Using EXCEL to solve a differential equation

Math 222-spring 2009

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March 27, 2009

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As the reaction proceeds, all B gets converted to A. How long does this take?

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Every time a reaction takes place, the ratio $x(t)$ increases, so

$$\frac{dx}{dt} \text{ is proportional to the reaction rate.}$$

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This is a calculus class, so let's assume $K = 1$.

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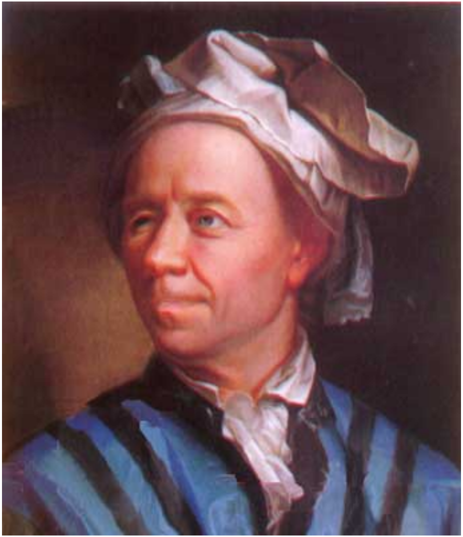
The solution is

$$x(t) = \frac{1}{1+49e^{-t}}.$$

(to be explained later this hour).

Leonhard "$e^{\pi i} + 1 = 0$" Euler (1707 - 1783)



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If you know $x(t)$ and h then you can solve this equation for $x(t+h)$.

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Knowing $x(h)$ you then can find $x(h+h) = x(2h)$,

and then $x(2h+h) = x(3h)$, $x(3h+h) = x(4h)$, etc....

Euler's (approximate) solution

Pick a small number $h > 0$, and compute

$$
\begin{aligned}
x(h) &= x(0) + h \times x(0)[1 - x(0)] \\
&\searrow \\
x(2h) &= x(h) + h \times x(h)[1 - x(h)] \\
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Now let's choose $h = 0.2$ and $x(0) = 0.02$, and compute $x(0.2)$, $x(0.4)$, $x(0.6)$, $x(0.8)$, $x(1.0)$, $\ldots$

Doing the calculations

Doing all these calculations is a drag of course. How did Euler do this? By hand!! (and with a lot of patience).

How do we do this in the 21st century? With a computer.

For more complicated diffeqs one should learn to program a computer, but for the example we've been looking at you can get Excel (or some other spreadsheet program like Open Office) to compute and plot the solutions.

What the spreadsheet computed

Here are the numbers, and graphs. The exact solution is $x(t) = 1/(1 + 49e^{-t})$.

h	t	x(t)	x'(t)	exact solution
0.2	0	0.020000	0.019600	0.020000
0.2	0.2	0.023920	0.023348	0.024320
0.2	0.4	0.028590	0.027772	0.029546
0.2	0.6	0.034144	0.032978	0.035853
0.2	0.8	0.040740	0.039080	0.043446
0.2	1	0.048556	0.046198	0.052559
0.2	1.2	0.057795	0.054455	0.063458
0.2	1.4	0.068686	0.063968	0.076434
0.2	1.6	0.081480	0.074841	0.091803
0.2	1.8	0.096448	0.087146	0.109894
0.2	2	0.113877	0.100909	0.131037
0.2	2.2	0.134059	0.116087	0.155537
0.2	2.4	0.157277	0.132541	0.183649
0.2	2.6	0.183785	0.150008	0.215545
0.2	2.8	0.213786	0.168082	0.251276
0.2	3	0.247403	0.186195	0.290734
0.2	3.2	0.284642	0.203621	0.333628
0.2	3.4	0.325366	0.219503	0.379465
0.2	3.6	0.369266	0.232909	0.427558
0.2	3.8	0.415848	0.242918	0.477061
0.2	4	0.464432	0.248735	0.527019
0.2	4.2	0.514179	0.249799	0.576441
0.2	4.4	0.564138	0.245886	0.624380
0.2	4.6	0.613316	0.237160	0.669999
0.2	4.8	0.660748	0.224160	0.712628
0.2	5	0.705580	0.207737	0.751790
0.2	5.2	0.747127	0.188928	0.787208
0.2	5.4	0.784913	0.168825	0.818791
0.2	5.6	0.818678	0.148445	0.846600
0.2	5.8	0.848367	0.128641	0.870815
0.2	6	0.874095	0.110053	0.891696
0.2	6.2	0.896105	0.093101	0.909552
0.2	6.4	0.914725	0.078003	0.924713
0.2	6.6	0.930326	0.064820	0.937508
0.2	6.8	0.943290	0.053494	0.948249
0.2	7	0.953989	0.043894	0.957229
0.2	7.2	0.962768	0.035846	0.964708
0.2	7.4	0.969937	0.029159	0.970920
0.2	7.6	0.975769	0.023644