Let $(\mathscr{L}, \mathscr{A})$ be a point of U_0. Then $q = q(\mathscr{L}, \mathscr{A}) \in \mathbb{C}$ and $\tilde{p} = \tilde{p}(\mathscr{L}, \mathscr{A}) \in (\mathscr{O}(2))_q = \mathbb{C}$. It is easy to see that there exists an isomorphism $\mathscr{S} : \mathscr{O} \oplus \mathscr{O}(-1) \xrightarrow{\sim} \mathscr{L}$, unique up to a multiplicative constant, such that the matrix of the d-connection $\mathscr{A}$ relative to $\mathscr{S}$ is

$$A = \begin{bmatrix} a_{11} = \tilde{p} & a_{12} \\ a_{21} = z - q & a_{22} \end{bmatrix}, \qquad \begin{array}{l} a_{22} \in \Gamma\big(\mathbb{P}^1, \mathscr{O}(2)\big), \\ a_{12} \in \Gamma\big(\mathbb{P}^1, \mathscr{O}(3)\big). \end{array} \tag{4.10}$$

Essentially, (4.10) serves as a normal form of d-connections $(\mathscr{L}, \mathscr{A})$ (provided that $q \neq \infty$). The conditions (4.4) and (4.5) now become equations on a_{12}, a_{22}. Explicitly,

a_{12} and a_{22} are determined by their coefficients

$$a_{12} = a_{12,3}z^3 + a_{12,2}z^2 + a_{12,1}z + a_{12,0},$$
$$a_{22} = a_{22,2}z^2 + a_{22,1}z + a_{22,0},$$

and (4.4) and (4.5) are a system of polynomial equations on $a_{i2,j}$, $\tilde{p}$, and q. Solving these equations, we find polynomial (in $\tilde{p}$ and q) formulas for all $a_{i2,j}$ except for $r := a_{22,0}$. The equation on r looks as follows:

$$\tilde{p}r = F(\tilde{p}, q), \tag{4.11}$$

where $F(\tilde{p}, q)$ is a polynomial. Thus, U_0 is identified with the zero locus of the equation (4.11) in the three-dimensional space with coordinates $\tilde{p}$, q, and r.

Besides, $F(0, q) = c(q - a_1)(q - a_2)(q - a_3)(q - a_4)$ for some $c \in \mathbb{C} - \{0\}$. Therefore, the map $(\tilde{p}, q) : U_0 \to \mathbb{A}^2$ identifies U_0 with the complement to the proper preimage of the q-axis $\{(0, q)\}$ in the blowup of $\mathbb{A}^2$ at the four points $(\tilde{p}, q) = (0, a_i)$, $i = 1, \ldots, 4$. This complement is exactly $\widetilde{U}_0$.

A similar approach works for U_∞. For $(\mathscr{L}, \mathscr{A}) \in U_\infty$, set $\omega := (q(\mathscr{L}, \mathscr{A}))^{-1} \in \mathbb{C}$, $\pi := \tilde{p}(\mathscr{L}, \mathscr{A})/(q(\mathscr{L}, \mathscr{A})^2) \in \mathbb{C}$, where the denominator is understood as the value of $z^2 \in \Gamma(\mathbb{P}^1, \mathscr{O}(2))$ at $z = q$. One can think of ω and π as coordinates on the complement to the zero locus of q in $\widetilde{K}$. Then there is a unique up to a multiplicative constant choice of $\mathscr{S} : \mathscr{O} \oplus \mathscr{O}(-1) \xrightarrow{\sim} \mathscr{L}$ such that the matrix of $\mathscr{A}$ is

$$A = \begin{bmatrix} \pi z^2 & a_{12} \\ 1 - \omega z & a_{22} \end{bmatrix}.$$

Again, we get a system of polynomial equations on the coefficients of a_{i2}. Solving the equations, we find polynomial (in π and ω) formulas for all $a_{i2,j}$ except for $r = a_{22,0}$. In this case, the equation on r is

$$\pi\omega^2 r = G(\pi, \omega), \tag{4.12}$$

where $G(\pi, \omega)$ is a polynomial. Therefore, U_∞ is the zero locus of equation (4.12) in the three-dimensional space with coordinates π, ω, and r. Again, from the formula for $G(\pi, \omega)$, one easily sees the isomorphism $U_\infty \xrightarrow{\sim} \widetilde{U}_\infty$.

For instance, let us consider the neighborhood of $\omega = 0$. (The complement of $\omega = 0$ is covered by U_0.) One can check that $G(\pi, 0) = (\pi - \rho_1)(\pi - \rho_2)$, so when $\omega = 0$, either $\pi = \rho_1$ or $\pi = \rho_2$. Consider the neighborhood of the set $\omega = 0, \pi = \rho_1$ in U_∞. It follows that $\pi_1 := (\pi - \rho_1)/\omega$ is a regular function on the neighborhood (π_1 is a coordinate on the blowup of the ω-π plane at $(\omega, \pi) = (0, \rho_1)$). We can then rewrite (4.12) in variables π_1, ω, and r:

$$(\omega\pi_1 + \rho_1)r\omega = H(\pi_1, \omega),$$

where $H(\pi_1, \omega)$ is a polynomial such that $H(\pi_1, 0) = (\rho_2 - \rho_1)(\pi_1 - \rho_1 d_1)$; therefore, r is essentially a coordinate on the blowup of the ω-π_1 plane at $(\omega, \pi_1) = (0, \rho_1 d_1)$. Of course, the neighborhood of the set $\omega = 0$, $\pi = \rho_2$ in U_∞ has a similar description. □

Remark 4.6
Theorems A and 4.4 can also be proved in a more geometric way, in the spirit of [3, Theorem 3].

4.4. Proof of Theorem B

The proof of Theorem B is also based on calculations. The calculations are simplified by the observation that it suffices to check the formulas (2.7) on a dense subset of M_θ; we can therefore assume that $q, q' \neq \infty$.

Take $(\mathscr{L}, \mathscr{A}) \in M_\theta$, and set $(\mathscr{L}', \mathscr{A}') := \mathrm{dPV}(\mathscr{L}, \mathscr{A})$. Let us assume that $q(\mathscr{L}, \mathscr{A}) \neq \infty$ (i.e., $(\mathscr{L}, \mathscr{A}) \in U_0$); then there is an isomorphism $\mathscr{S} : \mathscr{O} \oplus \mathscr{O}(-1) \xrightarrow{\sim} \mathscr{L}$ such that the matrix of $\mathscr{A}$ relative to $\mathscr{S}$ is of the form (4.10). Using the formula $\tilde{p} = p(q - a_3)(q - a_4)$, we can write the matrix as

$$A = \begin{bmatrix} p(q-a_3)(q-a_4) & a_{12} \\ z - q & a_{22} \end{bmatrix}, \quad \begin{array}{l} a_{22} \in \Gamma\big(\mathbb{P}^1, \mathscr{O}(2)\big), \\ a_{12} \in \Gamma\big(\mathbb{P}^1, \mathscr{O}(3)\big). \end{array} \tag{4.13}$$

Recall also that a_{12}, a_{22} are polynomials of z whose coefficients are rational functions of p, q.

Similarly, if we assume that $q(\mathscr{L}', \mathscr{A}') \neq 0$, there exists an isomorphism $\mathscr{S}' : \mathscr{O} \oplus \mathscr{O}(-1) \xrightarrow{\sim} \mathscr{L}'$ such that the matrix of $\mathscr{A}'$ relative to $\mathscr{S}'$ is of the form

$$A' = \begin{bmatrix} p'(q'-a_3)(q'-a_4) & a'_{12} \\ z - q' & a'_{22} \end{bmatrix}, \quad \begin{array}{l} a'_{22} \in \Gamma\big(\mathbb{P}^1, \mathscr{O}(2)\big), \\ a'_{12} \in \Gamma\big(\mathbb{P}^1, \mathscr{O}(3)\big). \end{array} \tag{4.14}$$

By the definition of dPV, the matrix A' is the d-gauge transformation of A:

$$A'(z) = R(z+1)^{-1} A(z) R(z), \tag{4.15}$$

where R is the matrix of the rational map $\mathscr{R} : \mathscr{L}' \dashrightarrow \mathscr{L}$ (from the definition of dPV) with respect to the bases $\mathscr{S}, \mathscr{S}'$. It follows from the properties of modifications (see Section 3.2) that $\mathscr{R}$ induces a regular map $\mathscr{L}' \to \mathscr{L} \otimes \mathscr{O}(1)$ whose determinant has simple zeros at a_1, a_2 and no other zeros. In other words, R is of the form

$$R = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}, \qquad \begin{array}{l} r_{11}, r_{22} \in \Gamma\big(\mathbb{P}^1, \mathscr{O}(1)\big), \\ r_{21} \in \mathbb{C}, r_{12} \in \Gamma\big(\mathbb{P}^1, \mathscr{O}(2)\big), \end{array}$$

such that

$$\det(R) = c(z - a_1)(z - a_2) \quad (c \in \mathbb{C} - \{0\}). \tag{4.16}$$

Condition (4.16) yields polynomial equations on the coefficients of r_{11}, r_{12}, r_{21}, and r_{22}; the condition that (4.15) gives a matrix A' of the form (4.14) also gives such equations. The resulting system determines R up to a multiplicative constant. From (4.15), we now obtain a formula for the matrix A' in terms of p and q; in particular, we can derive (2.7). □

5. Difference PV and classical PVI

5.1. *Geometry of* PVI

Let us recall the description of the surface $M_{(x,\lambda)}$. We suppose that

$$\sum_{i=1}^{4} (\lambda_i^- + \lambda_i^+) = 1. \tag{5.1}$$

It is easy to see that $M_{(x,\lambda)}$ depends only on the classes of $\lambda_i^\pm$ in $\mathbb{C}/\mathbb{Z}$ (because of modifications of bundles with connections), so our assumption does not restrict the generality.

Suppose that $x \in X$, $\lambda \in \Lambda$, and let K_x be the total space of the line bundle $\Omega_{\mathbb{P}^1}(x_1 + \cdots + x_4)$. Let $b_i \subset K_x$ be the fiber over $x_i \in \mathbb{P}^1$. Notice that the residue of 1-forms identifies the fiber of $\Omega(x_1 + \cdots + x_4)$ over x_i with $\mathbb{C}$, so we get a canonical isomorphism $\mathrm{res}_i : b_i \xrightarrow{\sim} \mathbb{A}^1$. Denote by $\tilde{M}_{(x,\lambda)}$ the blowup of K_x at the eight points $(\mathrm{res}_i)^{-1}(\lambda_i^\pm)$, $i = 1, \ldots, 4$, and let $M'_{(x,\lambda)} \subset \tilde{M}_{(x,\lambda)}$ be the complement to the proper preimages of $b_i \subset K_x$.

PROPOSITION 5.1
There exists an isomorphism $M_{(x,\lambda)} \xrightarrow{\sim} M'_{(x,\lambda)}$.

Proposition 5.1 is a slight generalization of [3, Theorem 3] (see also [17, Theorem 2.2]); [3] works only with SL(2)-bundles, which corresponds to assuming that $\lambda_i^- + \lambda_i^+ = 0$ $(i = 2, 3, 4)$. However, the general case is easily reduced to this special case. Let us sketch the construction of the map $M_{(x,\lambda)} \xrightarrow{\sim} M'_{(x,\lambda)}$.

Given $(\mathscr{L}, \nabla) \in M_{(x,\lambda)}$, one can show that $\mathscr{L} \simeq \mathscr{O} \oplus \mathscr{O}(-1)$. (This is similar to Corollary 3.13.) If we fix an isomorphism $\mathscr{O} \oplus \mathscr{O}(-1) \xrightarrow{\sim} \mathscr{L}$, the connection ∇ is determined by its matrix

$$M(z) = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, \qquad \begin{aligned} &m_{11}, m_{22} \in \Gamma\big(\mathbb{P}^1, \Omega_{\mathbb{P}^1}(x_1 + \cdots + x_4)\big), \\ &m_{12} \in \Gamma\big(\mathbb{P}^1, \Omega_{\mathbb{P}^1}(x_1 + \cdots + x_4) \otimes \mathscr{O}(1)\big), \\ &m_{21} \in \Gamma\big(\mathbb{P}^1, \Omega_{\mathbb{P}^1}(x_1 + \cdots + x_4) \otimes \mathscr{O}(-1)\big). \end{aligned}$$

It can be proved that m_{21} is not identically zero (because $(\mathscr{L}, \nabla)$ is irreducible; this is similar to Lemma 3.12). Therefore, m_{21} has a single zero on $\mathbb{P}^1$; denote it by q^{PVI}. Set $p^{\mathrm{PVI}} := m_{11}(q^{\mathrm{PVI}})$. Note that p^{PVI} belongs to the fiber of $\Omega_{\mathbb{P}^1}(x_1 + \cdots + x_4)$ over $q^{\mathrm{PVI}} \in \mathbb{P}^1$. In other words, p^{PVI} is a point of the total space K_x. (In the notation of Section 4.2, the point is $(q^{\mathrm{PVI}}, p^{\mathrm{PVI}}) \in K_x$.) One can check that q^{PVI} and p^{PVI} depend only on $(\mathscr{L}, \nabla)$ and not on the choice of $\mathscr{O} \oplus \mathscr{O}(-1) \xrightarrow{\sim} \mathscr{L}$. Therefore, we obtain a regular map $M_{(x,\lambda)} \to K_x$. Proposition 5.1 claims that the map induces an isomorphism $M_{(x,\lambda)} \xrightarrow{\sim} M'_{(x,\lambda)}$.

Proof of Theorem C

Let $\theta \in \Theta_4^\sharp$, $x \in X$, and $\lambda \in \Lambda$ be as in Theorem C; we define the isomorphism $M_\theta \to M_{(x,\lambda)}$ by explicit formulas. Let $q, p : M_\theta \to \mathbb{P}^1$ be the coordinates from Theorem A. Consider the expression

$$p^{\mathrm{PVI}} := (z^{-1} dz)_{z=p} q,$$

where $(z^{-1}dz)_{z=p} \in (\Omega_{\mathbb{P}^1}(x_1 + \cdots + x_4))_p$ is the value of $z^{-1}dz \in \Gamma(\mathbb{P}^1, \Omega_{\mathbb{P}^1}(x_1 + \cdots + x_4))$ at $z = p$. Then $p^{\mathrm{PVI}} \in (\Omega_{\mathbb{P}^1}(x_1 + \cdots + x_4))_p$, provided that $q \neq \infty$. Let us also set $q^{\mathrm{PVI}} := p$.

If $q \neq \infty$, we have $(q^{\mathrm{PVI}}, p^{\mathrm{PVI}}) \in K_x$; in this manner, we get a rational map

$$M_\theta \dashrightarrow K_x : (q, p) \mapsto (q^{\mathrm{PVI}}, p^{\mathrm{PVI}}).$$

Using Theorem A and Proposition 5.1, it is easy to see that the map is actually regular and that it lifts to an isomorphism $M_\theta \to M_{(x,\lambda)}$. □

5.2. *Classical* PVI

The isomonodromy deformation of bundles with connections gives a system of differential equations on the coordinates q^{PVI}, p^{PVI} (the usual PVI). Here q^{PVI}, p^{PVI} are viewed as functions of $x_1, \ldots, x_4$, while $\lambda_i^{\pm}$ are fixed parameters. Let us recall the explicit formulas (which we adapted from [16]).

For simplicity, we assume, in addition to (5.1), that $x_4 = \infty$. Define the new parameters by $\kappa_i := \lambda_i^+ - \lambda_i^-$, $i = 1, \ldots, 4$, and let us replace the variable p^{PVI} with

$$\tilde{p}^{\mathrm{PVI}} := \left(\frac{p^{\mathrm{PVI}}}{dz}\right) - \sum_{i=1}^{3} \frac{\lambda_i^-}{z - x_i}.$$

Since $p^{\mathrm{PVI}} \in (\Omega_{\mathbb{P}^1}(x_1 + \cdots + x_4))_{q^{\mathrm{PVI}}}$, the ratio p^{PVI}/dz (if it is defined) is a number. The advantage of $\tilde{p}^{\mathrm{PVI}}$ is that the differential equations for q^{PVI}, $\tilde{p}^{\mathrm{PVI}}$ involve fewer parameters: κ_i's rather than $\lambda_i^{\pm}$'s.

Set also

$$\kappa_0 := \frac{1}{2}\Big(1 - \sum_{i=1}^{4} \kappa_i\Big),$$

and set $q_i := q^{\mathrm{PVI}} - x_i$, $i = 1, 2, 3$. Define the Hamiltonians h_i, $i = 1, 2, 3$, by

$$h_i := \frac{(q_1 q_2 q_3)(\tilde{p}^{\mathrm{PVI}})^2 - ((\kappa_i - 1) q_j q_k + \kappa_j q_i q_k + \kappa_k q_i q_j)\tilde{p}^{\mathrm{PVI}} + \kappa_0(\kappa_0 + \kappa_4)}{(x_i - x_j)(x_i - x_k)}.$$

The equations can then be written in the Hamiltonian form as

$$\frac{\partial q^{\mathrm{PVI}}}{\partial x_i} = \frac{\partial h_i}{\partial \tilde{p}^{\mathrm{PVI}}}, \qquad \frac{\partial \tilde{p}^{\mathrm{PVI}}}{\partial x_i} = -\frac{\partial h_i}{\partial q^{\mathrm{PVI}}} \quad (i = 1, 2, 3). \tag{5.2}$$

The system (5.2) can be reduced to the usual form of PVI as follows. Set

$$y := \frac{q^{\mathrm{PVI}} - x_1}{x_2 - x_1}, \qquad x := \frac{x_3 - x_1}{x_2 - x_1}.$$

Then (5.2) implies that y depends only on x, not on x_1, x_2, x_3, and that y satisfies the PVI equation

$$\begin{aligned}\frac{d^2 y}{dx^2} = {} & \frac{1}{2}\Big(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\Big)\Big(\frac{dy}{dx}\Big)^2 - \Big(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\Big)\frac{dy}{dx} \\ & + \frac{y(y-1)(y-x)}{x^2(x-1)^2}\Big(\kappa_4^2 - \kappa_1^2\frac{x}{y^2} + \kappa_2^2\frac{x-1}{(y-1)^2} + (1-\kappa_3^2)\frac{x(x-1)}{(y-x)^2}\Big).\end{aligned} \tag{5.3}$$

5.3. Isomonodromy deformation of d-connections

Let us prove Theorem D(1). Informally, we need to show that given $\theta \in \Theta_{2n}^{\sharp}$ and $\rho_1', \rho_2' \in \mathbb{C}$, any d-connection of type θ has a natural first-order deformation that is of type

$$\theta^{\epsilon} := (a_1, \ldots, a_{2n}; \rho_1 + \epsilon\rho_1', \rho_2 + \epsilon\rho_2', d_1, d_2; n).$$

Here ϵ is the parameter of the deformation, and all calculations are done modulo ϵ^2, that is, over the ring of dual numbers $\mathbb{C}^{\epsilon} := \mathbb{C}[\epsilon]/(\epsilon^2)$. First, let us prove a statement for formal power series.

PROPOSITION 5.2

Suppose that the matrix $A(z) = \sum_{i \le n} A_i z^i$ over $\mathbb{C}((z^{-1}))$ has formal type $(\rho_1, \ldots, \rho_m; d_1, \ldots, d_m; n)$ at infinity (see Proposition 3.1). For any collection $\rho_1', \ldots, \rho_m' \in$

$\mathbb{C}$, there exists a gauge matrix $R^\epsilon(z) = R(z) + \epsilon R'(z)$, where $R(z)$ is as in Proposition 3.1 (i.e., $R(z)$ is an invertible $(m \times m)$-matrix over $\mathbb{C}[[z^{-1}]]$), and $R'(z)$ is an $(m \times m)$-matrix over the ring of formal Laurent series $\mathbb{C}((z^{-1}))$ such that

$$R^\epsilon(z+1)^{-1} A(z) R^\epsilon(z) = \operatorname{diag}\left((\rho_1 + \rho_1'\epsilon)(z^n + d_1 z^{n-1}), \dots, (\rho_m + \epsilon\rho_m')(z^n + d_m z^{n-1})\right). \tag{5.4}$$

The matrix $R^\epsilon(z)$ is unique up to right multiplication by a diagonal matrix with entries in $\mathbb{C}^\epsilon$.

Proof
Condition (5.4) is equivalent to the two conditions

$$R(z+1)^{-1} A(z) R(z) = \operatorname{diag}(\rho_1 z^n + \rho_1 d_1 z^{n-1}, \dots, \rho_m z^n + \rho_m d_m z^{n-1}), \tag{5.5}$$

$$\begin{aligned} &R(z+1)^{-1} A(z) R'(z) - R(z+1)^{-1} R'(z+1) R(z+1)^{-1} A(z) R(z) \\ &\quad = \operatorname{diag}(\rho_1' z^n + \rho_1' d_1 z^{n-1}, \dots, \rho_m' z^n + \rho_m' d_m z^{n-1}). \end{aligned} \tag{5.6}$$

As $A(z)$ has formal type $(\rho_1, \dots, \rho_m; d_1, \dots, d_m; n)$ at infinity, there exists a matrix $R(z)$ satisfying (5.5); moreover, $R(z)$ is unique up to right multiplication by a constant diagonal matrix (see Proposition 3.1). Once (5.5) is satisfied, (5.6) can be rewritten as

$$B(z)S(z) - S(z+1)B(z) = \operatorname{diag}(\rho_1' z^n + \rho_1' d_1 z^{n-1}, \dots, \rho_m' z^n + \rho_m' d_m z^{n-1}), \tag{5.7}$$

where we set $B(z) := \operatorname{diag}(\rho_1 z^n + \rho_1 d_1 z^{n-1}, \dots, \rho_m z^n + \rho_m d_m z^{n-1})$ and $S(z) := R(z)^{-1} R'(z)$. One can view (5.7) as a difference equation on the matrix $S(z)$; it is easy to see that the only solutions whose matrix elements are Laurent series are given by $S(z) = \operatorname{diag}((\rho_1'/\rho_1)z + c_1, \dots, (\rho_m'/\rho_m)z + c_m)$, where c_i's are arbitrary constants. This implies the statement. □

Proposition 5.2 allows us to construct the natural first-order deformation and thus proving Theorem D(1). The construction is most easily described using the following well-known statement.

LEMMA 5.3
Let $\mathscr{L}$ be a vector bundle on $\mathbb{P}^1$, and let $\mathscr{S}(z) : \mathbb{C}^2 \xrightarrow{\sim} \mathscr{L}_z$ be a trivialization of $\mathscr{L}$ in the punctured formal neighborhood of ∞ (so $\mathscr{S}(z)$ is essentially a matrix whose entries belong to $\mathbb{C}((z^{-1}))$). Then there exists a unique vector bundle $\mathscr{L}^{\mathscr{S}}$ such that $\mathscr{L}$ and $\mathscr{L}^{\mathscr{S}}$ have equal restrictions to $\mathbb{P}^1 - \{\infty\}$ and that the map

$$\mathscr{S}(z) : \mathbb{C}^2 \to \mathscr{L}_z = (\mathscr{L}^{\mathscr{S}})_z$$

extends to a trivialization of $\mathscr{L}^{\mathscr{S}}$ in the formal neighborhood of ∞.

Notice that Lemma 5.3 still works when $\mathscr{S}$ depends on parameters. In this case, the modification $\mathscr{L}^{\mathscr{S}}$ also depends on the parameters.

Proof of Theorem D(1)
Take $\rho = (\rho_1, \rho_2) \in P$, and take $(\mathscr{L}, \mathscr{A}) \in M_{\theta(\rho)}$. Take a tangent vector $\tau = \rho_1' \frac{\partial}{\partial \rho_1} + \rho_2' \frac{\partial}{\partial \rho_2}$ to P at ρ. Let us construct a natural lifting of τ to a tangent vector τ_M to M at $(\mathscr{L}, \mathscr{A}) \in M$.

Choose a trivialization $\mathscr{S}(z) : \mathbb{C}^2 \xrightarrow{\sim} \mathscr{L}_z$ on the neighborhood of $\infty \in \mathbb{P}^1$. The matrix

$$A(z) := \mathscr{S}^{-1}(z+1)\mathscr{A}(z)\mathscr{S}(z)$$

of $\mathscr{A}$ relative to $\mathscr{S}$ satisfies the assumption of Proposition 5.2. Let us set $\mathscr{S}^\epsilon(z) := \mathscr{S}(z)R^\epsilon(z)$, where the matrix $R^\epsilon(z)$ is given by Proposition 5.2. We can view $\mathscr{S}^\epsilon(z)$ as a trivialization of $\mathscr{L}$ in the punctured formal neighborhood of $\infty \in \mathbb{P}^1$, which depends on $\epsilon \in \mathbb{C}^\epsilon$. Lemma 5.3 defines a vector bundle $\mathscr{L}^\epsilon := \mathscr{L}^{\mathscr{S}^\epsilon}$, which depends on ϵ.

$\mathscr{L}^\epsilon$ and $\mathscr{L}$ coincide on $\mathbb{P}^1 - \{\infty\}$ (for any value of the parameter ϵ); thus, the d-connection $\mathscr{A}$ on $\mathscr{L}$ induces a d-connection $\mathscr{A}^\epsilon$ on $\mathscr{L}^\epsilon$. Notice also that when $\epsilon = 0$, we have $\mathscr{L}^\epsilon = \mathscr{L}$, $\mathscr{A}^\epsilon = \mathscr{A}$. The pair $(\mathscr{L}^\epsilon, \mathscr{A}^\epsilon)$ defines a tangent vector τ_M to M at $(\mathscr{L}, \mathscr{A})$. The vector τ_M does not depend on the choice of R^ϵ. It is easy to see that as τ and $(\mathscr{L}, \mathscr{A})$ vary, the lifting τ_M defines a flat algebraic connection on $M \to P$. □

5.4. Isomonodromy deformation for $2n = 4$
Suppose now that $2n = 4$, $\deg(\theta) = -1$. Then the construction of Section 5.3 can be reformulated more explicitly. Instead of working with d-connections, let us consider their matrices (i.e., we think of M_θ as a quotient X_θ/G; see Corollary 4.2).

Let $(\mathscr{L}, \mathscr{A})$ and $(\mathscr{L}^\epsilon, \mathscr{A}^\epsilon)$ be as in the proof of Theorem D(1). Choose a trivialization $\mathscr{S}^\epsilon : \mathscr{O} \oplus \mathscr{O}(-1) \xrightarrow{\sim} \mathscr{L}^\epsilon$ (depending on ϵ). When $\epsilon = 0$, $\mathscr{S}^\epsilon$ becomes a trivialization $\mathscr{S} : \mathscr{O} \oplus \mathscr{O}(-1) \xrightarrow{\sim} \mathscr{L}$. Let A be the matrix of $\mathscr{A}$ relative to $\mathscr{S}$, and let A^ϵ be the matrix of $\mathscr{A}^\epsilon$ relative to $\mathscr{S}^\epsilon$. Let us summarize the properties of A^ϵ in the following.

PROPOSITION 5.4
The matrix $A^\epsilon(z) = A(z) + \epsilon A'(z)$, where

$$A' = \begin{bmatrix} a_{11}' & a_{12}' \\ a_{21}' & a_{22}' \end{bmatrix}, \qquad \begin{aligned} &a_{11}', a_{22}' \in \Gamma\big(\mathbb{P}^1, \mathscr{O}(2)\big), \\ &a_{12}' \in \Gamma\big(\mathbb{P}^1, \mathscr{O}(3)\big), \\ &a_{21}' \in \Gamma\big(\mathbb{P}^1, \mathscr{O}(1)\big), \end{aligned} \tag{5.8}$$

satisfies the following conditions.

(1) *For some* (2×2)*-matrix* $S^\epsilon(z) = 1 + \epsilon S'(z)$*, where the entries of* $S'(z)$ *are polynomials in* z *(of arbitrary degree), we have* $A^\epsilon(z) = S^\epsilon(z+1)^{-1} A(z) S^\epsilon(z)$.

(2) *For some* (2×2)*-matrix*

$$R^\epsilon(z) = \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \bigl(T(z^{-1}) + \epsilon T'(z^{-1})\bigr),$$

where T, T' *are* (2×2)*-matrices over* $\mathbb{C}[[z^{-1}]]$ *and* T *is invertible (i.e.,* $\det(T|_{z^{-1}=0}) \neq 0$*), we have*

$$R^\epsilon(z+1)^{-1} A^\epsilon(z) R^\epsilon(z) = \operatorname{diag}\bigl((\rho_1 + \epsilon\rho_1')(z^2 + d_1 z), (\rho_2 + \epsilon\rho_2')(z^2 + d_2 z)\bigr).$$

Conversely, a matrix A^ϵ with such properties corresponds to the continuous isomonodromy deformation of Theorem D(1). Actually, we can reformulate Theorem D(1) (for $2n = 4$, $\deg(\theta) = -1$) as the following statement.

PROPOSITION 5.5
Let $A(z) \in X_\theta$*, let* $\theta \in \Theta_4^\sharp$*, and let* $\deg(\theta) = -1$.

(1) *There is a deformation* $A^\epsilon(z)$ *which satisfies the conditions of Proposition 5.4.*

(2) $A^\epsilon(z)$ *is unique up to a d-gauge transformation*

$$A^\epsilon(z) \mapsto R^\epsilon(z+1)^{-1} A^\epsilon(z) R^\epsilon(z)$$

for a gauge matrix $R^\epsilon(z) = 1 + \epsilon R'(z)$*, where* $R'(z)$ *is of the form*

$$R'(z) = \begin{bmatrix} r'_{11} & r'_{12} \\ 0 & r'_{22} \end{bmatrix}, \qquad \begin{array}{l} r'_{11}, r'_{22} \in \mathbb{C}, \\ r'_{12} \in \Gamma\bigl(\mathbb{P}^1, \mathscr{O}(1)\bigr). \end{array}$$

5.5. *Isomonodromy deformation of d-connections as* PVI

Let us now use coordinates q, p on M_θ to write the connection of Theorem D(1) as a system of differential equations on p and q. Suppose that $(\mathscr{L}, \mathscr{A}) \in M_\theta$, and let $A \in X_\theta$ be the matrix of $\mathscr{A}$ relative to some trivialization $\mathscr{S} : \mathscr{O} \oplus \mathscr{O}(-1) \xrightarrow{\sim} \mathscr{L}$. We need to find a matrix A^ϵ which satisfies the conditions of Proposition 5.4. As in Section 4.4, it suffices to do so when $(\mathscr{L}, \mathscr{A})$ belong to a dense subset of M_θ; we can thus assume that $q(\mathscr{L}, \mathscr{A}) \neq \infty$. We can then pick $\mathscr{S}$ such that A is of the form (4.13).

We look for A^ϵ in the form

$$A^\epsilon(z) = \begin{bmatrix} a_{11}^\epsilon & a_{12}^\epsilon \\ a_{21}^\epsilon & a_{22}^\epsilon \end{bmatrix} = S^\epsilon(z+1)^{-1} A(z) S^\epsilon(z)$$

for the gauge matrix

$$S^{\epsilon}(z) = 1 + \begin{bmatrix} s'_{11} & s'_{12} \\ s'_{21} & s'_{22} \end{bmatrix} \epsilon, \qquad \begin{aligned} & s'_{11}, s'_{22} \in \Gamma\big(\mathbb{P}^{1}, \mathcal{O}(1)\big), \\ & s'_{12} \in \Gamma\big(\mathbb{P}^{1}, \mathcal{O}(2)\big), \\ & s'_{21} \in \mathbb{C}. \end{aligned}$$

(Actually, the proof of Proposition 5.2 shows that A^{ϵ} is necessarily of this form.) Then A^{ϵ} automatically satisfies Proposition 5.4(1), so we only need to make sure that Proposition 5.4(2) is satisfied. From Lemma 4.1 (which still holds for d-connections that depend on ϵ), we see that Proposition 5.4(2) is equivalent to the equations

$$\det(A^{\epsilon}) = (z-a_1)(z-a_2)(z-a_3)(z-a_4)\rho_1^{\epsilon}\rho_2^{\epsilon},$$
$$a_{11}^{\epsilon} + a_{22}^{\epsilon}(1+z^{-1}) = (\rho_1^{\epsilon}+\rho_2^{\epsilon})z^2 + (d_1\rho_1^{\epsilon}+d_2\rho_2^{\epsilon})z + t(z^{-1}),$$

where $\rho_i^{\epsilon} = \rho_i + \rho_i'\epsilon$ and $t(z^{-1}) \in \mathbb{C}^{\epsilon}[[z^{-1}]]$ is a Taylor series in z^{-1} with coefficients in $\mathbb{C}^{\epsilon}$. Solving these equations, we can find formulas for q', p' in terms of ρ_1', ρ_2', and θ; here q' and p' are determined by the conditions

$$a_{21}^{\epsilon}(q+\epsilon q') = 0 \in \mathbb{C}^{\epsilon}, \qquad a_{11}^{\epsilon}(q+\epsilon q') = (p+\epsilon p')(q+\epsilon q'-a_3)(q+\epsilon q'-a_4).$$

The formulas for q' and p' can then be viewed as a system on differential equations on q and p (considered as functions of ρ_i):

$$\begin{aligned} dq &= \frac{\rho_1 d\rho_2 - \rho_2 d\rho_1}{\rho_1-\rho_2}\Big(\frac{p(q-a_3)(q-a_4)}{\rho_1\rho_2} - \frac{(q-a_1)(q-a_2)}{p}\Big), \\ dp &= p\frac{d\rho_1 - d\rho_2}{\rho_1-\rho_2} + \frac{\rho_1 d\rho_2 - \rho_2 d\rho_1}{\rho_1-\rho_2}\Big(a_1+a_2-2q+\frac{p^2(a_3+a_4-2q)}{\rho_1\rho_2} \\ &\quad + \frac{p}{\rho_1\rho_2}\big(d_1\rho_1+d_2\rho_2+2q(\rho_1+\rho_2)\big)\Big). \end{aligned} \tag{5.9}$$

Proof of Theorem D(2)
We need to verify that (5.9) is obtained from the PVI (5.2) by plugging in the formulas for p^{PVI}, q^{PVI}, x_i's, and $\lambda_i^{\pm}$'s (from Theorem C and Section 5.1). This is a straightforward calculation. □

Remark 5.6
Theorem D(2) can also be proved by an indirect argument. Indeed, both PVI and (5.9) define algebraic connections on the family $M \to P$ from Theorem D. The difference between two such connections is a vector field on the moduli space M_{θ}; on the other hand, it is known that M_{θ} has no nonzero global vector fields (see [2, Theorem 3, Lemma 3], [28, Proposition 2.1]).

Still another, more geometric, proof of Theorem D(2) uses the Mellin transform described in Section 5.6. It is easy to see that, under the transform, the continuous isomonodromy deformation of d-connections (from Theorem D(1)) corresponds to the isomonodromy deformation of ordinary connections, which is described by the sixth Painlevé equation.

5.6. *Mellin transform*

In this section (which is completely independent from the rest of the article), we sketch the geometric construction underlying Theorem C. Fix $\theta \in \Theta_4^\sharp$, $x \in X$, and $\lambda \in \Lambda$ as in Theorem C.

Take $(\hat{\mathscr{L}}, \nabla) \in M_{(x,\lambda)}$. For any $z \in \mathbb{C}$, consider the connection

$$\nabla_z := \nabla - z\zeta^{-1} d\zeta : \hat{\mathscr{L}} \to \hat{\mathscr{L}} \times \Omega_{\mathbb{P}^1}(x_1 + x_2 + x_3 + x_4),$$

where we denote by ζ the coordinate on $\mathbb{P}^{\mathbf{1}}$. Recall that $x_1 = 0, x_4 = \infty$, so subtraction of $z\zeta^{-1}d\zeta$ from ∇ does not introduce new poles. Denote by $\hat{\mathscr{L}}_{*!} \supset \hat{\mathscr{L}}$ the smallest quasi-coherent sheaf that contains $\hat{\mathscr{L}}$ and such that $\nabla_z(\hat{\mathscr{L}}_{*!}) \subset \hat{\mathscr{L}}_{*!}$ for all $z \in \mathbb{C}$. (In terms of D-modules, $\hat{\mathscr{L}}_{*!}$ can be constructed by taking the intermediate extension of $(\hat{\mathscr{L}}, \nabla_z)$ from $\mathbb{P}^{\mathbf{1}} - \{x_1, x_2, x_3, x_4\}$ to $\mathbb{P}^{\mathbf{1}} - \{0, \infty\}$ and then extending to $\mathbb{P}^{\mathbf{1}}$.) Consider the first de Rham cohomology group $H^1_{DR}(\hat{\mathscr{L}}_{*!}, \nabla_z)$. Since $\hat{\mathscr{L}}_{*!}$ and $\hat{\mathscr{L}}_{*!} \otimes \Omega_{\mathbb{P}^1}$ have no higher cohomologies, it can be computed by the formula

$$H^1_{\mathrm{DR}}(\hat{\mathscr{L}}_{*!}, \nabla_z) = \operatorname{coker}\bigl(\nabla_z : \Gamma(\mathbb{P}^{\mathbf{1}}, \hat{\mathscr{L}}_{*!}) \to \Gamma(\mathbb{P}^{\mathbf{1}}, \hat{\mathscr{L}}_{*!} \otimes \Omega_{\mathbb{P}^1})\bigr).$$

$H^1_{\mathrm{DR}}(\hat{\mathscr{L}}_{*!}, \nabla_z)$ depends on z in an algebraic way; more precisely, it is the fiber over $z \in \mathbb{C}$ of a natural quasi-coherent sheaf $\mathscr{L}_{*!}$ on $\mathbb{P}^{\mathbf{1}} - \{\infty\}$. The sheaf $\mathscr{L}_{*!}$ is the Mellin transform of $\hat{\mathscr{L}}_{*!}$ in terms of [21].

Consider now the rational map $a : \hat{\mathscr{L}} \dashrightarrow \hat{\mathscr{L}} : s \mapsto \zeta s$. Note that a satisfies the relation $a \circ \nabla_z = \nabla_{z+1} \circ a$. It is also easy to see that a induces an automorphism of $\hat{\mathscr{L}}_{*!}$; therefore, it becomes an isomorphism of D-modules (i.e., quasi-coherent sheaves with connections) $(\hat{\mathscr{L}}_{*!}, \nabla_z) \xrightarrow{\sim} (\hat{\mathscr{L}}_{*!}, \nabla_{z+1})$. Hence, a yields an identification,

$$\widetilde{\mathscr{A}}(z) : H^1_{\mathrm{DR}}(\hat{\mathscr{L}}_{*!}, \nabla_z) \xrightarrow{\sim} H^1_{\mathrm{DR}}(\hat{\mathscr{L}}_{*!}, \nabla_{z+1}).$$

As $z \in \mathbb{C}$ varies, we can view $\widetilde{\mathscr{A}}(z)$ as a d-connection on the quasi-coherent sheaf $\mathscr{L}_{*!}$. One can check that $\mathscr{L}_{*!}$ contains a unique coherent locally free subsheaf of rank 2 (i.e., a rank 2 vector bundle) $\mathscr{L} \subset \mathscr{L}_{*!}$ such that

$$\mathscr{A}(z) := (z - a_3)(z - a_4)\widetilde{\mathscr{A}}(z)$$

is a d-connection of type θ on $\mathscr{L}$. The correspondence

$$(\hat{\mathscr{L}}, \nabla) \mapsto (\mathscr{L}, \mathscr{A})$$

gives a map $M_{(x,\lambda)} \to M_\theta$. Note that the scalar multiple $(z-a_3)(z-a_4)$ also appears in Remark 4.5.

To describe the inverse map $M_\theta \to M_{(x,\lambda)}$, let us reconstruct $(\hat{\mathscr{L}}, \nabla)$ from $(\mathscr{L}, \mathscr{A})$. For any $\zeta \in \mathbb{C} - \{0\}$, consider the d-connection

$$\widetilde{\mathscr{A}}_\zeta := \zeta^{-1} \frac{\mathscr{A}}{(z-a_3)(z-a_4)}$$

on $\mathscr{L}$. Let $\mathscr{L}_{*!}$ be the smallest quasi-coherent sheaf on $\mathbb{P}^1$ which contains $\mathscr{L}$ and such that $\widetilde{\mathscr{A}}_\zeta$ induces an isomorphism $(\mathscr{L}_{*!})_z \to (\mathscr{L}_{*!})_{z+1}$ for all z and ζ. (The quotient $\mathscr{L}_{*!}/\mathscr{L}$ is the direct sum of length 1 skyscraper sheaves supported at points $a_1, a_1-1, a_1-2, \dots; a_2, a_2-1, \dots; a_3+1, a_3+2, \dots; a_4+1, a_4+2, \dots$.) For any $\zeta \in \mathbb{C} - \{0\}$, we obtain a structure of a $\mathbb{Z}$-equivariant sheaf on $\mathscr{L}_{*!}$, where $1 \in \mathbb{Z}$ acts on $\mathbb{P}^1$ by $z \mapsto z+1$ and on $\mathscr{L}_{*!}$ by $\widetilde{\mathscr{A}}_\zeta$. (In some sense, $\mathscr{L}_{*!}$ is obtained from $\mathscr{L}$ by an intermediate extension for $\mathbb{Z}$-equivariant sheaves.) Consider the corresponding equivariant cohomology group $H^1_{\mathbb{Z}}(\mathscr{L}_{*!}, \widetilde{\mathscr{A}}_\zeta)$, which can be computed by the formula

$$H^1_{\mathbb{Z}}(\mathscr{L}_{*!}, \widetilde{\mathscr{A}}_\zeta) = \operatorname{coker}\left(\widetilde{\mathscr{A}}_\zeta - 1 : \Gamma(\mathbb{P}^1, \mathscr{L}_{*!}) \to \Gamma(\mathbb{P}^1, \mathscr{L}_{*!})\right).$$

$H^1_{\mathbb{Z}}(\mathscr{L}_{*!}, \widetilde{\mathscr{A}}_\zeta)$ is the fiber over $\zeta \in \mathbb{C} - \{0\}$ of the quasi-coherent sheaf $\hat{\mathscr{L}}_{*!}$ on $\mathbb{P}^1 - \{\infty, 0\}$.

For every $\zeta \in \mathbb{C} - \{0\}$, consider the rational map

$$\delta(\zeta) : \mathscr{L} \dashrightarrow \mathscr{L} : s \mapsto z\zeta^{-1} s.$$

The map $\delta(\zeta)$ induces a regular map $\mathscr{L}_{*!} \to \mathscr{L}_{*!}$ and, therefore, a map

$$\delta_*(\zeta) : \Gamma(\mathbb{P}^1, \mathscr{L}_{*!}) \to \Gamma(\mathbb{P}^1, \mathscr{L}_{*!}).$$

The map $\delta_*(\zeta)$ satisfies the commutativity relation

$$\delta_*(\zeta)\widetilde{A}_\zeta = \widetilde{A}_\zeta \delta_*(\zeta) - \frac{d\widetilde{A}_\zeta}{d\zeta}.$$

Now let us consider the trivial quasi-coherent sheaf over $\mathbb{P}^1 - \{0, \infty\}$ whose fiber over every point $\zeta \in \mathbb{P}^1 - \{0, \infty\}$ equals $\Gamma(\mathbb{P}^1, \mathscr{L}_{*!})$. The formula $\widetilde{\mathscr{A}}_\zeta - 1$ gives an endomorphism of this sheaf; the cokernel of the endomorphism is $\hat{\mathscr{L}}_{*!}$. Notice now that $\widetilde{\mathscr{A}}_\zeta - 1$ is horizontal with respect to the connection $\nabla = d + \delta_*(\zeta) d\zeta$ on the sheaf. Therefore, ∇ induces a connection $\hat{\mathscr{L}}_{*!} \to \hat{\mathscr{L}}_{*!} \otimes \Omega_{\mathbb{P}^1}$ (which we also denote by ∇). Finally, $\hat{\mathscr{L}} \subset \hat{\mathscr{L}}_{*!}$ can be reconstructed as the only coherent locally free subsheaf of rank 2 such that ∇ is a connection of type (x, λ) on $\hat{\mathscr{L}}$.

6. Difference PVI

In this section we study M_θ for $\theta \in \Theta_k^\flat$. We need suitable versions of several statements from Section 3.

6.1

PROPOSITION 6.1 (cf. Proposition 3.1)
Suppose that the matrix $A(z) = \sum_{i \le n} A_i z^i$ over $\mathbb{C}((z^{-1}))$ satisfies the following condition:

$$\begin{array}{l}\textit{The leading term } A_n \textit{ is a nonzero scalar matrix,}\\ \textit{while all eigenvalues of the next term } A_{n-1} \textit{ are distinct.}\end{array} \tag{6.1}$$

Then there exists a gauge matrix $R(z) = \sum_{i \le 0} R_i z^i$ with invertible R_0 such that

$$R(z+1)^{-1} A(z) R(z) = A'_n z^n + A'_{n-1} z^{n-1}, \tag{6.2}$$

where A'_n and A'_{n-1} are diagonal. $R(z)$ is uniquely determined up to right multiplication by a permutation matrix and a constant diagonal matrix.

As before, we denote the only eigenvalue of A'_n by $\rho = \rho_1 = \cdots = \rho_n$, and we denote the eigenvalues of A'_{n-1} by $\rho d_1, \ldots, \rho d_n$. It is easy to see that $A_n = A'_n$ (so ρ is also the eigenvalue of A_n) and that A_{n-1} is conjugate to A'_{n-1} (so $\rho d_1, \ldots, \rho d_n$ are also eigenvalues of A'_{n-1}; this can be thought of as a version of Remark 3.3).

PROPOSITION 6.2 (cf. Proposition 3.8)
Suppose that $\theta = (a_1, \ldots, a_k; \rho, \rho, d_1, d_2; n)$, and suppose that $d_1 \neq d_2$. Let $(\mathscr{L}', \mathscr{A}')$ be an elementary upper modification of $(\mathscr{L}, \mathscr{A}) \in M_\theta$ given by $(x \in \mathbb{P}^1; l \subset \mathscr{L}_x)$. Then the only cases when $(\mathscr{L}', \mathscr{A}')$ belongs to $M_{\theta'}$ for some $\theta' \in \Theta$ are as follows.

(1) *If $x = \infty$, then l must be an eigenspace of $\mathscr{A}_{n-1} : \mathscr{L}_\infty \to \mathscr{L}_\infty$ (the second term of $\mathscr{A} = \rho z^n + \mathscr{A}_{n-1} z^{n-1} +$ lower-order terms). If, for instance, $l = \ker(\mathscr{A}_{n-1} - \rho d_1) \subset \mathscr{L}_\infty$, then $\theta' = (a_1, \ldots, a_k; \rho_1, \rho_2, d_1 - 1, d_2; n)$, and an analogous formula holds when $l = \ker(\mathscr{A}_{n-1} - \rho d_2)$.*

(2) *If $x = a_i$ is a zero of $\mathscr{A}$ and $x - 1 \neq a_j$ is not, then l must be the kernel of $\mathscr{A}(x) : \mathscr{L}_x \to \mathscr{L}_{x+1}$; in this case, $\theta' = (a_1, \ldots, a_i - 1, \ldots, a_k; \rho_1, \rho_2, d_1, d_2; n)$.*

In either case, the elementary modifications define an isomorphism $M_\theta \xrightarrow{\sim} M_{\theta'}$.

COROLLARY 6.3
Suppose that $\theta \in \Theta_k$ satisfies (2.5) and (2.8). Then M_θ is naturally isomorphic to $M_{\theta'}$ whenever θ' is obtained from θ by adding integers to a_i's and d_i's.

LEMMA 6.4 (cf. Corollary 3.13)
Suppose that $(\mathscr{L}, \mathscr{A}) \in M_\theta$, and suppose that $\theta \in \Theta_{2n}$ satisfies (2.4) and (2.8). If $\mathscr{L} \simeq \mathscr{O}(n_1) \oplus \mathscr{O}(n_2)$, then $|n_1 - n_2| \leq n - 1$.

Proof
The proof repeats that of Corollary 3.13; the only difference is that the order of pole of α at ∞ cannot exceed $n-1$ (because the coefficient of z^n in α is an off-diagonal element of a scalar matrix, i.e., zero). □

6.2. Proofs of Theorems E, F
Proof of Theorem E
The proof of Theorem E follows the same ideas as the proof of Theorem A. Fix $\theta \in \Theta_6^\flat$, $\deg(\theta) = -1$. For any $(\mathscr{L}, \mathscr{A}) \in M_\theta$, Lemma 6.4 implies that $\mathscr{L} \simeq \mathscr{O} \oplus \mathscr{O}(-1)$. Choosing an isomorphism $\mathscr{S} : \mathscr{O} \oplus \mathscr{O}(-1) \xrightarrow{\sim} \mathscr{L}$, we can write $\mathscr{A}$ as a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad \begin{aligned} &a_{11}, a_{22} \in \Gamma\big(\mathbb{P}^1, \mathscr{O}(3)\big), \\ &a_{12} \in \Gamma\big(\mathbb{P}^1, \mathscr{O}(4)\big), \\ &a_{21} \in \Gamma\big(\mathbb{P}^1, \mathscr{O}(2)\big). \end{aligned} \tag{6.3}$$

Choosing a different isomorphism $\mathscr{S}$ replaces A with its d-gauge transformation (4.3), where the gauge matrix R is given by (4.2).

LEMMA 6.5 (cf. Lemma 4.1)
Let $\mathscr{A}$ be a d-connection on $\mathscr{O} \oplus \mathscr{O}(-1)$; its matrix A is of the form (6.3). We claim that $\mathscr{A}$ is of type θ if and only if A satisfies the following conditions:

$$a_{12} \in \Gamma\big(\mathbb{P}^1, \mathscr{O}(3)\big), \qquad a_{21} \in \Gamma\big(\mathbb{P}^1, \mathscr{O}(1)\big), \qquad a_{11} - \rho z^3, a_{22} - \rho z^3 \in \Gamma\big(\mathbb{P}^1, \mathscr{O}(2)\big),$$
$$\det(A) = (z-a_1)(z-a_2)(z-a_3)(z-a_4)(z-a_5)(z-a_6)\rho^2,$$
$$(a_{11} - \rho z^3)\big(a_{22}(1+z^{-1}) - \rho z^3\big) - a_{12}a_{21} = d_1 d_2 \rho^2 z^4 + \textit{lower-order terms}.$$

Remark. The last condition of the lemma can be more naturally written as

$$\det\big(R(z+1)^{-1} A R(z) - \rho z^3\big) = d_1 d_2 \rho^2 z^4 + \text{lower-order terms},$$

where $R(z) := \operatorname{diag}(1, z^{-1})$ is a trivialization of $\mathscr{O} \oplus \mathscr{O}(-1)$ near $\infty \in \mathbb{P}^1$.

We can now think of M_θ as the quotient of the space of all matrices (6.3) that satisfy Lemma 6.5 modulo d-gauge transformations with gauge matrices (4.2) (cf. Corollary 4.2). For any matrix (6.3) which satisfies Lemma 6.5, denote by $q \in \mathbb{P}^1$ the only zero of a_{21}, and set $\tilde{p} := a_{11}(q) \in (\mathscr{O}(3))_q$. It is easy to see that q and $\tilde{p}$ do not change under d-gauge transformations with gauge matrices (4.2); therefore, $\tilde{P} := (q, \tilde{p})$ can be viewed as a map $M_\theta \to \widetilde{K}$, where $\widetilde{K} := \mathbb{V}(\mathscr{O}(3)^\vee)$ is the total space of the line bundle $\mathscr{O}(3)$. We can now use the map $\tilde{P}$ for a geometric description of M_θ. (We are using the notation of Theorem 4.4.)

THEOREM 6.6

(1) *The map $\tilde{P} : M_\theta \to \widetilde{K}$ is a regular birational morphism of smooth algebraic surfaces.*

(2) *Let $\tilde{\sigma}_1 : \widetilde{K}_1 \to \widetilde{K}$ be the blowup of $\widetilde{K}$ at the following seven points: $(a_i, 0(a_i))$ $(i = 1, \dots, 6)$ and $(\infty, (\rho z^3)(\infty))$. Let $\sigma_2 : \widetilde{K}_2 \to \widetilde{K}_1$ be the blowup of $\widetilde{K}_1$ at the two points $(\infty, (\rho z^3 + \rho d_j z^2)'(\infty))$, $j = 1, 2$. (These points belong to the preimage $\tilde{\sigma}_1^{-1}(\infty, (\rho z^3)(\infty)) \subset \widetilde{K}_1$.) Then the map $\widetilde{P}$ induces an open embedding $\widetilde{P}_2 : M_\theta \hookrightarrow \widetilde{K}_2$.*

(3) *The complement to $\widetilde{P}_2(M_\theta)$ in $\widetilde{K}_2$ is the union of the proper preimages of the following curves: the zero section $\{(z, 0(z)) : z \in \mathbb{P}^1\} \subset \widetilde{K}$, the fiber at infinity $\{(\infty, az^3(\infty)) : a \in \mathbb{C}\} \subset \widetilde{K}$, and the exceptional curve $\tilde{\sigma}_1^{-1}(\infty, (\rho z^3)(\infty)) \subset \widetilde{K}_1$.*

The proof of Theorem 6.6 is completely analogous to that of Theorem 4.4 (see Section 4.3). Now Theorem E easily follows; we set

$$p := \frac{\tilde{p}}{(q - a_4)(q - a_5)(q - a_6)},$$

and it is not hard to check that the map $P := (q, p) : M_\theta \to (\mathbb{P}^1)^2$ (which is birational by Theorem 6.6) is regular and induces an embedding $M_\theta \hookrightarrow K_2$ with the required properties. □

Proof of Theorem F

The proof repeats the proof of Theorem B (given in Section 4.4) almost word for word. (Of course, the calculations involved are somewhat more complicated.) The only real difference is formulas (4.13) and (4.14); the corresponding formulas in our case are

$$A = \begin{bmatrix} z^3 - q^3 + p(q - a_4)(q - a_5)(q - a_6) & a_{12} \\ z - q & a_{22} \end{bmatrix}, \quad a_{22}, a_{12} \in \Gamma\big(\mathbb{P}^1, \mathscr{O}(3)\big),$$

$$A' = \begin{bmatrix} z^3 - (q')^3 + p'(q' - a_3)(q' - a_4)(q' - a_6) & a'_{12} \\ z - q' & a'_{22} \end{bmatrix}, \quad a'_{22}, a'_{12} \in \Gamma\big(\mathbb{P}^1, \mathscr{O}(3)\big).$$

□

6.3. Degeneration to difference PV

Given

$$\tilde{\theta} = (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4; \tilde{\rho}_1, \tilde{\rho}_2, \tilde{d}_1, \tilde{d}_2; 2) \in \Theta_4^\sharp,$$

let us define $\theta(t)$ for $t \in \mathbb{C} - \{0\}$ by

$$\theta(t) = \left(\tilde{a}_1, \tilde{a}_2, -\frac{\tilde{\rho}_1}{t}, -\frac{\tilde{\rho}_2}{t}, \tilde{a}_3, \tilde{a}_4; 1, 1, \tilde{d}_1 + \frac{\tilde{\rho}_1}{t}, \tilde{d}_2 + \frac{\tilde{\rho}_2}{t}; 3\right).$$

Clearly, $\theta(t) \in \Theta_6^\flat$ for all but countably many t. Denote the components of $\theta = \theta(t)$ by $a_i = a_i(t)$, $d_j = d_j(t)$. Formulas (2.9) define a family of equations depending on parameter $t \in \mathbb{C} - \{0\}$. Let us show that the difference PV (2.7) is the limit of this family as $t \to 0$.

Replace p with a new variable $\tilde{p} := (\tilde{\rho}_2 + qt)p$; accordingly, set $\tilde{p}' := (\tilde{\rho}_2 + q't)p'$. After we plug the formulas for $\theta(t)$, $\tilde{p}$, and $\tilde{p}'$ into (2.9), it becomes the system

$$\begin{cases} q + q' = \tilde{a}_3 + \tilde{a}_4 + \dfrac{\tilde{\rho}_1(\tilde{d}_1 + \tilde{a}_4 + \tilde{a}_5)}{\tilde{p} - \tilde{\rho}_1} + \dfrac{\tilde{\rho}_2(\tilde{d}_2 + \tilde{a}_4 + \tilde{a}_5)}{\tilde{p} - \tilde{\rho}_2} + O(t), \\ \tilde{p}\tilde{p}' = \dfrac{(q' - \tilde{a}_1 + 1)(q' - \tilde{a}_2 + 1)}{(q' - \tilde{a}_3)(q' - \tilde{a}_4)} \cdot \tilde{\rho}_1\tilde{\rho}_2 + O(t), \end{cases} \tag{6.4}$$

where $O(t)$ stands for a Taylor series in t with no constant term. This is exactly the difference PV equation (2.7).

Remark 6.7

The degeneration of (2.9) to (2.7) has a clear geometric meaning; let us sketch it. It is easy to construct a family of moduli spaces $\nu : N \to \mathbb{A}^{\mathbf{1}}$ such that the fiber $\nu^{-1}(t)$ over $t \in \mathbb{A}^{\mathbf{1}} - \{0\}$ equals $M_{\theta(t)}$ whenever $\theta(t) \in \Theta_6^\flat$, while $\nu^{-1}(0) = M_{\tilde{\theta}}$. Similarly, one can define a family $\nu' : N' \to \mathbb{A}^{\mathbf{1}}$ such that $(\nu')^{-1}(t) = M_{\theta'(t)}$ if $t \neq 0$, $\theta(t) \in \Theta_6^\flat$, and such that $(\nu')^{-1}(0) = M_{\tilde{\theta}'}$. Here

$$\tilde{\theta}' = (\tilde{a}_1 + 1, \tilde{a}_2 + 1, \tilde{a}_3, \tilde{a}_4; \tilde{\rho}_1, \tilde{\rho}_2, \tilde{d}_1 - 1, \tilde{d}_2 - 1; 2),$$

$$\theta'(t) = \left(\tilde{a}_1 + 1, \tilde{a}_2 + 1, -\frac{\tilde{\rho}_1}{t}, -\frac{\tilde{\rho}_2}{t}, \tilde{a}_3, \tilde{a}_4; 1, 1, \tilde{d}_1 + \frac{\tilde{\rho}_1}{t} - 1, \tilde{d}_2 + \frac{\tilde{\rho}_2}{t} - 1; 3\right).$$

The modification of d-connections defines a rational isomorphism $N \overset{\sim}{\dashrightarrow} N'$ which is regular over a neighborhood of $0 \in \mathbb{A}^{\mathbf{1}}$; this isomorphism is given by (2.9) if $t \neq 0$ and $\theta(t) \in \Theta_6^\flat$, and it is given by (2.7) if $t = 0$.

6.4. Degeneration to classical PVI

Let us now show how difference PVI (2.9) degenerates into the classical PVI. Fix

$$\tilde{\theta} = (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4; \tilde{\rho}_1, \tilde{\rho}_2, \tilde{d}_1, \tilde{d}_2; 2) \in \Theta_4^\sharp,$$

and set

$$\theta(t) := \left(-\frac{\tilde{\rho}_1}{t}, -\frac{\tilde{\rho}_2}{t}, \tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4; 1, 1, \tilde{d}_1 + \frac{\tilde{\rho}_1}{t}, \tilde{d}_2 + \frac{\tilde{\rho}_2}{t}; 3\right) \quad (t \in \mathbb{C} - \{0\}).$$

Again, $\theta(t) \in \Theta_6^\flat$ for all but countably many t. Let us also set

$$\theta'(t) := \Big(-\frac{\tilde\rho_1}{t} - 1, -\frac{\tilde\rho_2}{t} - 1, \tilde a_1, \tilde a_2, \tilde a_3, \tilde a_4; 1, 1, \tilde d_1 + \frac{\tilde\rho_1}{t} + 1, \tilde d_2 + \frac{\tilde\rho_2}{t} + 1; 3\Big),$$

so that dPVI is an isomorphism $M_{\theta(t)} \xrightarrow{\sim} M_{\theta'(t)}$. Note that the formula for $\theta'(t)$ is obtained from the formula for $\theta(t)$ if we substitute

$$\tilde\rho_i' := \tilde\rho_i + t \tag{6.5}$$

for $\tilde\rho_i$, $i = 1, 2$.

Let us replace p with $\tilde p := (q - \tilde a_2)tp$; accordingly, set $\tilde p' := (q' - \tilde a_2)tp'$. Then (2.9) can be written as

$$\begin{cases} \dfrac{q'-q}{t} = \dfrac{(q-\tilde a_3)(q-\tilde a_4)}{\tilde\rho_1\tilde\rho_2}\tilde p - \dfrac{(q-\tilde a_1)(q-\tilde a_2)}{\tilde p} + O(t), \\ \dfrac{\tilde p'-\tilde p}{t} = \tilde a_1 + \tilde a_2 - 2q + \dfrac{2(\tilde\rho_1+\tilde\rho_2)q + \tilde d_1\tilde\rho_1 + \tilde d_2\tilde\rho_2}{\tilde\rho_1\tilde\rho_2}\tilde p \\ \qquad\qquad + \dfrac{\tilde a_3 + \tilde a_4 - 2q}{\tilde\rho_1\tilde\rho_2}\tilde p^2 + O(t), \end{cases} \tag{6.6}$$

where (q, p) are the coordinates on $M_{\theta(t)}$ and (q', p') are the coordinates on $M_{\theta'(t)}$. As $t \to 0$, the left-hand sides tend to derivatives of q and p with respect to t. Similarly, (6.5) becomes the expression

$$\frac{d\tilde\rho_i}{dt} = 1 \quad (i = 1, 2);$$

all other parameters $\tilde a_1, \dots, \tilde a_4; \tilde d_1, \tilde d_2$ do not depend on t. Now it is easy to see that (6.6) is obtained from (5.9) (which is equivalent to the sixth Painlevé equation) by changing variables from $\tilde\rho_1, \tilde\rho_2$ to t.

The degeneration of (2.9) to (6.6) has a geometric interpretation similar to that given for the degeneration to (2.7) (see Remark 6.7). The details are left to the reader.

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References

[1] M. ADLER and P. VAN MOERBEKE, *Recursion relations for unitary integrals, combinatorics and the Toeplitz lattice,* Comm. Math. Phys. **237** (2003), 397–440. MR 1993333 516

[2] D. ARINKIN and S. LYSENKO, *Isomorphisms between moduli spaces of* SL(2)*-bundles with connections on* $\mathbb{P}^1 \setminus \{x_1, \ldots, x_4\}$, Math. Res. Lett. **4** (1997), 181 – 190. MR 1453052 547

[3] ———, *On the moduli of* SL(2)*-bundles with connections on* $\mathbb{P}^1 \setminus \{x_1, \ldots, x_4\}$, Internat. Math. Res. Notices **1997**, no. 19, 983 – 999. MR 1488348 523, 524, 534, 536, 540, 541

[4] J. BAIK, "Riemann-Hilbert problems for last passage percolation" in *Recent Developments in Integrable Systems and Riemann-Hilbert Problems (Birmingham, Ala., 2000)*, Contemp. Math. **326**, Amer. Math. Soc., Providence, 2003, 1 – 21. MR 1989002 516

[5] A. BORODIN, *Discrete gap probabilities and discrete Painlevé equations,* Duke Math. J. **117** (2003), 489 – 542. MR 1979052 516, 518

[6] ———, *Isomonodromy transformations of linear systems of difference equations,* Ann. of Math. (2) **160** (2004), 1141 – 1182. MR 2144976 516, 529, 530, 532

[7] A. BORODIN and D. BOYARCHENKO, *Distribution of the first particle in discrete orthogonal polynomial ensembles,* Comm. Math. Phys. **234** (2003), 287 – 338. MR 1962463 516

[8] P. J. FORRESTER and N. S. WITTE, *Application of the* τ*-function theory of Painlevé equations to random matrices: PIV, PII and the GUE,* Comm. Math. Phys. **219** (2001), 357 – 398. MR 1833807 516

[9] ———, *Application of the* τ*-function theory of Painlevé equations to random matrices:* P_{V}, P_{III}*, the LUE, JUE, and CUE,* Comm. Pure Appl. Math. **55** (2002), 679 – 727. MR 1885665

[10] ———, *Discrete Painlevé equations and random matrix averages,* Nonlinearity **16** (2003), 1919 – 1944. MR 2012848

[11] ———, *Application of the* τ*-function theory of Painlevé equations to random matrices:* P_{VI}*, the JUE, CyUE, cJUE and scaled limits,* Nagoya Math. J. **174** (2004), 29 – 114. MR 2066104 520

[12] ———, *Discrete Painlevé equations, orthogonal polynomials on the unit circle, and* N*-recurrences for averages over* $U(N)$—$P_{III'}$ *and* P_{V} τ*-functions,* Int. Math. Res. Not. **2004**, no. 4, 160 – 183. MR 2040326 516

[13] B. GRAMMATICOS, F. W. NIJHOFF, and A. RAMANI, "Discrete Painlevé equations" in *The Painlevé Property*, CRM Ser. Math. Phys., Springer, New York, 1999, 413 – 516. MR 1713581 515

[14] B. GRAMMATICOS, Y. OHTA, A. RAMANI, and H. SAKAI, *Degeneration through coalescence of the* q*-Painlevé VI equation,* J. Phys. A **31** (1998), 3545 – 3558. MR 1626199 518

[15] B. GRAMMATICOS, A. RAMANI, and Y. OHTA, *A unified description of the asymmetric* q*-*P_{V} *and* d*-*P_{IV} *equations and their Schlesinger transformations,* J. Nonlinear Math. Phys. **10** (2003), 215 – 228. MR 1976382 520

[16] M.-A. INABA, K. IWASAKI, and M.-H. SAITO, *Bäcklund transformations of the sixth Painlevé equation in terms of Riemann-Hilbert correspondence,* Int. Math. Res. Not. **2004**, no. 1, 1 – 30. MR 2036953 542

[17] ———, *Moduli of stable parabolic connections, Riemann-Hilbert correspondence and geometry of Painlevé equation of type VI, part I,* preprint, arXiv:math.AG/0309342 523, 541

[18] M. JIMBO and T. MIWA, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients, II,* Phys. D **2** (1981), 407–448. MR 0625446 518

[19] M. JIMBO and H. SAKAI, *A q-analog of the sixth Painlevé equation,* Lett. Math. Phys. **38** (1996), 145–154. MR 1403067 515, 518, 520

[20] I. M. KRICHEVER, *Analytic theory of difference equations with rational and elliptic coefficients and the Riemann-Hilbert problem,* Uspekhi Mat. Nauk **59** (2004), no. 6, 111–150; English translation in Russian Math. Surveys **59** (2004), 1117–1154. MR 2138470 516

[21] G. LAUMON, *Transformation de Fourier généralisée,* preprint, arXiv:alg-geom/9603004 526, 548

[22] M. NOUMI and Y. YAMADA, *Affine Weyl groups, discrete dynamical systems and Painlevé equations,* Comm. Math. Phys. **199** (1998), 281–295. MR 1666847 515

[23] Y. OHTA, A. RAMANI, B. GRAMMATICOS, and K. M. TAMIZHMANI, *From discrete to continuous Painlevé equations: A bilinear approach,* Phys. Lett. A **216** (1996), 255–261. MR 1396258 515

[24] K. OKAMOTO, *Sur les feuilletages associés aux équations du second ordre à points critiques fixes de P. Painlevé,* Japan. J. Math. (N.S.) **5** (1979), 1–79. MR 0614694 516, 523, 525

[25] ———, *Isomonodromic deformation and Painlevé equations, and the Garnier system,* J. Fac. Sci. Univ. Tokyo Sect. IA Math. **33** (1986), 575–618. MR 0866050 516, 523, 525

[26] A. RAMANI, B. GRAMMATICOS, and Y. OHTA, *A geometrical description of the discrete Painlevé VI and V equations,* Comm. Math. Phys. **217** (2001), 315–329. MR 1821225 520

[27] M.-H. SAITO and T. TAKEBE, *Classification of Okamoto-Painlevé pairs,* Kobe J. Math. **19** (2002), 21–50. MR 1980990 524

[28] M.-H. SAITO, T. TAKEBE, and H. TERAJIMA, *Deformation of Okamoto-Painlevé pairs and Painlevé equations,* J. Algebraic Geom. **11** (2002), 311–362. MR 1874117 524, 547

[29] H. SAKAI, *Rational surfaces associated with affine root systems and geometry of the Painlevé equations*. Comm. Math. Phys. **220** (2001), 165–229. MR 1882403 515, 516, 518, 520

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