Sobolevology

1. Definitions and Notation

1.1. The domain. Ω is an open subset of $\mathbb{R}^n$.

1.2. Hölder seminorm. For $\alpha \in (0,1]$ the *Hölder seminorm of exponent* α of a function is given by

$$[f]_\alpha = \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

When $\alpha = 1$ this seminorm is the same as the Lipschitz constant of the function f.

2. Inequalities

2.1. Poincaré. If $\varphi \in C_c^\infty(\Omega)$ and if Ω has width L (i.e. $\Omega \subset (0,L)\times\mathbb{R}^{n-1}$) then

$$\int_\Omega \varphi(x)^2 dx \le C_P(\Omega) \int_\Omega |\nabla\varphi|^2 dx$$

where the Poincaré constant $C_P(\Omega)$ is bounded by

$$C_P(\Omega) \le \frac{1}{L^2}.$$

2.2. Sobolev ($p=1$). For any $f \in C_c^1(\mathbb{R}^n)$ one has

$$\Big(\int_{\mathbb{R}^n} |f|^{n/(n-1)} dx\Big)^{(n-1)/n} \le C_n \int_{\mathbb{R}^n} |\nabla f| dx.$$

The constant $C_n \le 1$.

2.3. Sobolev ($1 \le p < n$). For any $f \in C_c^1(\mathbb{R}^n)$ and any $1 \le p < n$ one has

$$\Big(\int_{\mathbb{R}^n} |f|^{np/(n-p)} dx\Big)^{(n-p)/np} \le C_{n,p}\Big(\int_{\mathbb{R}^n} |\nabla f|^p dx\Big)^{1/p}.$$

(See problems for the constant $C_{n,p}$)

2.4. Morrey. For any $f \in C_c^1(\mathbb{R}^n)$ and any $p > n$ one has

$$[f]_{1-n/p} \le C_{n,p}\Big(\int_{\mathbb{R}^n} |\nabla f|^p dx\Big)^{1/p}.$$

3. Spaces

3.1. The space $W^{1,p}(\Omega)$. For $1 \le p \le \infty$ the space $W^{1,p}(\Omega)$ consists of all $f \in L^p(\Omega)$ for which the distributional partial derivatives $\frac{\partial f}{\partial x_i}$ belong to $L^p(\Omega)$. The norm on $W^{1,p}(\Omega)$ is

$$\|f\|_{W^{1,p}} = \Big(\int_\Omega \{|f|^p + |\nabla f|^p\}\, dx\Big)^{1/p}$$

The space $W_0^{1,p}(\Omega)$ is by definition the closure in $W^{1,p}(\Omega)$ of $C_c^\infty(\Omega)$.

3.2. $H^1(\Omega)$ and $H_0^1(\Omega)$. When $p = 2$ the following notation is commonly used

$$W^{1,2}(\Omega) = H^1(\Omega), \qquad W_0^{1,2}(\Omega) = H_0^1(\Omega).$$

These spaces are Hilbert spaces, and there are various inner products on them which define equivalent norms. On $H^1(\Omega)$ one defines

$$(u, v)_{H^1} = \int_\Omega \big\{uv + \nabla u \cdot \nabla v\big\} dx.$$

This expression also defines an inner product on the subspace $H_0^1(\Omega)$.

The following expression defines an inner product on $H_0^1(\Omega)$ but not on $H^1(\Omega)$:

$$(u, v)_{H_0^1} = \int_\Omega \nabla u \cdot \nabla v \, dx.$$

The corresponding norm on $H_0^1(\Omega)$ is

$$\|u\|^2_{H_0^1(\Omega)} = \int_\Omega |\nabla u(x)|^2 \, dx.$$

It follows from the Poincaré inequality that this norm is equivalent with the H^1 norm defined above.

3.3. $H^{-1}(\Omega)$. The space of all distributions on Ω which can be written as

$$g = \nabla \cdot \vec{f} = \frac{\partial f_1}{\partial x_1} + \cdots + \frac{\partial f_n}{\partial x_n}$$

with $f_1, \ldots, f_n \in L^2(\Omega)$ is defined to be $H^{-1}(\Omega)$.

The quantity

$$\|g\|^2_{H^{-1}} = \inf_{g = \nabla\cdot\vec{f}} \int_\Omega |\vec{f}|^2 dx$$

defines a norm on $H^{-1}(\Omega)$.

3.4. $H^{-1}(\Omega)$ is the dual of $H_0^1(\Omega)$. Let g be any distribution on Ω. Then the functional

$$\varphi \mapsto \langle g, \varphi\rangle$$

extends to a bounded linear functional on $H_0^1(\Omega)$ if and only if $g \in H^{-1}(\Omega)$. Furthermore, the H^{-1} norm of g defined in §3.3 is also given by

$$\|g\|_{H^{-1}} = \sup_{\|v\|_{H_0^1} \le 1} \langle g, v\rangle.$$

3.5. Theorem. *If $f \in L^2(\Omega)$ then*

$$v \mapsto \int_\Omega f(x)v(x)\,dx$$

defines a bounded linear functional on $H_0^1(\Omega)$. Identifying the function f with the linear functional it defines, we may think of f as an element of $H^{-1}(\Omega)$. One has

$$\|f\|_{H^{-1}} \le C_P(\Omega)\|f\|_{L^2},$$

where $C_P(\Omega)$ is the Poincaré constant of Ω.

Proof. Use the Poincaré inequality, which says that for any $v \in H_0^1(\Omega)$ one has $\|v\|_{L^2} \le C_P\|v\|_{H_0^1}$, to bound $\int fvdx$ as follows:

$$\int fvdx \le \|f\|_{L^2}\|v\|_{L^2} \le C_P\|f\|_{L^2}\|v\|_{H_0^1}.$$

QED

4. Solving the Poisson equation

4.1. The Poisson equation and boundary value problem. Poisson's equation is

$$-\Delta u = f, \qquad u|_{\partial\Omega} = 0.$$

A function $u \in H_0^1(\Omega)$ is a weak solution, or a solution in the sense of distributions if

$$\forall v \in C_c^\infty(\Omega) \quad \int_\Omega \nabla u \cdot \nabla v\,dx = \int_\Omega fv\,dx.$$

4.2. Theorem. *Let Ω be bounded. Then for any $f \in L^2(\Omega)$ there is a unique solution of*

$$-\Delta u = f, \qquad u \in H_0^1(\Omega).$$

This is a special case of the following theorem.

4.3. Theorem. *Let Ω be bounded. Then for any $g \in H^{-1}(\Omega)$ there is a unique solution of*

$$-\Delta u = g, \qquad u \in H_0^1(\Omega).$$

Proof. The definitions imply that $u \in H_0^1(\Omega)$ is a weak solution iff

$$\forall\ v \in H_0^1 \quad (u, v)_{H_0^1} = \langle g, v\rangle \tag{1}$$

The right-hand side defines a bounded linear functional on $H_0^1(\Omega)$, because, by definition $\langle g, v\rangle \le \|g\|_{H^{-1}}\|v\|_{H_{01}}$ for all $v \in H_0^1$. Every bounded linear functional on a Hilbert space such as $H_0^1(\Omega)$ is of the form $v \mapsto (u, v)_{H_0^1}$ for one and only one $u \in H_0^1$. QED

4.4. Theorem. *If $u \in H_0^1(\Omega)$ is the solution to $-\Delta u = g$ with $g \in H^{-1}(\Omega)$, then*

$$\|u\|_{H_0^1} \le \|g\|_{H^{-1}}.$$

Proof. u satisfies (1) for all $v \in H_0^1$. Choose $v = u$ and you get

$$\|u\|_{H_0^1}^2 = (u, u)_{H_0^1} = \langle g, u\rangle \le \|g\|_{H^{-1}}\|u\|_{H_0^1}.$$

Cancel $\|u\|_{H_0^1}$s left and right. QED

5. Problems

5.1. Membership in Sobolev spaces. Let B be the open unit ball, and define for each $a > 0$ the function $f_a(x) = |x|^{-a}$. For which values of $a > 0$ does one have $f_a \in W^{1,p}(B)$?

5.2. Bad (nowhere continuous) Sobolev functions. Let $1 \le p < n$, and let B be the open unit ball in $\mathbb{R}^n$. Let $\{x_i \in B\}_{i\in\mathbb{N}}$ be a dense sequence of points. Find numbers $b_i > 0$ ($i \in \mathbb{N}$) and $a > 0$ so that the series $f(x) = \sum_{i=1}^{\infty} b_i|x - x_i|^{-a}$ converges in $W^{1,p}(B)$.

Background to the next three problems. *The Sobolev inequality says for any $p < n$ that $\nabla f \in L^p(\Omega)$ implies $f \in L^{np/(n-p)}(\Omega)$. If you let $p \nearrow n$ in this statement you would get: "$\nabla f \in L^n \implies f \in L^\infty$". This turns out not to be true however (for a counterexample, if you insist, try functions of the form $f(x) = |\log x|^\alpha$ for appropriate $\alpha > 0$.)*

5.3. Sobolev constant. Derive the Sobolev inequality with $1 < p < n$ from the Sobolev inequality with $p = 1$, and find an explicit upper bound for $C_{n,p}$ (assuming that $C_n \le 1$).

5.4. Sobolev inequality when $|\nabla f| \in L^n(\Omega)$. When $\Omega \subset \mathbb{R}^n$ and $\int_\Omega |\nabla f|^n dx < \infty$ neither the Sobolev inequality nor the Morrey inequality apply. Prove that for any $f \in W^{1,n}(\Omega)$ and any $p < \infty$ one has

$$\|f\|_{L^p(\Omega)} \le C\Big(\int |\nabla f|^n \, dx\Big)^{1/n}.$$

The constant depends on p and $|\Omega|$ (you must assume that Ω has finite volume).

(Hint: if $|\nabla f| \in L^n$ and $|\Omega| < \infty$ then Hölder's inequality implies that $|\nabla f| \in L^r(\Omega)$ for any $r \in [1, n)$. Pick the right r for the given p, and use $C_{n,p}$ from the previous problem.)

5.5. Sobolev inequality when $|\nabla f| \in L^n(\Omega)$, the sequel. Use the result from the previous problem to show that if $|\nabla f| \in L^n(\Omega)$ for some bounded domain $\Omega \subset \mathbb{R}^n$, then there is a constant $c > 0$ such that

$$\int_\Omega e^{c|f(x)|} \, dx < \infty.$$

Hint: use the Taylor expansion $e^{c|f|} = \sum_{k=0}^{\infty} c^k|f|^k/k!$ and estimate the integral of the terms in this expansion using the previous problem.

5.6. Membership in $H^{-1}(\Omega)$. In §3.5 we showed that $L^2(\Omega) \subset H^{-1}(\Omega)$ for any bounded open domain $\Omega \subset \mathbb{R}^n$.

A. For which $q \in [1, \infty]$ does one have $L^q(\Omega) \subset H^{-1}(\Omega)$? (The answer will depend on n; you need the Sobolev inequalities to find the appropriate range of q; the cases $n = 1$ and $n = 2$ are a bit different from the case $n > 3$.)

B. Suppose the domain Ω contains the origin. For which $a > 0$ does the boundary value problem

$$-\Delta u = |x|^{-a}, \qquad u \in H^1_0(\Omega)$$

have a solution? (hint: if $|x|^{-a}$ belongs to $H^{-1}(\Omega)$ then Theorem 4.3 applies. Use problem 5.1).

5.7. Poincaré from Sobolev. Let $\Omega \subset \mathbb{R}^n$ be an open subset *with finite volume*. Use the Sobolev inequality to prove that Poincaré's inequality holds for Ω:

$$\forall u \in C_c^\infty(\Omega) \quad \int_\Omega u^2 dx \leq C|\Omega|^{2/n} \int_\Omega |\nabla u|^2 dx$$

where $|\Omega|$ is the volume of Ω.