Riemann Surfaces and Dynamics

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Outline

1. Billiards and unfolding
2. Moduli spaces
3. The illumination problem

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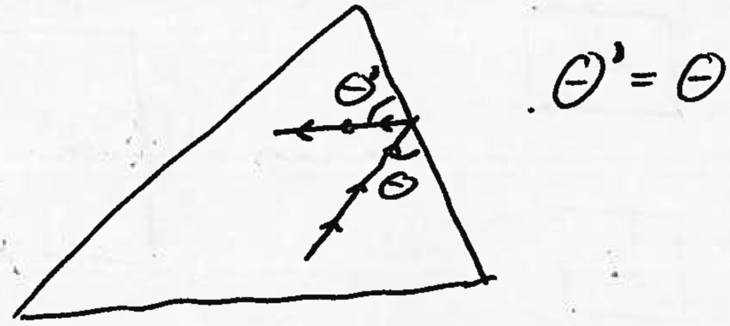

Unfolding a Table

We don't want to keep track of a sequence of angles! We unfold the table at every point of incidence, so that we have a mirror-version of our table where the path continues in a straight line

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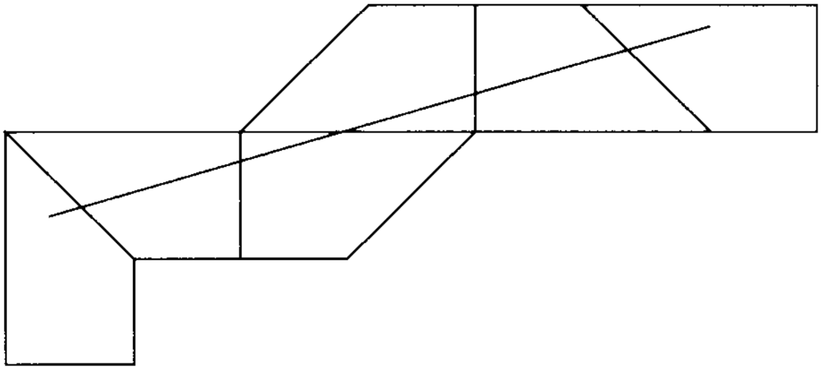

Fig. 2. Unfolding a billiard trajectory.

Unfolding a Table

Let G be the subgroup of $O(2)$ generated by the reflections through the sides of P. We want the angles of P to all be rational multiples of π, so that G is a finite group. The unfolding of P is the surface obtained by gluing together the reflected copies gP of P for every $g \in G$.

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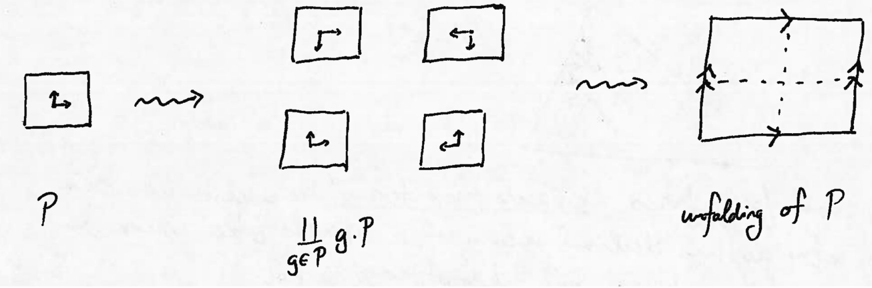
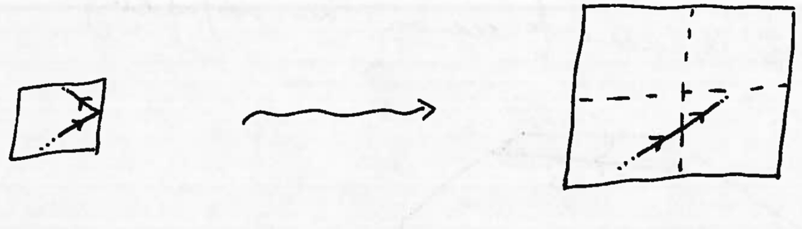


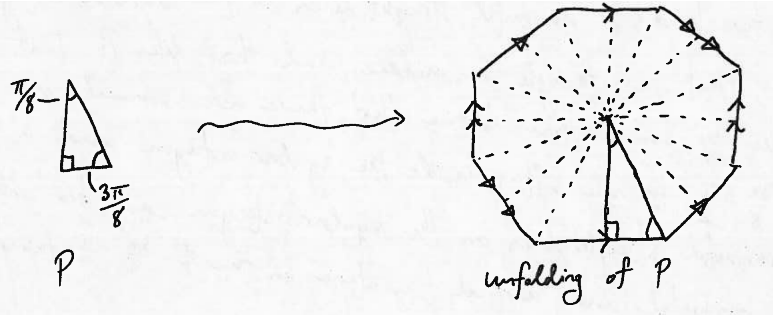
The Unfolded Square

The square table unfolds to a torus $\mathbb{R}^2/\mathbb{Z}^2$. All straight-line trajectories are given by the image of a straight line in $\mathbb{R}^2$ under the quotient $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$. Straight-line trajectories are therefore characterized by their slope: those of rational slope are periodic, and those of irrational slope are dense.



The Unfolded $\frac{\pi}{8} - \frac{3\pi}{8} - \frac{\pi}{2}$ Triangle

The triangular table with interior angles $\frac{\pi}{8}$, $\frac{3\pi}{8}$, and $\frac{\pi}{2}$ unfolds to the genus 2 surface given by identifying opposite sides of the regular octagon.



Translation Surfaces

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Definition (Translation surface, first interpretation)

A translation surface is a closed oriented topological surface S, along with a flat Riemannian metric on S, except at a finite set Σ of cone singularities, with trivial holonomy.

Translation Surfaces

- Let us consider for a moment a holomorphic 1-form ω on some Riemann surface X. On some coordinate chart U that does not include the zeroes of ω, we may write in coordinates $\omega = f(z)dz$, where f is a holomorphic function that vanishes nowhere on U.

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- Consider now two overlapping coordinate charts U_1 and U_2 with coordinates w_1 and w_2, respectively, where $\omega = dw_1$ on U_1 and $\omega = dw_2$ on U_2. Then on $U_1 \cap U_2$, we have $dw_1 = \omega = dw_2$, and hence $w_2 = w_1 + C$ for some $C \in \mathbb{C}$.

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- Translations $w_2 = w_1 + C$ preserve the Euclidean metric on $\mathbb{C} = \mathbb{R}^2$, and preserve the notion of "upward-pointing-vector."

Translation Surfaces

Definition (Translation surface, second interpretation)

A translation surface is a closed Riemann surface X, along with a holomorphic 1-form ω.

Moduli of Translation Surfaces

- Any translation surface can be drawn as a collection of polygons with opposite sides identified, as we saw with the torus (a single square) and the genus 2 surface (a single octagon).

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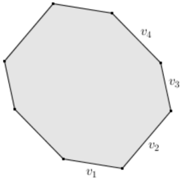
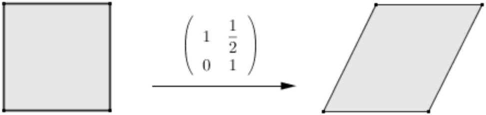
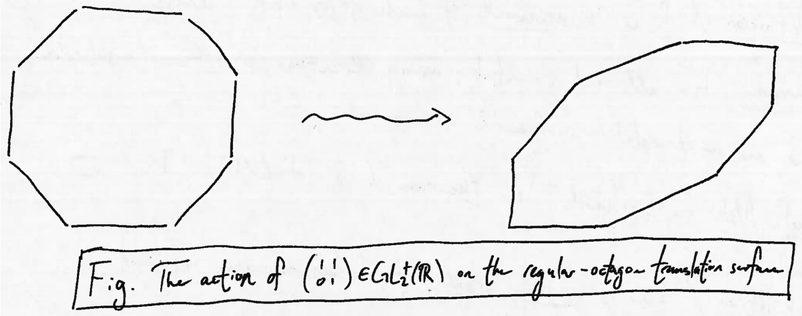


FIGURE 1.9. An octagon with opposite edges parallel may be specified by four complex numbers $v_1, v_2, v_3, v_4 \in \mathbb{C}$. (Not all choices give a valid octagon without self crossings, but there is an open set of valid choices.)

Moduli of Translation Surfaces

These polygonal pictures lie in the plane, and hence are acted upon by $\mathrm{GL}_2^+(\mathbb{R})$.





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Definition (Stratum of translation surfaces)

Let $\kappa = (k_1, \ldots, k_n)$ satisfy $\sum k_j = 2g - 2$. We denote by $\mathcal{H}(\kappa)$ the topological space that parametrizes all translation surfaces with zeroes of multiplicities κ. We have seen that $\mathcal{H}(\kappa)$ admits a natural action $\mathrm{GL}_2^+(\mathbb{R}) \curvearrowright \mathcal{H}(\kappa)$.

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Theorem (Magic Wand of EMM + Theorem of Filip)

Any closed $\mathrm{GL}_2^+(\mathbb{R})$-invariant subset of $\mathcal{H}(\kappa)$ is an algebraic $\mathbb{C}$-variety.

Moduli of Translation Surfaces

- Every translation surface can be drawn with polygons in infinitely many different ways. For example, any parallelogram with integral endpoints and area 1 gives the same translation surface $\mathbb{R}^2/\mathbb{Z}^2$.

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- In general, we take $\{v_j\}_j$ to be a collection of edges whose lengths and directions uniquely determine those of all the other edges. We call the resulting system of coordinates $(z_1, z_2, \ldots, z_{2g+n-1})$ period coordinates on $\mathcal{H}(\kappa)$.

The Illumination Problem

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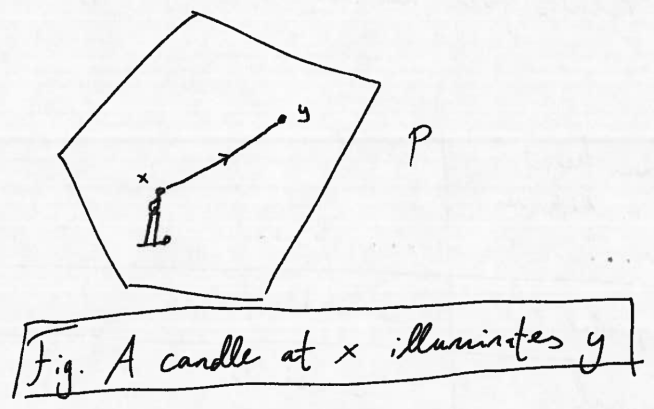
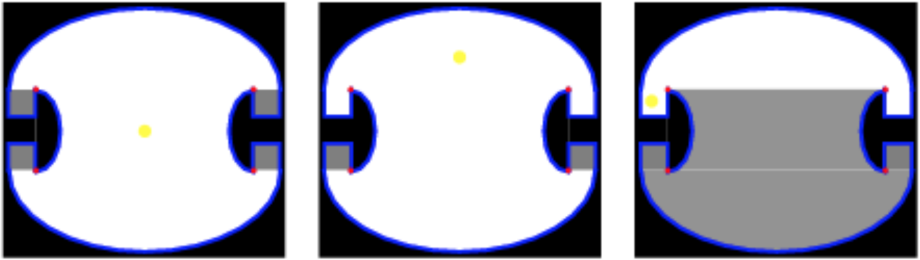


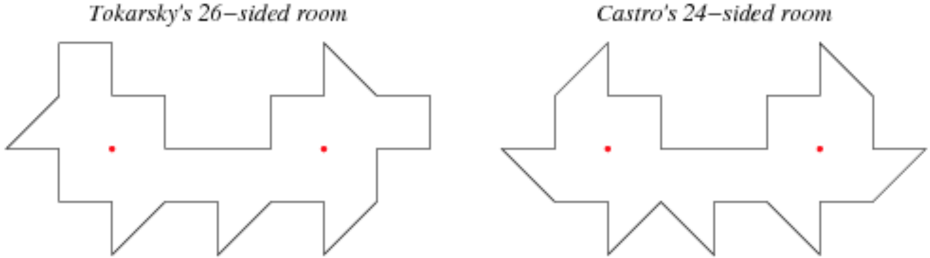
Roger Penrose's Non-Polygonal Room

In 1958, Roger Penrose gave an example of a non-polygonal room in which no point illuminates every other.



Tokarsky's and Castro's Polygonal Rooms

In 1995 and 1997, respectively, Tokarsky and Castro constructed polygonal rooms where the indicated points do not illuminate each other.



Cofinite Illuminability in Rational Polygonal Rooms

Theorem (Lelièvre-Monteil-Weiss, 2014)

In a polygonal room P, whose angles are rational multiples of π, every point fails to illuminate at most finitely many other points.

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Proof: Consider any such polygon P. We unfold P to a translation surface $(X_0, \omega_0) \in \mathcal{H}(\kappa)$ and choose any a point $x_0 \in X_0$. We consider x_0 as a marked point, or equivalently as a "zero" of ω_0 of multiplicity 0, so that $(X_0, \omega_0, x_0) \in \mathcal{H}(\kappa, 0)$.

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Define

$$\mathcal{N} = \{(X, \omega, x, y) \in \mathcal{H}(\kappa, 0, 0) \mid x \text{ does not illuminate } y\}$$
$$\mathcal{X}_0 = \{(X_0, \omega_0, x_0, y) \in \mathcal{H}(\kappa, 0, 0) \mid y \neq x_0\}.$$

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If I have any (X, ω, x, y) in the complement of $\mathcal{N}$, then x does illuminate y. If I vary this picture slightly, or by any element of $\mathrm{GL}_2^+(\mathbb{R})$ at all, we see that x continues to illuminate y. Therefore $\mathcal{N}$ is closed and $\mathrm{GL}_2^+(\mathbb{R})$-invariant, and hence algebraic over $\mathbb{C}$ by the Magic Wand and the Theorem of Filip.

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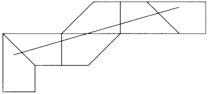
Since $\mathcal{X}_0$ is a copy of $X_0 \setminus x_0$, it is also algebraic over $\mathbb{C}$. Notice $\mathcal{X}_0 \not\subset \mathcal{N}$ and $\dim_{\mathbb{C}} \mathcal{X}_0 = 1$. Thus $\dim_{\mathbb{C}}(\mathcal{X}_0 \cap \mathcal{N}) = 0$. Thus the set of $y \in X_0$ not illuminated by x_0 is a 0-dimensional variety, which is a finite set of points.

$\square$

Image References

- 
Fig. 2. Unfolding a billiard trajectory.
Masur, H & Tabachnikov S. *Rational Billiards and Flat Structures*. Handbook of Dynamical Systems. vol. 1A, ch. 13, pg. 1019. Edited by B. Hasselblatt and A. Katok. Elsevier Science B.V., 2002.
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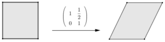
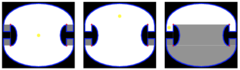
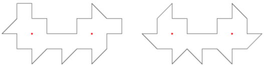
and  Wright, Alex. *Translation surfaces and their orbit closures: An introduction for a broad audience*. Pages 11 and 27. arXiv: 1411.1827.
-  and 

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Theorem References

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