The Ahlfors-Rauch variational formula

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June 5, 2020

Outline

1. Measuring the difference between Riemann surfaces

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2. The Torelli space

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3. The Ahlfors-Rauch variational formula

Measuring the difference between Riemann surfaces

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For an orientation-preserving diffeomorphism $f : X \to Y$, we would like to say how close f is to being a biholomorphism (holomorphic map with holomorphic inverse).

To do this, we will define a differential form μ_f on X that measures how far f is from being a biholomorphism.

Measuring the difference between Riemann surfaces

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Thus we may start with a more down-to-earth goal: Define a quantity μ_T measuring how far a linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ is from being $\mathbb{C}$-linear.

Measuring the difference between Riemann surfaces

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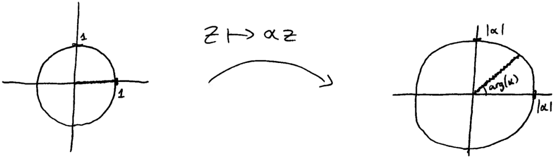
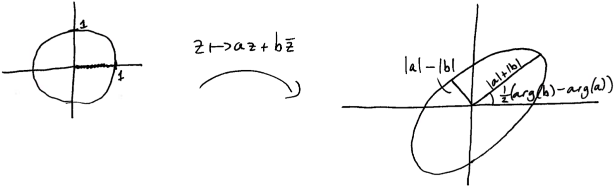
$\mathbb{C}$-linear $\iff$ Maps circles to circles



Failure to be $\mathbb{C}$-linear $\iff$ How much you stretch circles into ellipses



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When T is $\mathbb{C}$-linear, we have $T(z) = az$, and so $\mu_T = 0$. The more T depends on $\overline{z}$, the greater b is, and hence the greater μ_T is.

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When T is orientation-preserving, we have $\left|\frac{b}{a}\right| < 1$.

Measuring the difference between Riemann surfaces

Goal: Define a form μ_f on X measuring how far $f : X \to Y$ is from being a biholomorphism.

Definition (Beltrami differential at a point)

Let $f : X \to Y$ be an orientation-preserving diffeomorphism of Riemann surfaces, and let $p \in X$. Fix coordinate systems about p and $f(p)$, giving isomorphisms $T_pX \cong \mathbb{C}$ and $T_{f(p)}Y \cong \mathbb{C}$. Then we have $(df)_p = \left(\frac{\partial f}{\partial z}(p)\right) z + \left(\frac{\partial f}{\partial \overline{z}}(p)\right) \overline{z}$. We define

$$\mu_f(p) = \mu_{(df)_p} = \frac{\frac{\partial f}{\partial \overline{z}}(p)}{\frac{\partial f}{\partial z}(p)}.$$

Measuring the difference between Riemann surfaces

Exercise

Let $f : X \to Y$ be an orienation-preserving diffeomorphism of Riemann surfaces, and let $p \in X$.

1. Fix a coordinate system about p. Show that $\mu_f(p)$ does not depend on the choice of coordinate system about $f(p)$.
2. Show that if z and w are local coordinates about p, with $z = \varphi(w)$, then

$$\mu_f(p)_{\text{w.r.t. } w} = \frac{\frac{\partial(f\circ\varphi)}{\partial\overline{w}}(p)}{\frac{\partial(f\circ\varphi)}{\partial w}(p)} = \frac{\frac{\partial f}{\partial\overline{z}}(p)}{\frac{\partial f}{\partial z}(p)} \cdot \frac{\overline{\frac{\partial\varphi}{\partial w}(p)}}{\frac{\partial\varphi}{\partial w}(p)} = \mu_f(p)_{\text{w.r.t. } z} \cdot \frac{\overline{\frac{\partial\varphi}{\partial w}(p)}}{\frac{\partial\varphi}{\partial w}(p)}.$$

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The second exercise shows that μ_f can be understood as a C^∞ section of $\overline{K} \otimes K^*$, where K is the holomorphic cotangent bundle of X, and $\overline{K}$ and K^* are its complex conjugate and linear dual, respectively. In local coordinates, we write $\mu_f = \mu(z)\frac{\overline{dz}}{dz}$ for some local C^∞ function μ.

Measuring the difference between Riemann surfaces

Definition

Let X be a Riemann surface. We define the vector space $\mathrm{Bel}(X)$ of C^∞ Beltrami differentials to be the set of C^∞ sections of $\overline{K} \otimes K^*$.

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Theorem (Global C^∞ Riemann mapping theorem)

Let $\mathrm{Bel}_1(X)$ be the set of $\mu \in \mathrm{Bel}(X)$ with $|\mu(p)| < 1$ for every $p \in X$. For every $\mu \in \mathrm{Bel}_1(X)$, there exists a Riemann surface X_μ and a diffeomorphism $f : X \to X_\mu$ such that $\mu_f = \mu$.

The surface X_μ is unique up to biholomorphism, and the map f is unique up to postcomposition by some automorphism of X_μ.

The Torelli space

Recall that a Torelli marking on a Riemann surface X is a choice of basis $\mathscr{B} = \{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ for $H_1(X; \mathbb{Z})$ so that $a_i \cdot b_j = \delta_{ij}$ and $a_i \cdot a_j = b_i \cdot b_j = 0$ for all i, j.

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Definition

Fix $g > 0$. The Torelli space for genus g Riemann surfaces is

$$\mathcal{U}_g = \{(X, \mathscr{B}) \,|\, X \text{ a genus } g \text{ Riemann surface with Torelli marking } \mathscr{B}\}/\sim,$$

where $(X, \mathscr{B}) \sim (Y, \mathscr{C})$ if there is there is a biholomorphism $f : X \to Y$ with $f_* \mathscr{B} = \mathscr{C}$.

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Let $(X, \mathscr{B}) \in \mathcal{U}_g$, and let $\mu \in \mathrm{Bel}(X)$. For small enough $t \in \mathbb{R}$, we have $t\mu \in \mathrm{Bel}_1(X)$, and so by the global C^∞ Riemann mapping theorem we have a diffeomorphism $f_{t\mu} : X \to X_{t\mu}$. By the definition of the Torelli space, there is a well-defined point $(X_{t\mu}, (f_{t\mu})_*\mathscr{B}) \in \mathcal{U}_g$, irrespective of the choice of $X_{t\mu}$ and $f_{t\mu}$.

The Torelli space

Theorem

The map

$$\begin{aligned} \mathrm{Bel}(X) &\to T_{(X,\mathscr{B})}\mathcal{U}_g \\ \mu &\mapsto \frac{\partial}{\partial t}\Big|_{t=0}(X_{t\mu}, (f_{t\mu})_*\mathscr{B}) \end{aligned}$$

is a linear surjection. We may therefore understand every tangent vector to $\mathcal{U}_g$ as an equivalence class $[\mu]$ of Beltrami differentials.

The Torelli space

Recall that every $(X, \mathscr{B}) \in \mathcal{U}_g$ has a dual basis $\omega_1, \ldots, \omega_g \in \Omega(X)$ satisfying

$$\int_{a_i} \omega_j = \delta_{ij}, \qquad \forall 1 \leq i, j \leq g.$$

We define the period matrix $\tau(X, \mathscr{B})_{i,j=1}^{g} = \left(\int_{b_i} \omega_j \right)_{i,j=1}^{g}$.

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Let $f : X \to Y$ be a biholomorphism with $\mathscr{C} = f_* \mathscr{B}$. Then the formula $\int_{f_* \gamma} \omega = \int_\gamma f^* \omega$ implies that $(f^{-1})^* \omega_1, \ldots, (f^{-1})^* \omega_g$ is a dual basis for $(Y, \mathscr{C})$, and that $\tau(X, \mathscr{B}) = \tau(Y, \mathscr{C})$.

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Therefore we have a well defined map

$$\begin{aligned} \tau : \mathcal{U}_g &\to \mathfrak{S}_g \subset \mathbb{C}^{g^2} \\ (X, \mathscr{B}) &\mapsto \tau(X, \mathscr{B}), \end{aligned}$$

where $\mathfrak{S}_g$ is the space of symmetric $g \times g$ complex matrices with positive-definite imaginary part, called the Siegel upper half-space.

The Ahlfors-Rauch variational formula

Theorem (Ahlfors-Rauch variational formula)

The derivative $(d\tau)_{(X,\mathscr{B})} : T_{(X,\mathscr{B})}\mathcal{U}_g \to T_{\tau(X,\mathscr{B})}\mathfrak{S}_g$ *is given by*

$$(d\tau)_{(X,\mathscr{B})}([\mu])_{ij} = \int_X (\omega_i \otimes \omega_j)\mu, \qquad \forall \mu \in \mathrm{Bel}(X).$$

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It is not immediately evident that the expression $(\omega_i \otimes \omega_j)\mu$ denotes the sort of thing that can be integrated. If we write in local coordinates $\omega_i = c_i(z)dz$, $\omega_j = c_j(z)dz$, and $\mu = \mu(z)\frac{\overline{dz}}{dz}$, then one often sees the deceptively simple algebraic manipulation

$$(c_i(z)dz)(c_j(z)dz)\left(\mu(z)\frac{\overline{dz}}{dz}\right) = c_i(z)c_j(z)\mu(z)\frac{(dz)^2\overline{dz}}{dz} = c_i(z)c_j(z)\mu(z)dz\overline{dz},$$

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In the exercises, we will obtain this manipulation as a sequence of vector bundle isomorphisms.

The Ahlfors-Rauch variational formula

We proceed to prove the variational formula

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For small enough $t \in \mathbb{R}$, we again have $f_{t\mu} : X \to X_{t\mu}$. Let $\omega_{1,t}, \ldots, \omega_{g,t}$ be the dual basis for $X_{t\mu}$. Then

$$(d\tau)_{(X,\mathscr{B})}([\mu])_{ij} = \lim_{t\to 0} \frac{1}{t}\left(\tau(X_{t\mu}, (f_{t\mu})_*\mathscr{B})_{ij} - \tau(X,\mathscr{B})_{ij}\right).$$

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Fix j, and let $\psi_t = f_{t\mu}^* \omega_{j,t} - \omega_j$. Then $\tau(X_{t\mu}, (f_{t\mu})_*\mathscr{B})_{ij} - \tau(X, \mathscr{B})_{ij} = \int_{b_i} \psi_t$. The variational formula then becomes

$$\int_{b_i} \psi_t = t \int_X (\omega_i \otimes \omega_j)\mu + O(t^2).$$

The Ahlfors-Rauch variational formula

Since $T^*_{\mathbb{C}}X = K \oplus \overline{K}$, we may decompose $\psi_t = f^*_{t\mu}\omega_{j,t} - \omega_j$ as the sum of its K-part ψ_t^K and its $\overline{K}$-part $\psi_t^{\overline{K}}$. Let z be a local coordinate on X and z_t be a local coordinate on $X_{t\mu}$. Then, writing $\omega_j = c_j(z)dz$ and $\omega_{j,t} = c_{j,t}(z_t)dz_t$, we have

$$\psi_t^K = \left((c_{j,t} \circ f_{t\mu}) \cdot \tfrac{\partial f_{t\mu}}{\partial z} - c_j \right) dz,$$
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\psi_t^{\overline{K}} &= (c_{j,t} \circ f_{t\mu}) \cdot \tfrac{\partial f_{t\mu}}{\partial \overline{z}} \overline{dz} \\
&= (c_{j,t} \circ f_{t\mu}) \cdot t\mu \cdot \tfrac{\partial f_{t\mu}}{\partial z} \overline{dz},
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Exercise

Use Riemann's bilinear relations to show that

$$\int_{b_i} \psi_t = \int_X \omega_i \wedge \psi_t^{\overline{K}}.$$

The Ahlfors-Rauch variational formula

Since we are only interested in integrating $\omega_i \wedge \psi_t^{\overline{K}}$, it suffices to consider this form outside a set of measure 0. Let $U \subset X$ be a (contractible) coordinate chart on X so that $X \setminus U$ has measure 0, and let z be a coordinate on U.

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We may now write

$$\omega_i \wedge \psi_t^{\overline{K}} = \left(c_i \cdot (c_{j,t} \circ f_{t\mu}) \cdot t\mu \cdot \frac{\partial f_{t\mu}}{\partial z} \right) dz \wedge \overline{dz}$$

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We have reduced our problem to showing that the integral of $\omega_i \wedge \psi_t^{\overline{K}} - t(\omega_i \otimes \omega_j)\mu$ is $O(t^2)$. Note that

$$\omega_i \wedge \psi_t^{\overline{K}} - t(\omega_i \otimes \omega_j)\mu = c_i \cdot t\mu \cdot \left((c_{j,t} \circ f_{t\mu}) \cdot \frac{\partial f_{t\mu}}{\partial z} - c_j \right) dz \wedge \overline{dz}$$

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We have reduced our problem to showing that $t \int_X \left(\omega_i \otimes \psi_t^K\right) \mu = O(t^2)$.
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Exercise

Use Riemann's bilinear relations to show that

$$-\frac{i}{2}\int_X \psi_t \wedge \overline{\psi}_t = 0.$$

Since $\psi_t = \psi_t^K + \psi_t^{\overline{K}}$, it follows that $-\frac{i}{2}\int_X \psi_t^K \wedge \overline{\psi_t^K} = \frac{i}{2}\int_X \psi_t^{\overline{K}} \wedge \overline{\psi_t^{\overline{K}}}$.

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We have reduced our problem to showing that $t \int_X \left(\omega_i \otimes \psi_t^K\right) \mu = O(t^2)$. Recall $\psi_t^K = \left((c_{j,t} \circ f_{t\mu}) \cdot \frac{\partial f_{t\mu}}{\partial z} - c_j\right) dz$.

Exercise

Use Riemann's bilinear relations to show that

$$-\frac{i}{2} \int_X \psi_t \wedge \overline{\psi}_t = 0.$$

Since $\psi_t = \psi_t^K + \psi_t^{\overline{K}}$, it follows that $-\frac{i}{2} \int_X \psi_t^K \wedge \overline{\psi_t^K} = \frac{i}{2} \int_X \psi_t^{\overline{K}} \wedge \overline{\psi_t^{\overline{K}}}$.

In coordinates, this equation becomes

$$\int_U \left|(c_{j,t} \circ f_{t\mu}) \cdot \tfrac{\partial f_{t\mu}}{\partial z} - c_j\right|^2 dx \wedge dy = \int_U \left|(c_{j,t} \circ f_{t\mu}) \cdot t\mu \cdot \tfrac{\partial f_{t\mu}}{\partial z}\right|^2 dx \wedge dy.$$

The Ahlfors-Rauch variational formula

We therefore have

$$\frac{1}{4}\left|t\int_X\left(\omega_i\otimes\psi_t^K\right)\mu\right|^2\leq|t|^2\left(\int_U|c_i\cdot t\mu|^2\,dx\wedge dy\right.$$
$$\left.+\int_U\left|c_j-(c_{j,t}\circ f_{t\mu})\cdot\frac{\partial f_{t\mu}}{\partial z}\right|^2dx\wedge dy\right)$$

The Ahlfors-Rauch variational formula

We therefore have

$$\frac{1}{4}\left|t\int_X \left(\omega_i \otimes \psi_t^K\right)\mu\right|^2 \leq |t|^2\left(\int_U |c_i \cdot t\mu|^2\,dx\wedge dy + \int_U \left|c_j - (c_{j,t}\circ f_{t\mu})\cdot \frac{\partial f_{t\mu}}{\partial z}\right|^2 dx\wedge dy\right)$$

$$= |t|^2\left(\int_U |c_i \cdot t\mu|^2\,dx\wedge dy + \int_U \left|(c_{j,t}\circ f_{t\mu})\cdot t\mu\cdot \frac{\partial f_{t\mu}}{\partial z}\right|^2 dx\wedge dy\right)$$

The Ahlfors-Rauch variational formula

We therefore have

$$
\begin{aligned}
\frac{1}{4}\left|t\int_X \left(\omega_i \otimes \psi_t^K\right)\mu\right|^2 \leq |t|^2 \Bigg(&\int_U \left|c_i \cdot t\mu\right|^2 dx \wedge dy \\
&+ \int_U \left|c_j - (c_{j,t} \circ f_{t\mu}) \cdot \tfrac{\partial f_{t\mu}}{\partial z}\right|^2 dx \wedge dy\Bigg) \\
= |t|^2 \Bigg(&\int_U \left|c_i \cdot t\mu\right|^2 dx \wedge dy \\
&+ \int_U \left|(c_{j,t} \circ f_{t\mu}) \cdot t\mu \cdot \tfrac{\partial f_{t\mu}}{\partial z}\right|^2 dx \wedge dy\Bigg) \\
= |t|^4 \Bigg(&\int_U \left|c_i \cdot \mu\right|^2 dx \wedge dy \\
&+ \int_U \left|(c_{j,t} \circ f_{t\mu}) \cdot \mu \cdot \tfrac{\partial f_{t\mu}}{\partial z}\right|^2 dx \wedge dy\Bigg).
\end{aligned}
$$

We therefore have

$$
\begin{aligned}
\frac{1}{4}\left| t \int_X \left(\omega_i \otimes \psi_t^K\right) \mu \right|^2 &\leq |t|^2 \left(\int_U |c_i \cdot t\mu|^2 \, dx \wedge dy \right. \\
&\qquad \left. + \int_U \left| c_j - (c_{j,t} \circ f_{t\mu}) \cdot \tfrac{\partial f_{t\mu}}{\partial z} \right|^2 dx \wedge dy \right) \\
&= |t|^2 \left(\int_U |c_i \cdot t\mu|^2 \, dx \wedge dy \right. \\
&\qquad \left. + \int_U \left| (c_{j,t} \circ f_{t\mu}) \cdot t\mu \cdot \tfrac{\partial f_{t\mu}}{\partial z} \right|^2 dx \wedge dy \right) \\
&= |t|^4 \left(\int_U |c_i \cdot \mu|^2 \, dx \wedge dy \right. \\
&\qquad \left. + \int_U \left| (c_{j,t} \circ f_{t\mu}) \cdot \mu \cdot \tfrac{\partial f_{t\mu}}{\partial z} \right|^2 dx \wedge dy \right).
\end{aligned}
$$

Taking square roots, we conclude that $t \int_X \left(\omega_i \otimes \psi_t^K\right) \mu = O(t^2)$.

References

- My notes on the complex structure of Teichmüller space and classical Teichmüller theory on my website
- Y. Imayoshi and M. Taniguchi. *An introduction to Teichmüller spaces*. Chapter 1 and Appendix A
- S. Nag. *The complex-analytic theory of Teichmüller spaces*. Section 4.1