Veech Surfaces

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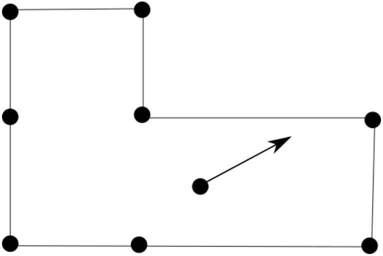
On a square torus:

1. Rational lines are closed
2. Irrational lines are dense

Definition

A flow is **minimal** if every **orbit** is dense. A flow is **periodic** if every orbit is closed.



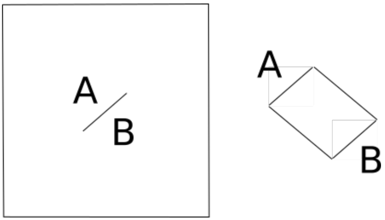
Directions are well defined on a flat surface.

The flow is not well defined when you hit a singularity!

Definition

A **saddle connection**, aka saddle, is a straight line segment connecting two singularities. A **separatrix** is a ray coming out of a singularity.

Is every direction minimal or periodic? No!



To what extent this can happen?

A closed orbit must be on the interior of a **cylinder**.

The boundary of the cylinder is made up of saddles.

Corollary

If you find one closed billiard trajectory (not going through a singularity), close enough trajectories in the same direction are also closed.

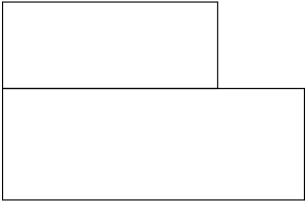
Proposition

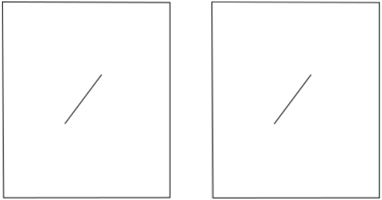
Fix any direction, and cut the surface along all saddle connections in that direction. On every connected component straight line flow is either minimal or periodic. In the latter case, the component is a cylinder.

Corollary

For a compact translation surface of genus $g \geq 2$, a direction without any saddle connections is minimal.

Examples





We actually know that every irrational orbit on the torus equidistributes i.e. spends equal time in every part of the torus.

Definition

A flow ϕ_t on a space S is **uniquely ergodic** with respect to a measure μ if for EVERY $x_0 \in S$, and any function $f \in L^1(S)$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(\phi_t x_0) dt = \frac{1}{\mu(S)} \int_S f(x) d\mu(x)$$

Plugging in the indicator function $f = \chi_A$, we have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \chi_A(\phi_t x) dt = \frac{\mu(A)}{\mu(S)}$$

The Lebegue measure on a translation surface ω makes sense. (Exercise!)

Uniquely ergodic with respect to Lebesgue measure implies minimal.

Theorem (Kerckhoff, Masur, Smillie '86)

For every translation surface ω and for almost every direciton $\theta \in S^1$ (with respect to the uniform measure on S^1), the straight line flow in the direction θ is uniquely ergodic (with respect to the Lebesgue measure on ω).

Theorem (Masur '92)

For every translation surface ω, the set of nonuniquely ergodic directions has Hausdorff dimension at at most $1/2$.

Are there directions that are minimal but not uniquely ergodic? Yes!

This is an active area of research.

Theorem (Athreya, Chaika '14)

For almost every surface $\omega \in \mathcal{H}(2)$, the set of directions that are minimal but not uniquely ergodic has Hausdorff dimension $1/2$.

Summary

For a translation surface:

1. The set of uniquely ergodic directions is full measure.
2. The set of minimal but not uniquely ergodic directions has Hausdorff dimension $\leq 1/2$. (For almost all translation surfaces the Hausdorff dimension is > 0.)
3. There are countably many directions with a saddle connection. (Exercise!)

Two dynamical systems:

1). straight line flow on a translation surface ω

2). $GL^+(2,\mathbb{R})$ action on a stratum $\mathcal{H}$

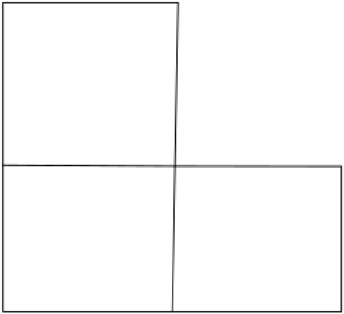
MAIN IDEA: Understanding the $GL^+(2,\mathbb{R})$ action on ω tells you about the surface ω.

Definition

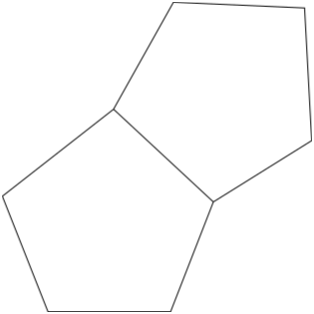
A **Veech surface** is a translation surface ω with a **closed** $GL^+(2,\mathbb{R})$-orbit.

Examples

Square tiled surfaces:



Regular polygons:



Theorem (Veech Dichotomy)

Let ω be a Veech surface. Any direction that contains a saddle connection is periodic. Any direction that does not contain a saddle connection is uniquely ergodic.

Veech surfaces have no minimal non uniquely ergodic direections.

You can find closed billiards on a regular n-gon by finding saddles.

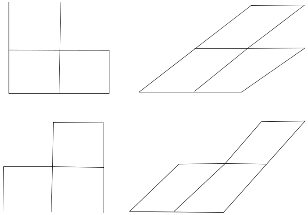
How can you tell if a surface is Veech?

Definition

An **isomorphism** between translation surfaces $(X, \omega) \cong (X', \omega')$ is a homeomorphism that is a translation in every coordinate chart. (Equivalently, it is a isomorphism of Riemann surfaces $\phi : X \cong X'$ such that $\phi^* \omega' = \omega$.)

Definition

The **Veech group** of (X, ω), denoted $SL(X, \omega)$, is the set of matrices $A \in SL(2, \mathbb{R})$ such that $A(X, \omega) \cong (X, \omega)$.



Example

Proposition

The Veech group of the square torus is $SL(2, \mathbb{Z})$.

Proof.

1. Check that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL(X, \omega)$
2. These two matrices generate $SL(2, \mathbb{Z})$.
3. The Veech group must preserve the **periods** of ω.
4. The periods of the square torus is $\mathbb{Z}^2$.

□

Definition

The **relative periods** $\Gamma < \mathbb{C}$ of (X, ω) is $\{\int_\gamma \omega : \gamma \in H^1(X, \Sigma; \mathbb{C})\}$.

Aside: $GL^+(2,\mathbb{R})$ or $SL(2,\mathbb{R})$?

Let $\omega \in \mathcal{H}(0)$ be the unit area square torus.

$GL^+(2,\mathbb{R})\omega = \mathcal{H}(0)$.

$GL^+(2,\mathbb{R}) \cong SL(2,\mathbb{R}) \times \mathbb{R}_+$

$SL(2,\mathbb{R})$ preserves the area of a translation surface.

$SL(2,\mathbb{R})\omega = \mathcal{H}_1(0) :=$ the area one surfaces in $\mathcal{H}(0)$.

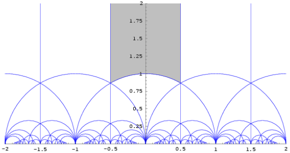
By orbit stabilizer, $\mathcal{H}_1(0) \cong SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ is a homeomorphism.

Definition

Let G be a Lie group (such as $\mathbb{C}$ or $SL(2,\mathbb{R})$). A **lattice** $\Gamma < G$ is a *discrete* subgroup such that the quotient G/Γ has *finite volume*.

Examples:

1. $\mathbb{Z}^2 < \mathbb{C}$ is a lattice
2. $SL(2,\mathbb{Z}) < SL(2,\mathbb{R})$ is a lattice.
   Proof: It suffices to show $\mathbb{H}/SL(2,\mathbb{Z})$ has finite volume.



Theorem

(X,ω) has closed $GL^+(2,\mathbb{R})$-orbit iff $SL(X,\omega)$ is a lattice. (Usually the latter property is the definition of a Veech surface.)

We have a map $\pi : \mathcal{H}(\kappa) \to \mathcal{M}_g$ given by $(X, \omega) \mapsto X$.

Example: $\mathcal{H}(0) \cong GL^+(2, \mathbb{R})/SL(2, \mathbb{Z}) \mapsto \mathcal{M}_1 \cong \mathbb{H}/SL(2, \mathbb{Z})$.

This map is not injective and may not be surjective. E.g $SO(2)$ does not change the Riemann surface structure.

Definition

A **Teichmüller curve** is an isometrically immersed $\mathbb{H} \to \mathcal{M}_g$. It can be thought of as a "complex geodesic".

Theorem

The Teichmüller curves are exactly the images of closed $GL^+(2, \mathbb{R})$-orbits projected to $\mathcal{M}_g$.

Proposition

The quotient $SL(2,\mathbb{R})/SL(X,\omega)$ is not compact. (True for all translation surfaces.)

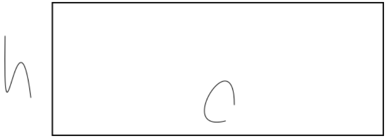
The orbit $SL(2,\mathbb{R})(X,\omega)$ is an immersed copy of $SL(2,\mathbb{R})/SL(X,\omega)$. Thus, it suffices to show the orbit is noncompact.

Theorem (Masur's compactness criterion)

A closed subset S of a stratum is compact iff there is a positive lower bound of all saddles of all surfaces in S.

Proof of proposition.

Choose any saddle connection on ω. Rotate the surface so that saddle connection is vertical i.e. there is some matrix $A \in SO(2)$ such that $A\omega$ has a vertical saddle connection. Then for large t, $\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} A\omega$ has arbitrarily small saddle. □

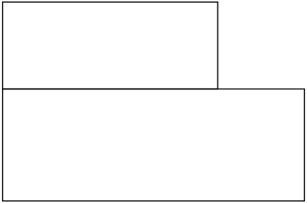


Let C be the above cylinder. The matrix $u_{c/h} = \begin{pmatrix} 1 & c/h \\ 0 & 1 \end{pmatrix}$ satisfies $u_{c/h}C = C$.

$u_{c/h}$ acts on C as a Dehn twist

Definition

The ratio $m = h/c$ is called the **modulus** of a cylinder. $u_{1/m}$ is a Dehn twist on a cylinder with modulus m.



Now we have two cylinders with modulus m_1, m_2.

If m_1/m_2 is a rational number (we say m_1, m_2 are **rationally related**), then there is a common multiple u_s between the Dehn twists $u_{1/m_1}, u_{1/m_2}$. u_s a parabolic in the Veech group of the above surface.

Proposition

Let (X, ω) be a translation surface. A direction θ is periodic with cylinders with rationally moduli iff there is a parabolic element $P \in SL(X, \omega)$ such that the direction of the eigenvector of P points in the direction θ. Such a direction is called **parabolic**.

Theorem (Veech Dichotomy '89)

Let (X, ω) be a Veech surface. Every direction with a saddle is parabolic and otherwise it is uniquely ergodic.

Converse is open!

Proof of easy direction.

1. Let θ be a direction with a saddle connection. Rotate the surface, so the saddle is vertical.
2. Under the action of $\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$ this saddle becomes very small.
3. By Masur compactness, a small saddle means this surface leaves all compact sets
4. It is a classical fact that if you go off to infinity in a certain direction in $SL(2, \mathbb{R})/\Lambda$, for Λ a lattice, then there is a parabolic $P \in \Lambda$ in that direction.

□

Trace Field

Definition

The **trace field** of a Veech surface (X, ω) is $\mathbb{Q}(\operatorname{tr} A : A \in SL(X, \omega))$ is the smallest field extension of $\mathbb{Q}$ that contains the trace of all elements of $SL(X, \omega)$.

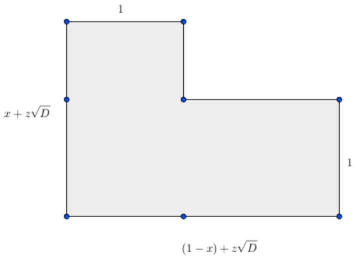
1. For an appropriately scaled surface, the periods live in the trace field.
2. Fix any parabolic direction. The ratio of circumferences of cylinders c_i/c_j are in the trace field. These ratios generate the trace field.
3. For any $A \in SL(X, \omega)$ that satisfies satisfies $\operatorname{tr} A > 2$ (i.e. A is hyperbolic), then $\operatorname{tr} A$ generates the trace field.
   Such an A corresponds to a closed geodesic in $SL(2, \mathbb{R})/SL(X, \omega)$ of length $\log \lambda$ where λ is the largest eigenvalue of A. We have $\operatorname{tr} A = \lambda + 1/\lambda$.

Concluding Remarks

Square-tiled surfaces are dense in each stratum.

The regular 10-gon is the only Veech surface in $\mathcal{H}(1,1)$ that is not a square-tiled surface.

Calta and McMullen's examples in $\mathcal{H}(2)$:



Open question: Which lattices are Veech groups of Veech surfaces?