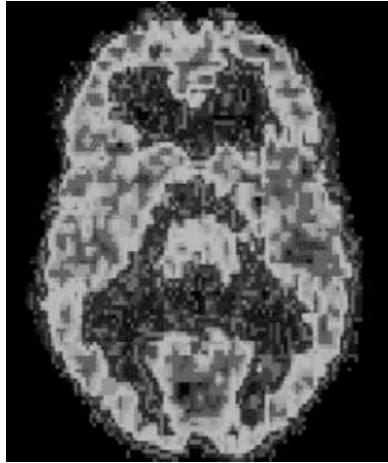


Uniformization,
the Monge-Kantorovich Problem
and Medical Imaging

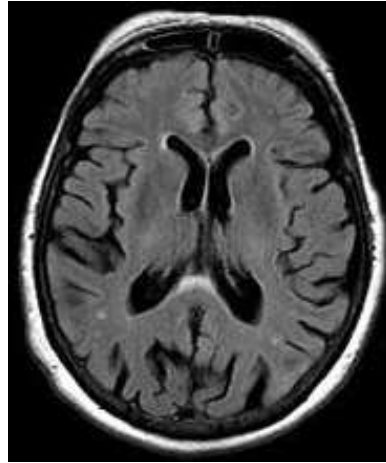
Sigurd Angenent & Steve Haker & Ron Kikinis &
Allen Tannenbaum

12 October, 2005

Medical Imaging produces pictures like these



PET



MRI

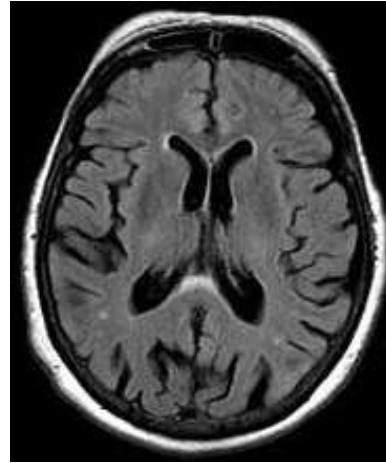


CT

Medical Imaging produces pictures like these



PET



MRI



CT

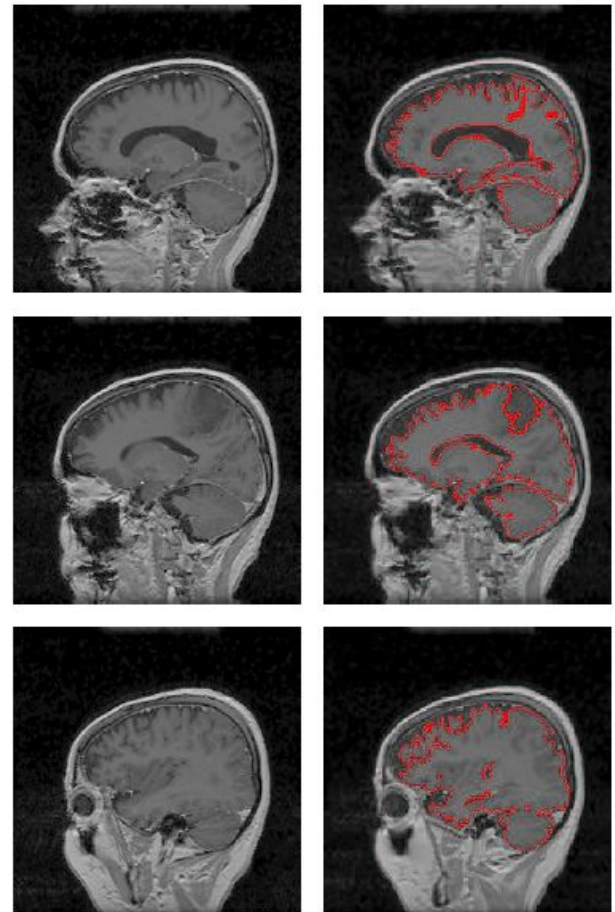
(These are actually 2D slices of 3D images)

“Segmentation”

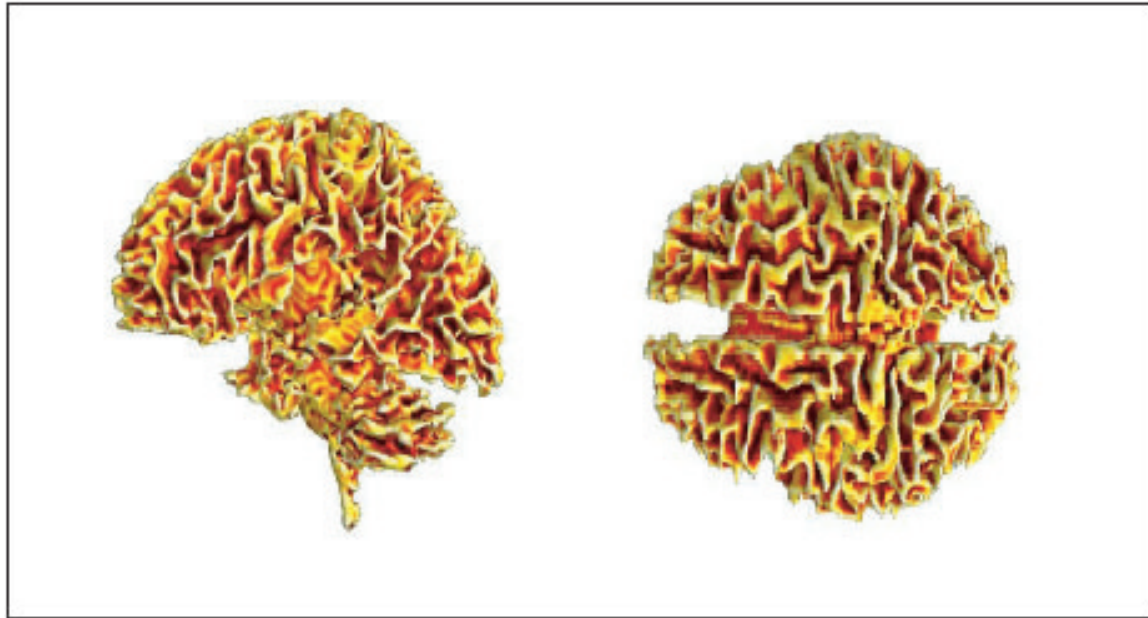
Extracting relevant
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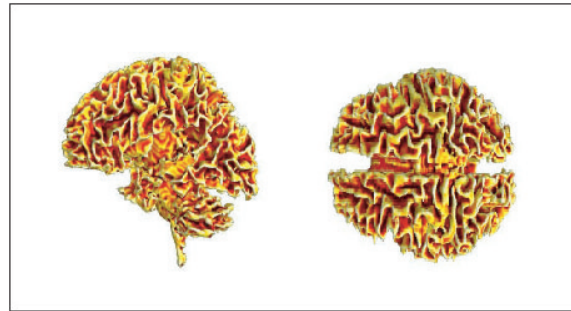
Putting together all the slices leads to a surface .



The segmented surface is only minimally smooth.

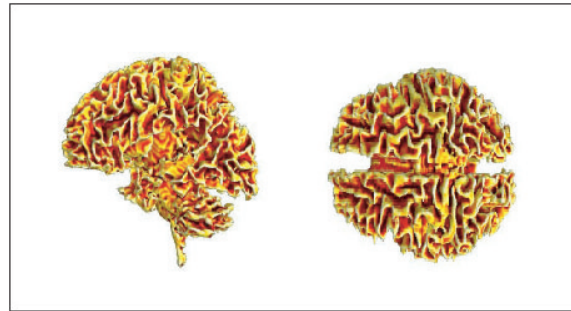
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Find a “canonical method” to map the given surface onto a standard surface, such as the unit sphere S^2



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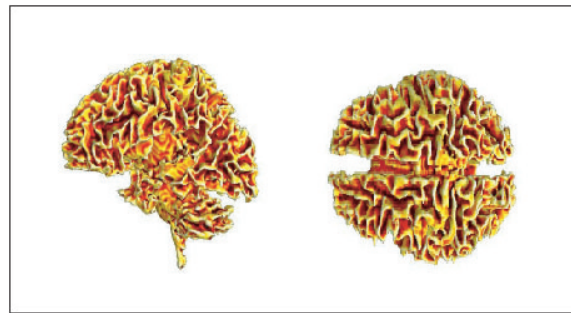
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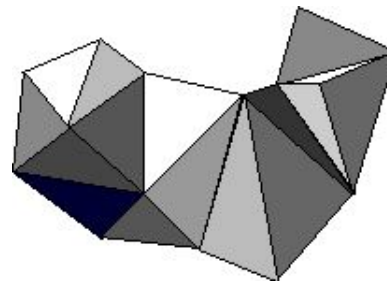
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Options: instead of looking for an isometry, try to find

conformal maps, or **area preserving maps**, or ...

Definition. A map $f : S \rightarrow \Sigma$ is **conformal** if “ f preserves angles,” or, if there is some positive function $\lambda : S \rightarrow \mathbb{R}_+$ such that

$$\langle df_p \cdot \vec{v}, df_p \cdot \vec{w} \rangle = \lambda(p) \langle \vec{v}, \vec{w} \rangle$$

holds for all points $p \in S$ and all tangent vectors $\vec{v}, \vec{w} \in T_p S$.

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Some examples. . .

Theorem. (Gauss) A sufficiently small neighborhood of any point p on a smooth surface $\Sigma \subset \mathbb{R}^3$ has a conformal parametrization $X: \mathcal{U} \rightarrow \Sigma$ ($\mathcal{U} \subset \mathbb{R}^2$ open).

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In conformal coordinates the metric is given by

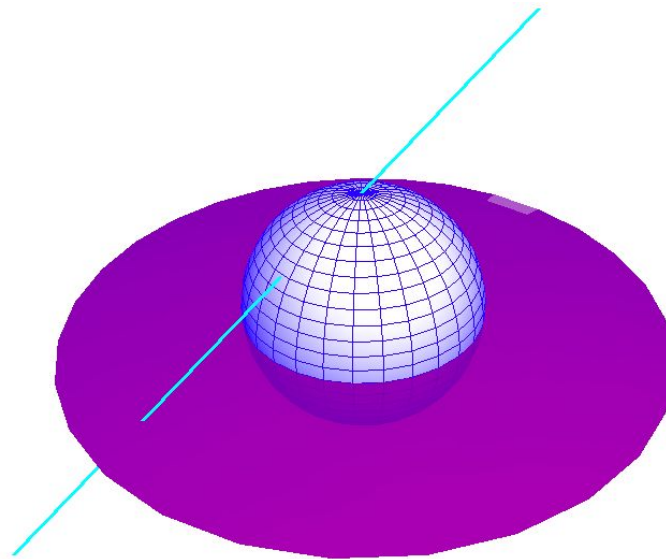
$$(ds)^2 = \lambda(u, v)\{(du)^2 + (dv)^2\}.$$

Example – Stereographic projection.

$$X(u, v) = \begin{pmatrix} 2u/(1 + u^2 + v^2) \\ 2v/(1 + u^2 + v^2) \\ (1 - u^2 - v^2)/(1 + u^2 + v^2) \end{pmatrix}$$

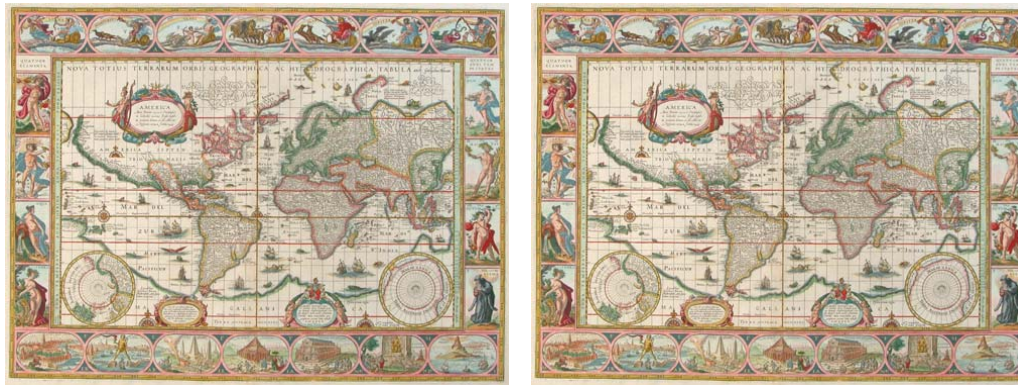
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- $\|df_p\| = \sqrt{\lambda}$.

Generalized stereographic projection. We call the composition

$$F : \Sigma \setminus \{p\} \xrightarrow{f} S^2 \setminus \{N\} \xrightarrow{x} \mathbb{R}^2 = \mathbb{C}$$

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Boundary condition near p ?

The PDE for F . Given a point $p \in \Sigma$, a scale $\lambda > 0$ and a unit vector $\vec{v} \in T_p \Sigma$.

Choose $\vec{w} \in T_p \Sigma$ so $\{\vec{v}, \vec{w}\}$ is orthonormal. Then F satisfies

$$\Delta_\Sigma F = \lambda \nabla_{\vec{v} + i\vec{w}} \{\delta_p(u, v)\} \quad (1)$$

in the sense of distributions.

Proof: In conformal coordinates $z = u + iv$ near p one has

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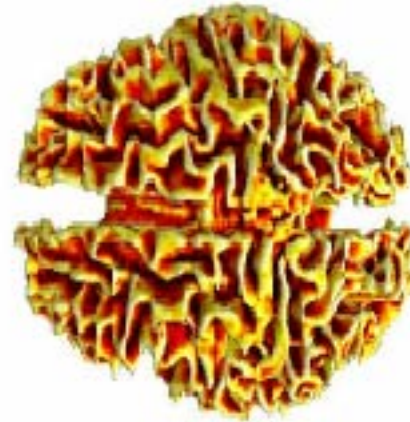
Solution by finite elements. For every $\phi \in C^\infty(\Sigma)$ one has

$$\left. \begin{aligned} \iint_{\Sigma} \nabla \phi \cdot \nabla F \, dA &= \langle -\Delta_{\Sigma} F, \phi \rangle \\ &= \langle -\lambda \nabla_{\vec{v} + i\vec{w}} \delta_p, \phi \rangle \\ &= \lambda \left[\nabla_{\vec{v} + i\vec{w}} \phi \right] (p) \end{aligned} \right\} \quad (2)$$

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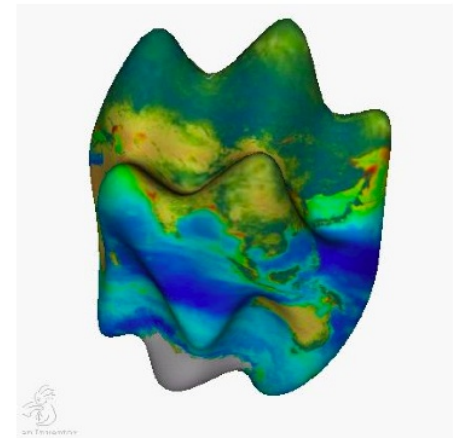
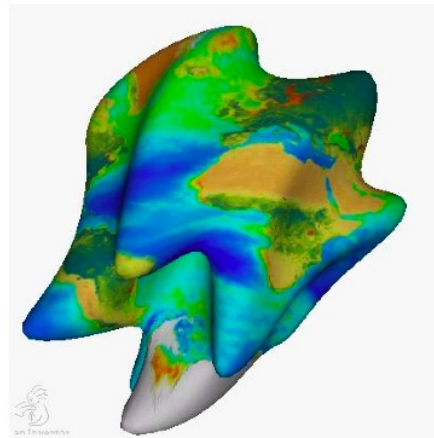
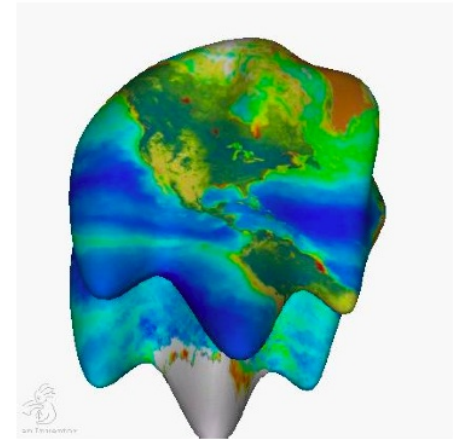
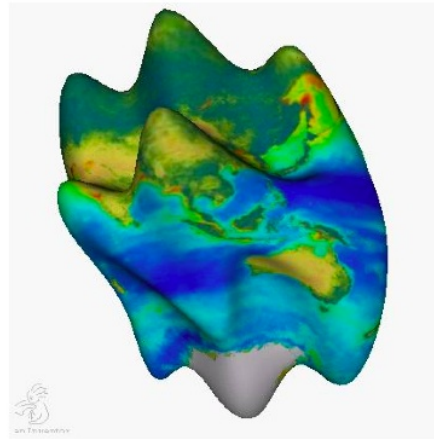
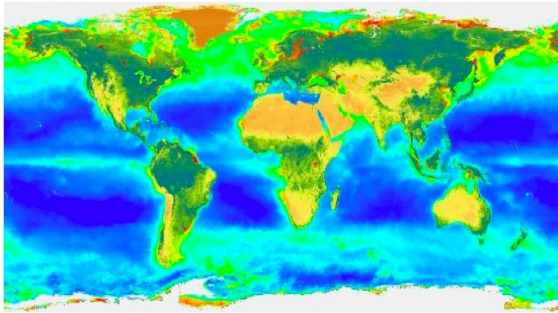
Since the surface Σ is approximately given through a triangulation, one can consider only piecewise linear functions on Σ . Then (2) leads to a (large but sparse) system of linear equations which can be solved by, e.g. the conjugate gradient method.



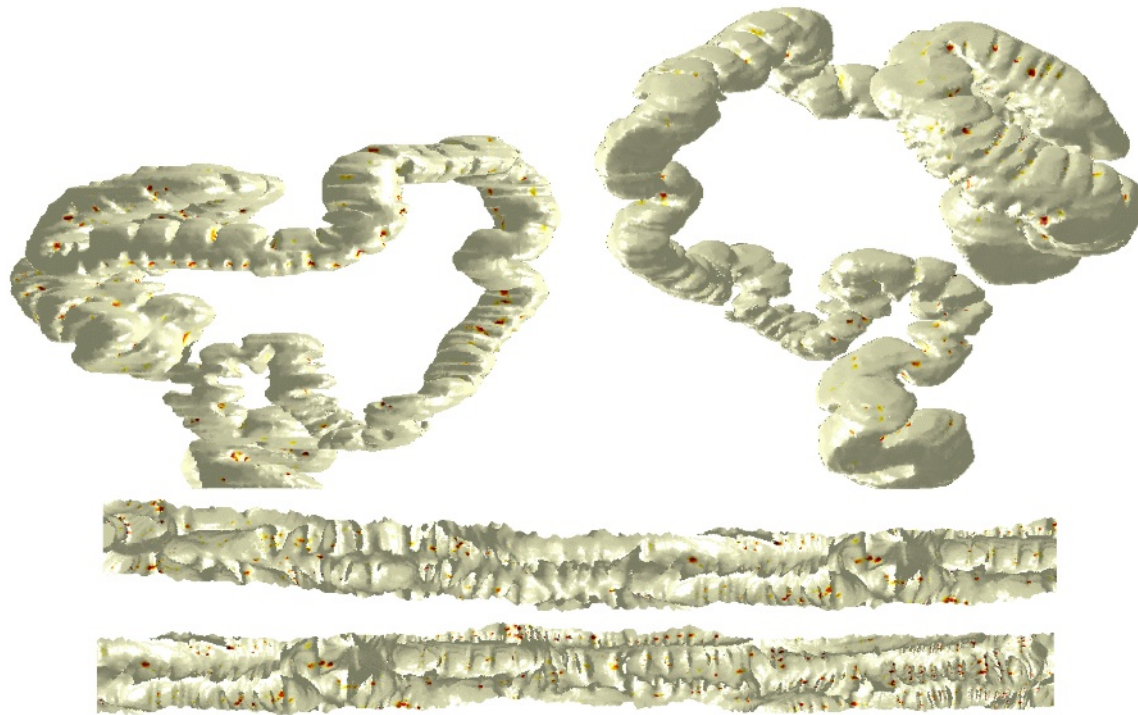
The result:



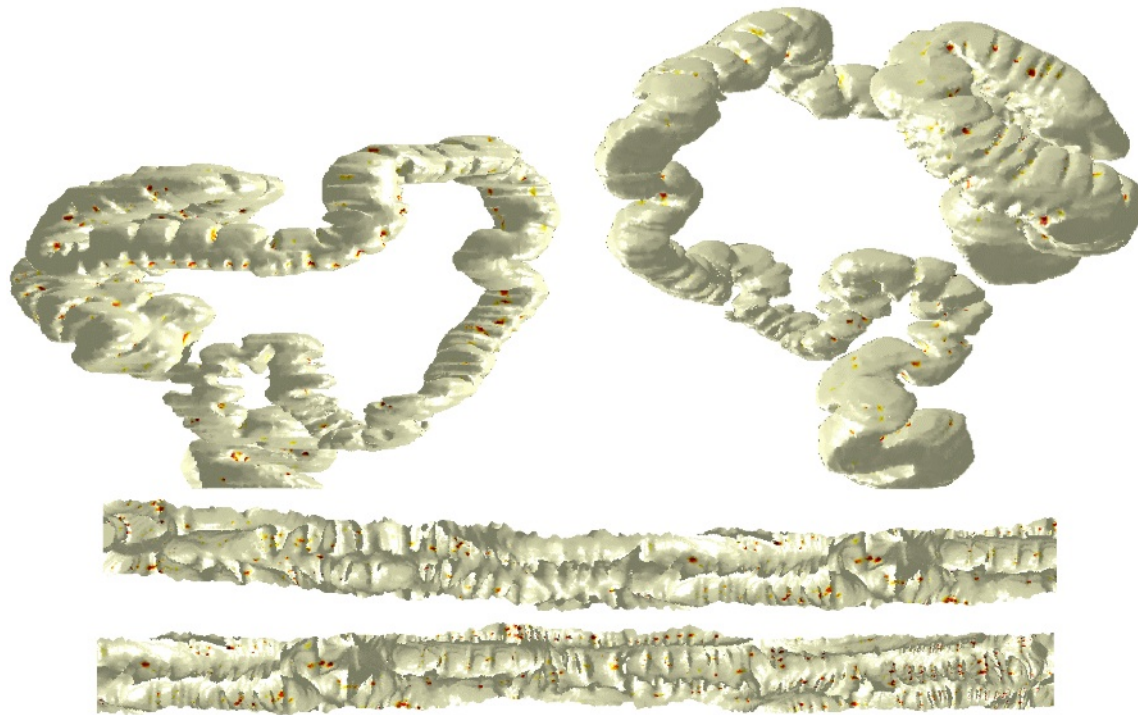
Synthetic image



Other applications – virtual colonoscopy



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This surface has the topology of a cylinder rather than a sphere.

Registration

Given two images
find a canonical map between them

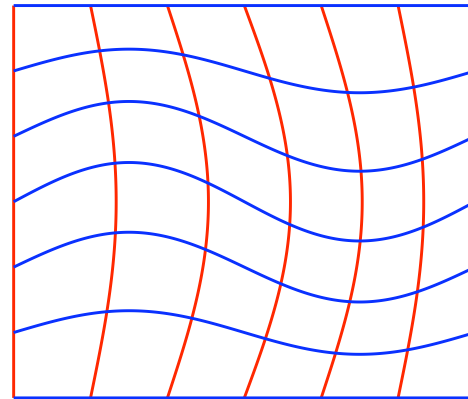
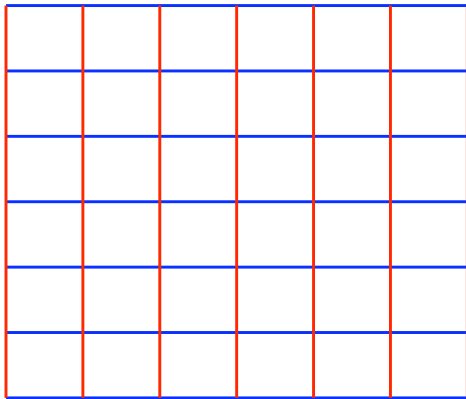
Initial image



Final image



A map from the rectangle R to itself:



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The difference is in the mapping behaviour

A map $\phi : \mathbb{R} \rightarrow \mathbb{R}$

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If $\mu = m(x) dL$ then
 $\phi^* \mu = \det(d\phi) m \circ \phi dL$.

Monge's transportation cost. Let $C : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a given function. Then Monge (1781) defined the cost of transporting the measure μ to $\phi^*\mu$ to be

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Monge's choice was $C(x, y) = |x - y|$. We will choose

$$C(x, y) = \frac{1}{2}|x - y|^2.$$

The Kantorovich solution. Given two measures $\mu_{1,2}$ on $\mathbb{R}$ with the same total mass, any measure preserving transformation $\phi : \mathbb{R} \rightarrow \mathbb{R}$ (i.e. $\mu_1 = \phi^*(\mu_2)$) determines a **reallocation measure** γ on $\mathbb{R} \times \mathbb{R}$ by the formula

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Minimization of $M(\gamma)$ with constraints (3) is a classical example of a linear programming problem.

Existence & uniqueness theorems.

A minimizing reallocation measure always exists.

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How to compute ϕ ?

Steepest descent for Monge's cost. To reduce the transportation cost of a given measure preserving map $\phi_0 : (R, \mu_1) \rightarrow (R, \mu_2)$, by deforming it through a family of m.p. maps $\phi_t : (R, \mu_1) \rightarrow (R, \mu_2)$, write the maps as

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The s_t are determined by their velocity fields,

$$\frac{\partial s_t}{\partial t} = v_t \circ s_t.$$

We will choose the maps s_t by choosing the velocities $\vec{v}_t$.

$$\frac{dM(\phi_t)}{dt} = \frac{d}{dt} \int_{\mathbb{R}} \frac{1}{2} |x - \phi_t(x)|^2 m_1(x) dx \quad (\text{set } x = s_t(\xi))$$

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&= \int_{\mathbb{R}} \vec{v}_t(x) \cdot (x - \phi_t(x)) d\mu_1(x)
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Thus

$$\frac{dM(\phi_t)}{dt} = \int_{\mathbb{R}} \vec{v}_t(x) \cdot (x - \phi_t(x)) m_1(x) dx$$

Since s_t preserves $\mu_1 = m_1(x)dL(x)$, one has

$$\operatorname{div}(m_1(x)\vec{v}) = 0. \tag{5}$$

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We choose $\vec{v}_t$ so that

$$m_1(x)\vec{v}_t = \phi_t(x) - x + \nabla p_t(x)$$

where $\Delta p_t = -\Delta(\phi_t(x) - x)$. Then

$$\frac{dM(\phi_t)}{dt} = - \int_{\mathbb{R}} |x - \phi_t(x) + \nabla p|^2 dx$$

Differentiate $\phi_0 = \phi_t \circ s_t$ w.r.t. time to get the transport equation

$$\frac{\partial \phi_t}{\partial t} + \vec{v}_t \cdot \nabla \phi_t = 0.$$

Summary: Steepest descent is achieved when

$$m_1(x)\vec{v}_t = \phi_t(x) - x + \nabla p_t(x)$$

$$\Delta p_t = \Delta(x - \phi_t(x))$$

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Theorems.

- Smooth solutions: short-time existence and uniqueness,
- Weak solutions: global existence and convergence for $t \rightarrow \infty$ in the context of reallocation measures.

————— (show the movie clip) —————