Day 5 Exercises

June 5, 2020

1. Let $f : X \to Y$ be an orientation-preserving diffeomorphism of Riemann surfaces, and let $p \in X$. Recall that transition functions between coordinate systems are always holomorphic.

(i) Fix a coordinate system about p. Show that $\mu_f(p)$ does not depend on the choice of coordinate system about $f(p)$.

(ii) Show that if z is a local coordinate about p, and w is another with $z = \varphi(w)$, then

$$\mu_f(p)_{\text{w.r.t. } w} = \frac{\frac{\partial(f\circ\varphi)}{\partial \overline{w}}(p)}{\frac{\partial(f\circ\varphi)}{\partial w}(p)} = \frac{\frac{\partial f}{\partial \overline{z}}(p)}{\frac{\partial f}{\partial z}(p)} \cdot \frac{\overline{\frac{\partial \varphi}{\partial w}(p)}}{\frac{\partial \varphi}{\partial w}(p)} = \mu_f(p)_{\text{w.r.t. } z} \cdot \frac{\overline{\frac{\partial \varphi}{\partial w}(p)}}{\frac{\partial \varphi}{\partial w}(p)}.$$

Hint: Use the chain rule for $\frac{\partial}{\partial z}$ and for $\frac{\partial}{\partial \overline{z}}$, which may be derived from the identities $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ and $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$.

(iii) Show that $|\mu_f(p)|$ is independent of any choice of coordinates.

Problems 2-4 are part of the proof of the Ahlfors-Rauch variational formula. Recall our setup: For $\mu \in \mathrm{Bel}(X)$ and for small enough $t \in \mathbb{R}$, let $f_{t\mu} : X \to X_{t\mu}$ be the map given by the global C^∞ Riemann mapping theorem. Let $\mathscr{B}$ be a Torelli marking on X, let $\omega_1, \ldots, \omega_g$ be the associated dual basis, and let $\omega_{1,t}, \ldots, \omega_{g,t}$ be the dual basis on $X_{t\mu}$ for $(f_{t\mu})_*\mathscr{B}$. Fix j, and define $\psi_t = f_{t\mu}^*\omega_{j,t} - \omega_j$.

2. Let T^*X denote the real cotangent bundle of X, and let $T^*_{\mathbb{C}}X = T^*X \otimes_{\mathbb{R}} (X \times \mathbb{C})$. At every $p \in X$, we have $(T^*_{\mathbb{C}}X)_p = \mathbb{C}\langle dz, \overline{dz}\rangle$. Convince yourself that this can be extended to the global formula $T^*_{\mathbb{C}}X = K \oplus \overline{K}$, and then compute

$$\psi_t^K = \left((a_{j,t} \circ f_{t\mu}) \cdot \frac{\partial f_{t\mu}}{\partial z} - a_j\right) dz, \qquad \psi_t^{\overline{K}} = (a_{j,t} \circ f_{t\mu}) \cdot \frac{\partial f_{t\mu}}{\partial \overline{z}} \overline{dz}$$

for the K- and $\overline{K}$-parts of ψ_t, where $\omega_j = a_j(z)dz$ and $\omega_{j,t} = a_{j,t}(z_t)dz_t$.

3. Use Riemann's bilinear relations to show that

$$\int_{b_i} \psi_t = \int_X \omega_i \wedge \psi_t^{\overline{K}}.$$

4. Use Riemann's bilinear relations to show that

$$-\frac{i}{2}\int_X \psi_t \wedge \overline{\psi}_t = 0.$$

Problems 5 and 6 outline the relationship between today's topic and the Teichmüller geodesic flow for translation surfaces.

5. Let ω be a holomorphic 1-form on X, i.e. a section of K. In local coordinates, we may write $\omega = f(z)dz$. On any such coordinate chart, define the Beltrami differential

$$\frac{\overline{\omega}}{\omega} := \frac{\overline{f(z)}}{f(z)} \frac{\overline{dz}}{dz}.$$

Show that $\frac{\overline{\omega}}{\omega}$, when defined with respect to another coordinate w with $z = \varphi(w)$, satisfies the change-of-coordinates formula from Problem 1, part (ii). Conclude that $\frac{\overline{\omega}}{\omega}$ is a well-defined global section of $\overline{K} \otimes K^*$.

6. Let (X, ω) be a translation surface, expressed as a collection of polygons with $\omega = dz$. Then consider the action of $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ on this collection of polygons, and let (X_t, ω_t) be the translation surface given by these new polygons with $\omega_t = dz$.

(i) The local coordinate $z = x + iy$ on X is well-defined outside a set of measure 0. Verify that the local formula $f_t(x + iy) = e^t x + ie^{-t} y$ gives a well-defined homeomorphism $f_t : X \to X_t$, which is a diffeomorphism outside a set of measure 0.

(ii) The global C^∞ Riemann mapping theorem generalizes to a measurable Riemann mapping theorem, where Beltrami differentials need only be defined outside a set of measure 0. Therefore, compute

$$\mu_{f_t} = \tanh(t) \frac{\overline{dz}}{dz}.$$

(iii) Choose an arbitrary Torelli marking $\mathscr{B}$ on X, and consider the path $\gamma(t) = (X_t, (f_t)_* \mathscr{B})$ in $\mathcal{U}_g$. Conclude that

$$\gamma'(0) = \left[\frac{\overline{\omega}}{\omega} \right].$$

Problems 7 and 8 are meant to help explicate the algebraic manipulation

$$(c_i(z)dz)(c_j(z)dz) \left(\mu(z) \frac{\overline{dz}}{dz} \right) = c_i(z)c_j(z)\mu(z) \frac{(dz)^2 \overline{dz}}{dz} = c_i(z)c_j(z)\mu(z) dz \overline{dz} \qquad (\star)$$

and the subsequent identificaton

$$c_i(z)c_j(z)\mu(z) dz \overline{dz} = c_i(z)c_j(z)\mu(z) dz \wedge \overline{dz} \qquad (\star\star)$$

in the language of differential topology.

7. ($\star$) Let X be a Riemann surface, and let K be its holomorphic cotangent bundle. Holomorphic 1-forms ω_i, ω_j are sections of K, and Beltrami differentials μ are sections of $\overline{K} \otimes K^*$. Then $\omega_i \otimes \omega_j \otimes \mu$ is a section of $K \otimes K \otimes \overline{K} \otimes K^*$.

(i) Show that $K \otimes K^*$ is canonically isomorphic to the trivial bundle $X \times \mathbb{C}$.

Warmup: Let V be a 1-dimensional complex vector space, given without any preferred isomorphism $V \cong \mathbb{C}$. Nonetheless, demonstrate an explicit isomorphism $V \otimes V^* \cong \mathbb{C}$.

(ii) Conclude that we have a bundle isomorphism $K \otimes K \otimes \overline{K} \otimes K^* \xrightarrow{\sim} K \otimes \overline{K}$ given in local coordinates by

$$(c_i(z)dz) \otimes (c_j(z)dz) \otimes \left(\mu(z)\overline{dz} \otimes \tfrac{\partial}{\partial z}\right) \mapsto c_i(z)c_i(z)\mu(z)dz \otimes \overline{dz}.$$

8. ($\star\star$)

(i) Show that every section of $K \otimes \overline{K}$ acts on pairs of real tangent vectors as neither an alternating nor symmetric 2-form, but indeed has nontrivial alternating and symmetric parts.

Hint: Use the identities $dz = dx + idy$ and $\overline{dz} = dx - idy$.

(ii) Conclude that we have a bundle isomorphism $K \otimes \overline{K} \xrightarrow{\sim} \bigwedge^2 T^*_{\mathbb{C}}X$ given in local coordinates by

$$c_i(z)c_j(z)\mu(z)dz \otimes \overline{dz} \mapsto c_i(z)c_j(z)\mu(z)dz \wedge \overline{dz},$$

where $T^*_{\mathbb{C}}X$ is the complex cotangent bundle of X. That is,

$$T^*_{\mathbb{C}}X = T^*X \otimes_{\mathbb{R}} (X \times \mathbb{C}),$$

where T^*X is the real cotangent bundle of X.