1 Exercises

1. There are finitely many separatrices/ saddles in a give direction. Show that straight line flow is undefined for a finite set of separatrices and saddles as above.
2. Show there are at most finitely many saddle connections of length $\leq L$. Thus, there are countably many saddle connections on a flat surface.
3. Show that a closed straight line orbit (that does not go through singularities) must lie on the interior of a cylinder. The boundary of the cylinder must be saddles.
4. Find a translation surface (X, ω) with a nontrivial automorphism such that X has genus ≥ 2.
5. Show that for any curve, there is a geodesic segment in that homotopy class (with the same endpoints). Every closed curve has a geodesic representative in its free homotopy class. It is not unique iff it is the core curve of a cylinder.
6. Compute the Veech group of the following square-tiled surface with 3 squares.
7. A square-tiled surface (X, ω) can be thought of as a branched cover of the square torus with one branch point p. Some preimages of p are singularities of the flat metric and some are not. For this problem we will consider all preimages either singularities or marked points.
   (a) Show that $SL(X, \omega) < SL(2, \mathbb{Z})$.
   (b) Show that it is a finite index subgroup.
8. Let $A \in SL(2, \mathbb{Z})$. How is the Veech group of $A(X, \omega)$ related to the Veech group of (X, ω)?
9. We outline the proof of the proposition that after cutting all the saddles in a given direction, all the resulting components are periodic or minimal. Let ω be a compact translation surface and θ a direction.
   (a) Assume the straight-line orbit of x in direction θ is closed. Show that for a small enough neighborhood U around x, the straight-line orbit is closed for all points in U.
   (b) Let $I \subset \omega$ be a straight line segment not pointing in the direction θ. Show that if you flow I in the direction θ, it will eventually hit itself.
   (c) Let a be the left endpoint of I. Prove that if a is not on a saddle connection, a will eventually hit I (after flowing in the direction θ).
   (d) Let x be any point with an orbit Ox that is not closed. Show the boundary of $\overline{Ox}$ is a union of saddles as follows. Assume by contradiction that $a \in \partial\overline{Ox}$ was not on a saddle. There is a small straight line perpendicular to θ whose left (or right) endpoint is a. Use part (b) to get a contradiction.
   (e) Conclude that every component is minimal or periodic.
10. We outline the proof that a direction that contains a parabolic element $P \in SL(X, \omega)$ is periodic with rationally related cylinders. This was used in the proof of the Veech dichotomy.
    (a) $P \in SL(X, \omega)$ means that there is a homeomorphism $\phi : X \to X$ that is of the form $z \mapsto Pz + c$ in every translation coordinate chart where c can be different constants.
    (b) Show that ϕ permutes the singularities, so some power ϕ^k fixes all singularities.
    (c) Justify that ϕ^k must fix the separatrices in the direction v of the eigenvector of P.
    (d) By your work in problem 9, each separatrix must be a saddle connection or is dense in a component. Show that if a dense separatrix were fixed, then ϕ^k would be the identity, and that this is a contradiction.

   (e) Conclude that the saddles cut the surface up into cylinders and the cylinders have rationally related moduli.

11. Check that the sum of the orders of singularities of a holormophic 1-form is $2g - 2$. In two ways:

    (a) Using Riemann-Roch

    (b) Using Poincare-Hopf

12. Show that the complex dimension of $H_1(X, \Sigma; \mathbb{C})$ is $2g + |\kappa| - 1$.

13. We outline the proof that if the diameter is large, then there is a large cylinder.

    (a) Fix a point $x \in \omega$. A consider the closed ball $B_r(x)$ of radius r centered at x in the flat metric. We choose r to be the smallest radius such that $B_r(x)$ intersects itself. Assume that there are no singularities of $B_r(x)$ (or its boundary), so that there must be a saddle connection.

    (b) Let there by k singularities and let the diameter of the surface be R. Show there is a point distance at least R/k from any singularity.

    (c) Let x be a point that is far away all singularities. Show that a large enough $B_r(x)$ must intersect itself.