EXERCISES TO ACCOMPANY "TRANSLATION SURFACES AND THEIR ORBIT CLOSURES"

1. Translation surfaces

(1) How many strata are their in genus 4?
(2) Which triangles unfold to surfaces of genus 1?
(3) Prove that square-tiled surfaces are dense in every stratum.
(4) Prove that a surface covers a torus iff the absolute periods span a rank 2 $\mathbb{Q}$-module. The absolute periods of (X, ω) are $\{\int_\gamma \omega : \gamma \in H_1(X, \mathbb{Z})\}$.
(5) Prove that every surface has countably many saddle connections.
(6) Prove that, for any $T > 0$, there are finitely many saddle connections of length at most T.
(7) Can a surface in $\mathcal{H}(3, 1)$ be hyperelliptic?
(8) Draw a flat picture of a surface in the stratum $\mathcal{H}(2, 1, 1, 1, 1)$, by starting with a surface in $\mathcal{H}(2)$ and gluing on slit tori. What genus is this surface?
(9) Prove that $\mathcal{H}(2)$ and $\mathcal{H}(1, 1)$ are both path connected. Bonus: Show that $\mathcal{H}(4)$ is not (there is one connected component which consists entirely of hyperelliptic surfaces, and another which does not).

2. Linear submanifolds

(1) Prove that the regular 8-gon is contained in a 2 dimensional linear submanifold. Write down the linear equations defining this in period coordinates.
(2) Consider all degree d covers of a tori, branched over k points whose sum is zero. Show that any connected component of this space is a linear submanifold.
(3) Verify the torsion condition for covers of tori branched at one point.
(4) Say that a translation surface is *primitive* if it does not cover a translation surface of lower genus, or if it is genus 1. Show that every translation surface covers a primitive surface, which is unique if it is not a torus. (This is a non-trivial theorem of Möller, but you might rediscover the proof if you think about natural maps from X to abelian varieties finitely covered by $\text{Jac}(X, \omega)$.)

(5) Prove that if a two complex dimensional linear submanifold contains a single cover of a torus, then every surface in the submanifold is a cover.

3. The $GL^+(2,\mathbb{R})$ action

(1) Fix $\epsilon > 0$. Show, using the results presented in this section, that the set of surfaces without triangles of area less than ϵ is a finite union of linear submanifolds. Here a triangle is required to be bounded by three saddle connections.
(2) Compute the $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ orbit closure for a surface in $\mathcal{H}(2)$ consisting of two horizontal cylinders. The answer depends on moduli of the two cylinders.
(3) Show that the stabilizer of regular 8-gon is a lattice in $PSL(2,\mathbb{R})$ (if you know about Fuchsian groups).

4. The straight line flow

(1) Depending on the stratum, what is the maximum number of parallel cylinders a surface can have? First consider $\mathcal{H}(2)$ and $\mathcal{H}(1,1)$.
(2) Using the slit torus construction, give an example of a genus 3 surface with three minimal components in some direction.
(3) It is known that the action of g_t is ergodic on the set of unit area surfaces in each linear submanifold. Using this and the Birkhoff Ergodic Theorem, show that almost every surface in a linear submanifold has full $GL^+(2,\mathbb{R})$ orbit closure.
(4) Prove that every surface has a cylinder.
(5) For the $\mathcal{M}'$ in Proposition 4.18, derive Proposition 4.20 from the Veech dichotomy for $(X,\omega) \in \mathcal{M}$, but show that the Veech Dichotomy itself need not hold.

5. Revisiting genus 2 with new tools

(1) Show that the field of definition is well defined, and that it is the field generated by the coefficients of the reduced row echelon form of any system of equations defining the subspace. Hint: first figure out the field of definition of a line in $\mathbb{C}^2$.
(2) Show rank 1 arithmetic implies torus cover.
(3) Prove that non-arithmetic closed orbits are dense in $\mathcal{H}(2)$, and eigenform loci are dense in $\mathcal{H}(1,1)$.
(4) Show that if a horizontally periodic surface has cylinders with appropriate moduli, then there is a cylinder deformation.
(5) Prove there is no 3 dimensional orbit closure in $\mathcal{H}(4)$.