Inflection Points, Extatic Points and Curve Shortening[1]

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If (M^2, g) is a surface with Riemannian metric, then a family of immersed curves $\{\mathcal{C}_t \mid 0 \le t < T\}$ on M^2 evolves by *Curve Shortening* if

$$\frac{\partial \mathcal{C}}{\partial t} = \kappa_g \vec{\nu}, \tag{1}$$

where κ_g is the geodesic curvature, and $\vec{\nu}$ is a unit normal to the curve. Since $\kappa_g \vec{\nu}$ can be written as $\frac{\partial^2 \mathcal{C}}{\partial s^2}$, where s is arclength along $\mathcal{C}$, (1) is essentially a parabolic equation, i.e. a nonlinear heat equation.

In [An2] it is shown that for any solution $\{\mathcal{C}_t \mid 0 \le t < T\}$ of (1) there is only a discrete set of times at which the immersed curve $\mathcal{C}_t$ will have self tangencies. Hence the number of self- intersections of $\mathcal{C}_t$ is always finite, and it was also shown in [An2] that this number decreases whenever the curve develops a self- tangency.

As the name suggests, Curve Shortening is a gradientflow for the length functional on the space of immersed curves in the surface M^2. One can therefore try to use Curve Shortening to prove existence of geodesics by variational methods. In my talk at S'Agarro I observed that geodesics always are curves without self-tangencies, and recalled that the space of such curves has many different connected components. I then discussed how one can try to exploit the nice behaviour of Curve Shortening with respect to self-intersections to prove existence of geodesics in each component.

The fact that Curve Shortening never increases the number of self- intersections of a curve is a consequence of a theorem of Sturm on linear parabolic equations, and instead of describing the contents of my talk I would like to point out that this theorem of Sturm can also be used to give alternative proofs of the following theorems of Arnol'd:

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Theorem A ("Tennis ball theorem"). *Any embedded curve in S^2 which divides the sphere into two parts of equal area has at least four inflection points.*

Theorem B. *Any noncontractible embedded curve in $\mathbb{RP}^2$ has at least three inflection points.*

Theorem C. *Any plane convex curve has at least four inflection points and six extatic points.*

Proofs of these theorems can be found in [Ar1] and in the preprints made available at the conference in S'Agarro. One of these preprints [Ar2] contains a fourth theorem on "flattening points" of space curves which Arnol'd puts in the same list of generalizations of the Morse inequalities. I have not been able to find a proof of this particular theorem along the same lines as the proofs of theorems A, B and C which I will give below.

The theorem of Sturm which we use to give alternative proofs of theorems A, B and C can be stated as follows:

Theorem ([Sturm 1836, An1]). *Let $u(x,t)$ satisfy a linear parabolic PDE of the type*

$$\frac{\partial u}{\partial t} = a(x,t)\frac{\partial^2 u}{\partial x^2} + b(x,t)\frac{\partial u}{\partial x} + c(x,t)u \qquad (2)$$

for $x \in \mathbb{R}$, $0 < t < T$, and assume that $u(x+1,t) \equiv u(x,t)$. Assume u, u_t, u_x, u_{xx}, a, a_t, a_x, a_{xx}, b, b_x and c are continuous. Assume furthermore that the coefficient $a(x,t)$ is strictly positive.

Then $z(t) = \#\{x \in \mathbb{R}/\mathbb{Z} \mid u(x,t) = 0\}$ is a finite and nonincreasing function of t. At any time t for which $u(\cdot,t)$ has a multiple zero $z(t)$ will decrease.

At the end of this note I will show how this theorem is similar to the well known fact that zeroes of a function of one complex variable always have positive degree.

The proofs of theorems A, B and C will more or less go like this: Given an arbitrary curve $\mathcal{C}$ we consider the maximal evolution $\{\mathcal{C}_t \mid 0 \le t < T\}$ of Curve Shortening with initial data $\mathcal{C}_0 = \mathcal{C}$ and determine the asymptotic behaviour of $\mathcal{C}_t$ as $t \uparrow T$. We then observe that the curvature $\kappa_{\mathcal{C}_t}$ satisfies a linear parabolic equation of the form (2), so that Sturm's theorem tells us that $\mathcal{C}$ has at least as many inflection points as $\mathcal{C}_t$ for t close to T. From the asymptotic behaviour of $\mathcal{C}_t$ for $t \uparrow T$ we then get the desired lower bound for the number of inflection points. To estimate the number of extatic points we allow the curve to evolve by "Affine Curve Shortening" and consider the affine curvature μ instead of the Euclidean curvature.

§1. The Tennis ball Theorem.

Let $\mathcal{C}_0 \subset S^2$ be an embedded curve which divides the sphere in two parts of equal area, and let $\{\mathcal{C}_t \mid 0 \le t < T\}$ be the corresponding evolution by Curve Shortening. If one parametrizes $\mathcal{C}_t$ by $\mathcal{C} : \mathbb{R}/\mathbb{Z} \times [0,T) \to S^2$ with $\partial_t \mathcal{C} \perp \partial_x \mathcal{C}$, then the curvature $\kappa(x,t)$ of $\mathcal{C}_t$ at $\mathcal{C}(x,t)$ satisfies

$$\frac{\partial \kappa}{\partial t} = \frac{\partial^2 \kappa}{\partial s^2} + (\kappa^2 + 1)\kappa \tag{3}$$

where $\partial/\partial s = |\partial_x \mathcal{C}|^{-1} \partial/\partial x$. This equation is of the form (2) so that the number of zeroes of $\kappa(\cdot, t)$, i.e. the number of inflection points of $\mathcal{C}_t$ does not increase with t.

Lemma. *The evolution $\mathcal{C}_t$ exists for all $t > 0$. At any time t the curve $\mathcal{C}_t$ divides the sphere into two parts of equal area. As $t \to \infty$ the curve $\mathcal{C}_t$ converges to a great circle.*

Denote by $\Omega(t)$ one of the two components of $S^2 \setminus \mathcal{C}_t$ and let $A(t)$ be the area of $\Omega(t)$. Then

$$A'(t) = \int_{\mathcal{C}_t} v\,ds = \int_{\mathcal{C}_t} \kappa\,ds,$$

where v is the normal velocity and κ is the curvature of $\mathcal{C}_t$ in the direction of the outward normal of Ω_t. The sphere has Gauss curvature $K \equiv +1$ so the Gauss-Bonnet theorem tells us that

$$A'(t) = \int_{\mathcal{C}_t} \kappa\,ds = -2\pi + \int\int_{\Omega_t} K\,dA = A(t) - 2\pi.$$

By our assumption $A(0) = 2\pi$ and it thus follows that $A(t) \equiv 2\pi$ for all $t \in [0, T)$.

By Grayson's theorem [Gr] the solution $\mathcal{C}_t$ either shrinks to a point in finite time or else exists for all $t \geq 0$. Since $\mathcal{C}_t$ bounds a region Ω_t of area 2π it cannot shrink to a point and hence must exist forever.

The ω-limit set of the evolution $\mathcal{C}_t$ consists of geodesics, and since the flow is real analytic an argument of Leon Simon implies that the $\mathcal{C}_t$ converge to a unique geodesic $\mathcal{C}_\infty$ of S^2. Such a geodesic must of course be a great circle. To determine how many inflection points $\mathcal{C}_t$ has for large t we choose coordinates and linearize Curve Shortening around the limit $\mathcal{C}_\infty$.

We may assume that $\mathcal{C}_\infty$ is the equator, i.e. the intersection of S^2 with the xy-plane. After removing north and south poles we can then project the sphere onto the cylinder $x^2 + y^2 = 1$, which gives us coordinates (ϕ, z). In these coordinates the equator is given by $z = 0$ and any C^1 nearby curve is the graph of a 2π periodic function $z = u(\phi)$. For instance, any great circle which is not a meridian is the graph of $u(\phi) = A\cos(\phi - \phi_0)$ for certain $A \in \mathbb{R}$, $\phi_0 \in \mathbb{R}/2\pi\mathbb{Z}$.

If t is large then $\mathcal{C}_t$ will be a graph $z = u(\phi, t)$. Curve Shortening implies that $u(\phi, t)$ satisfies

$$\frac{\partial u}{\partial t} = A(u, u_\phi)\left(\frac{\partial^2 u}{\partial \phi^2} + u\right) \tag{4}$$

where $A(u, u_\phi)$ is some smooth positive function with $A(0,0) = 1$. One can compute A explicitely, but the precise form of the equation is not important. Instead we observe that since great circles do not evolve under Curve Shortening, the functions $u = M\cos(\phi - \phi_0)$ must be steady states of (4) for any value of M, ϕ_0.

Our solution $u(\phi, t)$ together with its derivatives converge to $u \equiv 0$ as $t \to \infty$. From this one can deduce that u must be asymptotic to a solution of the linearized equation corresponding to (4), i.e. to

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial \phi^2} + v.$$

Thus

$$u(\phi, t) = Ce^{(1-n^2)t}\cos\left(n(\phi - \phi_0)\right) + o\left(e^{(1-n^2)t}\right) \tag{5}$$

for some $n \geq 2$ and some $C \neq 0$. Here $o(\cdots)$ represents some function which is small in C^k for any $k < \infty$.

The proof is now complete since the graph of $u(\cdot, t)$ will have at least $2n \geq 4$ inflection points.

Nearly the same argument gives us the theorem on inflection points of simple noncontractible curves in $\mathbb{RP}^2$. Indeed, if $\gamma_0 \subset \mathbb{RP}^2$ is such a curve then its lift $\mathcal{C}_0$ to the unit sphere is an embedded curve which divides the sphere into two parts of equal area. The lift $\mathcal{C}_0$ is also invariant under the antipodal map $x \mapsto -x$. As above the corresponding evolution $\{\mathcal{C}_t \mid t \geq 0\}$ by Curve Shortening will converge to a great circle, with asymptotics given by (5) for some $n \geq 2$. Since all $\mathcal{C}_t$ must also be invariant under the antipodal map $(\phi, z) \mapsto (\phi + \pi, -z)$ only odd values of n can occur in (5). Hence the lowest value of n which can appear is $n = 3$. For any $n \geq 3$ the curve with graph $\varepsilon \cos(n(\phi - \phi_0))$ has at least 6 inflection points. By Sturm's theorem $\mathcal{C}_0$ must have at least 6 inflection points, and the curve γ_0 in the projective plane must have at least 3 inflection points.

§2. Extatic Points

Let $\mathcal{C} \subset \mathbb{R}^2$ be a convex curve. For any point $P \in \mathcal{C}$ there will be a conic section

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

which has maximal order of contact with $\mathcal{C}$ at P. In general this order of contact will be 5. If the order of contact is 6 or more the point P is called *extatic*. We can also describe these points in terms of affine geometry.

Recall that the affine arc length on a convex curve is defined by

$$ds = (\omega(\mathcal{C}_x, \mathcal{C}_{xx}))^{1/3}\, dx$$

where ω is the area or symplectic form on $\mathbb{R}^2$. If one parametrizes $\mathcal{C}$ by affine arclength one has $\omega(\mathcal{C}_s, \mathcal{C}_{ss}) \equiv 1$, and hence after differentiation $\omega(\mathcal{C}_s, \mathcal{C}_{sss}) \equiv 0$. It follows that for some $\mu \in \mathbb{R}$ one has

$$\mathcal{C}_{sss} = -\mu \mathcal{C}_s.$$

The quantity μ is called the *affine curvature* of $\mathcal{C}$. Conic sections are exactly those curves which have constant affine curvature. One easily verifies the following:

Lemma. *$P \in \mathcal{C}$ is extatic if and only if* $\dfrac{\partial \mu}{\partial s}(P) = 0$.

We must therefore show that the affine curvature has at least 6 critical points on any convex curve. To do this we let the curve evolve by *affine curve* *shortening*, i.e. with normal velocity $v = (\kappa)^{1/3}$. In terms of a parametrization this is equivalent with the PDE

$$\frac{\partial \mathcal{C}}{\partial t} = \frac{\partial^2 \mathcal{C}}{\partial s^2}$$

which is formally similar to (1), but actually is different since here s is affine arclength, and thus $\frac{\partial}{\partial s} = \{\omega(\mathcal{C}_x, \mathcal{C}_{xx})\}^{-1/3} \frac{\partial}{\partial x}$.

Let $p(\theta, t)$ be the *support function* of $\mathcal{C}_t$, i.e.

$$p(\theta, t) = \sup\{x \cos\theta + y \sin\theta \mid (x, y) \in \mathcal{C}_t\}.$$

The support function p completely determines the curve $\mathcal{C}_t$, and the curvature, velocity and affine curvature are given by

$$\kappa = -\left(p + p_{\theta\theta}\right)^{-1}, \qquad v = -\left(p + p_{\theta\theta}\right)^{-1/3},$$
$$\mu = v^3 \left(v_{\theta\theta} + v\right). \tag{6}$$

Starting from $v = \frac{\partial p}{\partial t}$ one then computes that $v(\theta, t)$ and $\mu(\theta, t)$ evolve according to

$$3\frac{\partial v}{\partial t} = v^4 \left(v_{\theta\theta} + v\right) = v\mu, \tag{7}$$

$$3\frac{\partial \mu}{\partial t} = v^4 \mu_{\theta\theta} + 2v^3 v_\theta \mu_\theta + 4\mu^2, \tag{8}$$

and hence, after differentiating (8) with respect to θ, one finds that μ_θ satisfies a linear parabolic PDE of the type (2). Sturm's theorem therefore says that affine curve shortening does not increase the number of extatic points of a convex curve.

Recall that Sapiro and Tannenbaum [STa] showed that the $\mathcal{C}_t$ shrink to a point at some time $T > 0$, and that the $\mathcal{C}_t$, after rescaling by a factor $(T-t)^{-3/4}$, converge to an ellipse. After applying a translation and a special affine transformation we may assume that the limiting point is the origin, and that the limiting ellipse is a circle. Thus the curve $(T-t)^{3/4}\mathcal{C}_t$ converges to a circle, and the rescaled curvature $(T-t)^{3/4}\kappa(\theta, t)$ and rescaled velocity $(T-t)^{1/4}v(\theta, t)$ converge in C^∞ to constants.

We put $w(\theta, \tau) = (T-t)^{1/4}v(\theta, t)$, and $\tau = -\ln(T-t)$, and observe that (7) implies

$$\frac{\partial w}{\partial \tau} = \frac{w^4}{3}\left(w_{\theta\theta} + w\right) - \frac{w}{4}. \tag{9}$$

The constant to which v converges as $t \uparrow T$, i.e. when $\tau \to \infty$, must be a time independent solution of (9), from which one finds that $w(\theta, \tau) \to \left(\frac{3}{4}\right)^{1/4}$ $(\tau \to \infty)$.

To detect the oscillations of w for large τ we linearize (9), i.e. we put $w(\theta, \tau) = \left(\frac{3}{4}\right)^{1/4} + \psi(\theta, \tau)$, and after discarding higher order terms in ψ find that ψ satisfies

$$\frac{\partial \psi}{\partial \tau} = \tfrac{1}{4}\psi_{\theta\theta} + \psi. \tag{10}$$

For large values of τ any solution of this equation is asymptotically like

$$\psi(\theta, \tau) \sim C e^{(1-k^2/4)\tau} \cos(k(\theta - \theta_0)) \tag{11}$$

for some $k \in \mathbb{N}$, $C > 0$ and θ_0. Since ψ vanishes as $\tau \to \infty$ we must have $k \geq 3$.

If we now substitute (11) back in (6), we get an asymptotic expansion for $\mu(\theta, t)$,

$$\mu(\theta, t) \sim \frac{3}{4(T-t)} + C'(T-t)^{k^2/4-2} \cos k(\theta - \theta_0) + \cdots,$$

from which it follows that μ has at least $2k \geq 6$ critical points.

To prove the four vertex theorem one evolves a convex curve by (ordinary, Euclidean) Curve Shortening and applies exaclty the same argument. The analog of the Sapiro-Tannenbaum theorem is the result of Gage and Hamilton [GH], which says that after rescaling by $(T-t)^{-1/2}$ the curve converges smoothly to a circle of radius $\sqrt{2}$. The resulting proof is of course much more complicated than the textbook proof.

§3. Parabolic equations as degenerate Cauchy-Riemann equations.

One can present Sturm's theorem as an analog of a well known fact concerning analytic functions: Any nondegenerate zero of an analytic function has positive degree; by a small perturbiation of the function one can decompose a degenerate zero into several nondegenerate zeroes and conclude that any zero (simple or not) has positive degree.

To see the analogy, consider $u(x, t)$ a smooth function on $\Omega = \mathbb{R}/\mathbb{Z} \times [0, T]$ which satisfies (2). If $u(\cdot, t_0)$ has simple zeroes for some $t_0 \in [0, T]$, then the number of zeroes of $u(\cdot, t_0)$ is twice the winding number of the map $w : \Omega \to \mathbb{R}^2$

$$w(x,t) = \left(\frac{\partial u}{\partial x}(x,t), u(x,t)\right)$$

on the circle $\mathbb{R}/\mathbb{Z} \times \{t_0\}$. Thus the number of zeroes at time T minus the number of zeroes at time $t = 0$ is twice the degree of the map w on Ω.

Suppose now that $w = (v, u)$ satisfies a first order system of equations of the form

$$u_x = A(x,t)u + B(x,t)v \tag{12a}$$

$$u_t - \beta(x,t)v_x = C(x,t)u + D(x,t)v \tag{12b}$$

for certain functions β, A, B, C, D on Ω. For instance, this will be true if u satisfies (2) (choose $\beta(x,t) = a(x,t)$, $A = 0$, $B = 1$, $C(x,t) = -c(x,t)$ and $D(x,t) = -b(x,t)$.)

Positivity Lemma. *Assume $\beta(x,t) > 0$ on Ω. If w is non zero on $\partial\Omega$ and if w only has simple zeroes in Ω, i.e. if $\det Dw \neq 0$ at any zero of w, then the degree of $w : \Omega \to \mathbb{R}^2$ is nonnegative.*

The reason is simple: at any zero one has $u = v = 0$, so that (12a), (12b) imply that $u_x = 0$, and $u_t = \beta(x,t)v_x$. Hence

$$\det Dw(x,t) = \begin{vmatrix} v_x & v_t \\ u_x & u_t \end{vmatrix} = \begin{vmatrix} v_x & v_t \\ 0 & \beta v_x \end{vmatrix} = \beta(x,t)\,(v_x)^2 \geq 0.$$

If the determinant is non zero, it must therefore be positive.

If the system (12a), (12b) is such that any solution can be approximated by a solution with only simple zeroes (or even a solution of a system of the same type with simple zeroes) then one can drop the condition that w must have simple zeroes: any solution of (12a), (12b) which is nonzero on $\partial\Omega$ must have nonnegative degree.

The system (12a), (12b) is similar to the Cauchy-Riemann equations. Indeed, the same arguments can (and have of course been) applied to equations of the form

$$u_x + \alpha(x,t)v_t = A(x,t)u + B(x,t)v \tag{13a}$$

$$u_t - \beta(x,t)v_x = C(x,t)u + D(x,t)v \tag{13b}$$

with positive coefficients α, β, to yield the same conclusion (so, for $\alpha = \beta = 1$ and $A = B = C = D = 0$ one finds that an analytic function $v + iu$ of a complex variable $x + it$ always has nonnegative degree.)

One obtains the system (12a), (12b) which contains the linear parabolic equation (2) as a special case by letting the coefficient α tend to zero.

§4. Which equations satisfy the positivity lemma?

We will show that up to linear transformations the only systems of two equations in two functions of two variables for which the proof of the positivity lemma works, are the Cauchy Riemann equations and the Heat Equation written as a system.

Consider a system of ℓ PDEs

$$\mathcal{M}_i(u) = \sum_{i=1}^{n} a_i^j(x) u_j(x) \tag{14}$$

in n functions $u_1, \cdots, u_n$ of n variables $x_1, \cdots, x_n$, where $\mathcal{M}$ is a first order differential operator

$$\mathcal{M}_i(u) = \sum_{j,k} M_i^{jk} \frac{\partial u_j}{\partial x_i}.$$

Assume that the degree of any nondegenerate zero of any solution of (14) is positive, for the reasons given in the proof of the positivity lemma. In other words, we assume that any nonsingular matrix A_{jk} with $M_i^{jk} A_{jk} = 0$ actually has $\det A > 0$. What can we say about the differential operator $\mathcal{M}$?

One can transform the equations (14) in three different ways: one can make linear combinations of the equations, i.e. one can replace the differential operators $\mathcal{M}_i$ by $\mathcal{M}'_i = S_{ij}\mathcal{M}_j$ for an arbitrary $S \in \mathrm{GL}(n, \mathbb{R})$. The new and old equations will have the same solutions, so the positivity lemma will hold for one if and only if it holds for the other. Upto substitutions of this kind, the operators $\mathcal{M}_i$ are completely determined by their kernel, i.e. the subspace

$$\mathcal{K} = \{A \in \mathcal{L}_n \mid \sum_{j,k} M_i^{jk} A_{jk} = 0 \forall i\}$$

of the space of all $n \times n$ matrices $\mathcal{L}_n$

Thus we can rephrase our question as follows: for which subspaces $\mathcal{K} \subset \mathcal{L}_n$ is $\det A \geq 0$ for all $A \in \mathcal{K}$?

Since $\det(-A) = (-1)^n \det A$, the dimension n must be even. Examples of such subspaces $\mathcal{K} \subset \mathcal{L}_n$ in any even dimension are given by the kernel of the Cauchy Riemann equations.

The two other types of transformations which one can apply to the equations (14) are linear transformations of the dependent variables $u'_i = S_{ij} u_j$, and of the independent variables $x'_i = R_{ij} x_j$. The corresponding action on $\mathcal{L}_n$ is given by $A \mapsto S \cdot A \cdot R^{-1}$. If we require $\det S$, $\det R > 0$, then the determinant will be nonnegative on $\mathcal{K}$ if and only if it is nonnegative on $S\mathcal{K}R^{-1}$.

In two dimensions one can now easily classify all equations of the type (14) for which the positivity lemma applies.

Let $\mathcal{K} \subset \mathcal{L}_2$ be a linear subspace on which the determinant is nonnegative. If $\mathcal{K}$ is one dimensional, then it is spanned by a matrix with nonnegative determinant. Conversely, any subspace spanned by such a matrix has $\det \geq 0$. This situation corresponds to a system of three equations.

Assume that $\mathcal{K}$ is two dimensional. If $\mathcal{K}$ only contains singular matrices, then up to linear transformations it must be the subspace spanned by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$.

If $\mathcal{K}$ contains at least one nonsingular matrix A then we may assume that A is the identity matrix after replacing $\mathcal{K}$ with $A^{-1}\mathcal{K}$. Being two dimensional, $\mathcal{K}$ is spanned by I and some other matrix B, i.e. $\mathcal{K} = \{\lambda I + \mu B \mid \lambda, \mu \in \mathbb{R}\}$. After replacing $\mathcal{K}$ with $S\mathcal{K}S^{-1}$ for suitable S we may assume that B is in Jordan normal form. By subtracting a suitable multiple of I from B we may assume that B has trace zero. Let $\pm\beta$ be the eigenvalues of B. Then by assumption

$$\det(\lambda I + \mu B) = (\lambda + \mu\beta)(\lambda - \mu\beta) \geq 0$$

for all $\lambda, \mu \in \mathbb{R}$. This can only happen if $\beta = i\omega$, $\omega \geq 0$ is imaginary or zero.

If $\beta \neq 0$ we find that $\mathcal{K}$ consists of all matrices of the form $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$, and that the corresponding differential operator $\mathcal{M}$ is equivalent to the Cauchy-Riemann equations. If $\beta = 0$, then B must be the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, and $\mathcal{K}$ consists of all matrices of the form $\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)$. the corresponding equations are then equivalent to the system (12a), (12b) related to the heat equation.

For a geometric view of the preceding argument one should identify $\mathcal{L}_2 =$

$\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R}\}$ with $\mathbb{R}^4$, and identify the linear subspace $\mathcal{K}$ with the corresponding linear subspace $K \subset \mathbb{RP}^3$. The equation $\det \geq 0$, i.e. $ad - bc \geq 0$ defines a subset of $\mathbb{RP}^3$ whose boundary is a quadric Q. The question is therefore *which subspaces $K \subset \mathbb{RP}^3$ lie on one side of the quadric Q?* We found: points ($\mathcal{K}$ one-dimensional), lines disjoint from Q (Cauchy-Riemann equations), lines tangent to but not contained in Q (equations of parabolic type) and lines contained in Q ($\mathcal{K}$ only contains singular matrices and the positivity lemma is vacuous).

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