Teichmüller Geodesic Flow

Christopher Zhang

Masur's Criterion

Main idea: understanding dynamics of $GL^+(2,\mathbb{R})$ on the moduli space of translation surfaces gives us information about individual surfaces (X,ω).

Example: Veech dichotomy

Theorem (Masur's Criterion)

Let $g_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$ *(often called Teichmüller geodesic flow). If* ω *is a translation surface such that the vertical direction is not uniquely ergodic, then* $g_t\omega$ *leaves every compact set i.e. for any compact* $K \subset \mathcal{H}(\kappa)$ *there exists a time* T *s.t. if* $t > T$ *then* $g_t\omega \notin K$*.*

Equivalently, if g_t **recurs** to a compact set K, then the vertical direction of ω is uniquely ergodic. This means $g_{t_n}\omega \in K$ for some sequence $t_n \to \infty$.

Masur's Criterion

Theorem (Veech Dichtomy)

For a Veech surface, any direction with a saddle connection is parabolic and any other direction is uniquely ergodic.

Proof.

Yesterday we showed any direction with a saddle is parabolic. Now we will finish the proof using Masur's criterion.

1. Let θ be a direction that is not uniquely ergodic. Rotate the surface so this direction is vertical. We call the rotated surface ω. By Masur's criterion $g_t\omega$ **diverges**.
2. Fact: if $g_t\omega \in SL(2, \mathbb{R})/SL(X, \omega)$ diverges, then $SL(X, \omega)$ has a parabolic element in the vertical direction.
3. The original surface must be parabolic in the direction θ. In particular it has a saddle.

□

Masur's Criterion

Proposition

On each $\mathcal{H}_1(\kappa)$*, there exists a finite* $SL(2,\mathbb{R})$*-invariant measure* μ_{MV}*. This measure is called the* **Masur-Veech** *measure.*

Proposition (Poincare Recurrence)

If ϕ_t *is a flow on* S *and* μ *a finite invariant measure, then for any set* U *with* $\mu(U) > 0$*,* μ*-almost every* $x \in U$ *comes back to* U *infinitely often.*

Proof.

1. Let $U_0 \subset U$ be the set $\phi^t(U_0) \notin U$ for $t \geq 1$. $\phi^n(U_0) \cap \phi^m(U_0) = \emptyset$ for all $n \neq m \in \mathbb{N}$.
2. If $\mu(U_0) = \epsilon > 0$, then $\mu(S) \geq \mu(U) + \mu(\phi(U)) + \mu(\phi^2(U)) + \dots$ is not finite.
3. If x comes back to U only finitely many times, $x \in \phi^{-k}(U)$.

□

Masur's Criterion

Corollary

For almost all surface $\omega \in \mathcal{H}(\kappa)$, the vertical direction is uniquely ergodic.

Proof.

1. By Poincare recurrence, almost every surface ω, $g_t\omega$ recurs to some compact set K.
2. By Masur's criterion, such ω have uniquely ergodic vertical direction.

□

This is weaker than Kerckhoff, Masur, Smillie: on EVERY surface, almost every direction is uniquely direction.

Masur-Veech Measures

Let $(X, \omega) \in \mathcal{H}(\kappa)$. A holomorphic 1-form ω gives an element of $H^1(X, \Sigma; \mathbb{C}) \cong \mathbb{C}^d$ by integrating.

Proposition

Around every point in $(X, \omega) \in \mathcal{H}(\kappa)$, there is a neighborhood U such that $U \to H^1(X, \Sigma; \mathbb{C}) \cong \mathbb{C}^d$ is a homeomorphism onto its image. These are called **period coordinate** *charts. Two charts different by an element of $SL(d, \mathbb{Z})$.*

Analogy: any two isomorphisms $H_1(T^2; \mathbb{Z}) \cong \mathbb{Z}^2$ differ by an element of $SL(2, \mathbb{Z})$.

Proposition

If $(X, \omega) \in \mathcal{H}(\kappa)$, $H^1(X, \Sigma; \mathbb{C}) \cong \mathbb{C}^{2g+|\kappa|-1}$, so $2g + |\kappa| - 1$ is the dimension of $\mathcal{H}(\kappa)$.

Masur-Veech Measures

Proposition

The dimension of $\mathcal{H}(\kappa)$ is $2g + |\kappa| - 1$.

Example: the dimension of $\mathcal{H}(1,1)$ is 5 (genus $= 2$).

The lowest dimension stratum is $\mathcal{H}(2g-2)$ is dimension $2g$.

The highest dimension stratum is $\mathcal{H}(1,1,\dots,1)$ has $|\kappa| = 2g-2$ and dimension $= 4g-3$.

Fact: $\mathcal{M}_g$ has dimension $3g-3$, so there are stratum $\mathcal{H}(\kappa) \to \mathcal{M}_g$ is not surjective for many strata.

Sanity check: the Hodge bundle $\mathcal{H} \to \mathcal{M}_g$ has $3g-3$-dimensional base and g-dimensional fiber, so it has dimension $4g-3$, which is the dimension of the largest stratum.

Masur-Veech Measures

Let $\phi : U \to \mathbb{C}^d$ be a periodic coordinate chart. For any open set $V \subset U$ we define $\mu_{MV}(V)$ as the Lebesgue measure of $\phi(V)$.

If $V \subset U_1 \cap U_2$, $\mu_{MV}(V)$ is still well defined, because $SL(d, \mathbb{Z})$ preserves Lebesgue measure.

$\mu_{MV}(\mathcal{H}(\kappa))$ is infinite, but $\mu_{MV}(\mathcal{H}_1(\kappa))$ is finite.

Example:

1. $\mathcal{H}_1(0) \cong SL(2, \mathbb{R})/SL(2, \mathbb{Z})$
2. $\mathcal{H}(0) \cong SL(2, \mathbb{R})/SL(2, \mathbb{Z}) \times \mathbb{R}_+$

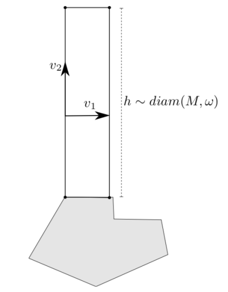
Masur-Veech Measures

Lemma

Let (X, ω) be a flat surface of area 1. There is some constants R, c such that if ω has diameter $\geq R$, then it must have a cylinder in ω of height $\geq cR$.

Sketch of proof that Masur-Veech measures are finite.

1. The set $S_0 \subset \mathcal{H}(\kappa)$ such that all relative periods are bounded is finite measure.
2. The set S_1 with "one long cylinder" is similar to $SL(2, \mathbb{R})/SL(2, \mathbb{Z})$
3. Only finitely of these S_n



□

Ergodicity

Definition

A flow ϕ_t on S is **ergodic** with respect to a finite measure μ if for almost every $x \in S$

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \chi_A(\phi_t x)dt = \frac{\mu(A)}{\mu(S)}$$

Theorem

Teichmüller geodesic flow g_t is ergodic with respect to μ_{MV}. It is also **mixing**.

Ergodicity

Theorem

For almost all $x \in \mathcal{H}_1(\kappa)$, and all $f \in C_c^0$

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(\phi_t x) = \frac{\mu_{MV}(A)}{\mu_{MV}(\mathcal{H}_1(\kappa))}$$

Sketch of proof.

1. Let $\hat{f}$ be the left hand side of the above equation. It is well-defined for almost every x. It suffices to show $\hat{f}$ is constant. $\hat{f}$ is constant along orbits of g_t.
2. There are expanding and contracting foliations $\mathcal{W}^u, \mathcal{W}^s$.
3. $\hat{f}$ is constant on each leaf of $\mathcal{W}^u, \mathcal{W}^s$.
4. There is a full measure set $E \subset \mathcal{H}_1(\kappa)$ such that you can get between any two points in E by traveling along g_t orbits or leaves of $\mathcal{W}^u, \mathcal{W}^s$.

□