Backward heat equations

[Black–Scholes equation - Wikipedia](https://en.wikipedia.org/wiki/Black%E2%80%93Scholes_equation)
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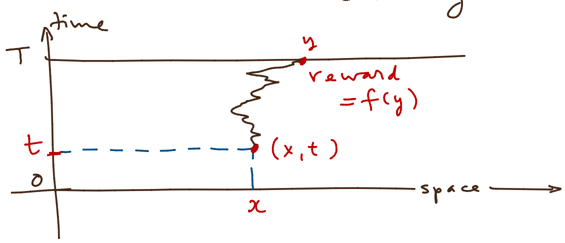
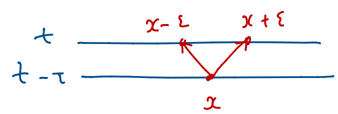
A random walker on $\mathbb{R}$ with stepsize $\varepsilon > 0$ every time interval $\tau > 0$, starting at $x \in \mathbb{R}$, at time $t \in [0, T]$. location $y \in \mathbb{R}$ at time T.

Upon arrival the walker collects a reward $f(y)$ where $f : \mathbb{R} \to \mathbb{R}$ is given.

Let $u(x,t) =$ average reward over all randomwalks starting at $x \in \mathbb{R}$ and at time t





$$u(x, t-\tau) = \frac{1}{2}\left[u(x+\varepsilon, t) + u(x-\varepsilon, t) \right]$$

$$\frac{u(x,t-\tau) - u(x,t)}{\tau} \cdot \frac{\tau}{\varepsilon^2} = \frac{1}{2} \frac{u(x+\varepsilon,t) - 2u(x,t) + u(x-\varepsilon,t)}{\varepsilon^2}$$

Let $\varepsilon \to 0$, $\tau \to 0$ with $\varepsilon^2 = 2D\tau$. Then you get

$$-\frac{\partial u}{\partial t}(x,t) = D \frac{\partial^2 u}{\partial x^2}$$

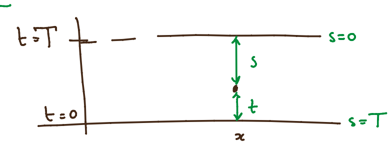
This is a backward heat equation. "final value problem"

$$u(x, T) = f(x) \quad \text{is given.}$$

Relation between forward and backward heat equations

Consider $v(x,s) = u(x, T-s)$ $\quad u(x,t) = v(x, T-t)$ $\quad (t = T-s)$

Then $v(x,s)$ satisfies



$$\begin{cases} \dfrac{\partial v}{\partial s} = D \dfrac{\partial^2 v}{\partial x^2} & 0 < s < T, \ x \in \mathbb{R} \\ v(x,0) = f(x) & x \in \mathbb{R} \end{cases}$$

Proof: $v(x,s) = u(x, \overbrace{T-s}^{=t}) \Rightarrow \frac{\partial^2 v}{\partial x^2}(x,s) = \frac{\partial^2 u}{\partial x^2}(x, T-s)$

$$\frac{\partial v}{\partial s}(x,s) = -\frac{\partial u}{\partial t}(x, T-s) = +D\frac{\partial^2 u}{\partial x^2}(x, T-s) = D\frac{\partial^2 v}{\partial x^2}(x,s)$$

$$v(x,0) = u(x, T-0) = f(x)$$

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$v(x,0) = u(x, T-0) = f(x)$ ////

Therefore $$v(x,s) = \frac{1}{\sqrt{4\pi s}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4s}} f(y)\, dy$$ assuming $D=1$

and hence $$u(x,t) = \frac{1}{\sqrt{4\pi(T-t)}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4(T-t)}} f(y)\, dy.$$ $u(x,t) = v(x, T-t)$

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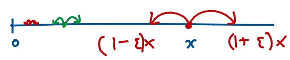
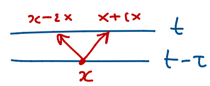
VARIATION ON THIS THEME.



Assume $x>0$ and in every time interval the random walker makes a random jump to $(1 \pm \varepsilon)\cdot x$

$u(x,t)$ = average prize at time T if you start from x at time t

The jumpsize is proportional to x.



$$u(x, t-\tau) = \tfrac{1}{2}\, u(x-\varepsilon x, t) + \tfrac{1}{2}\, u(x+\varepsilon x, t)$$

$$\frac{u(x,t-\tau) - u(x,t)}{\tau} \cdot \frac{\tau}{\varepsilon^2} = \frac{1}{2}\, \frac{u(x-\varepsilon x,t) - 2u(x,t) + u(x+\varepsilon x,t)}{\varepsilon^2}$$

$$\lim_{\varepsilon\to 0} \frac{u(x-\varepsilon x,t) - 2u(x,t) + u(x+\varepsilon x,t)}{\varepsilon^2} = \lim_{\varepsilon\to0} \frac{-x\, u_x(x-\varepsilon x,t) + x\, u_x(x+\varepsilon x,t)}{2\varepsilon}$$
$$= \lim_{\varepsilon\to0} \frac{(-x)^2 u_{xx}(x-\varepsilon x,t) + (x)^2 u_{xx}(x+\varepsilon x,t)}{2} = x^2 \frac{2u_{xx}(x,t)}{2} = x^2 u_{xx}(x,t)$$

Take limit with $\frac{\varepsilon^2}{\tau} = 2D$:

$$-\frac{\partial u}{\partial t} = \underbrace{D}_{} x^2 \frac{\partial^2 u}{\partial x^2}. \qquad 0 < x < \infty$$

diffusion rate depends on x

$$\frac{\partial u}{\partial t} + D x^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Reduce to standard heat equation by substituting

$$u(x,t) = v(\overbrace{\ln(x) + Dt}^{=\xi}, t) \qquad (\xi = \ln x + Dt)$$

$x \to (1\pm\varepsilon)x$

$\ln x \to \ln((1\pm\varepsilon)x) = \ln(1\pm\varepsilon) + \ln x$
$= \pm\varepsilon (+ \frac{\varepsilon^2}{2} + \dots) + \ln x$

$$-\frac{\partial u}{\partial t} = -\frac{\partial v}{\partial t}(\ln(x) + Dt, t) - D\frac{\partial v}{\partial \xi}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial \xi}(\ln x + Dt, t) \cdot \frac{\partial \ln x + Dt}{\partial x} = \frac{\partial v}{\partial \xi}(\dots)\cdot \frac{1}{x}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial \xi^2}(\ln x + Dt, t)\cdot\frac{1}{x^2} - \frac{\partial v}{\partial \xi}(\ln x + Dt, t)\frac{1}{x^2}$$

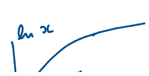
$$x^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial \xi^2}(\xi, t) - \frac{\partial v}{\partial \xi}(\xi, t)$$

So

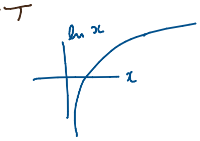
$$-\frac{\partial u}{\partial t} = D x^2 \frac{\partial^2 u}{\partial x^2}$$

$$\Leftrightarrow \quad -\frac{\partial v}{\partial t}(\xi,t) \cancel{- D\frac{\partial v}{\partial \xi}(\xi,t)} = D\frac{\partial^2 v}{\partial \xi^2} \cancel{- D\frac{\partial v}{\partial \xi}}$$

$$\Leftrightarrow \quad -\frac{\partial v}{\partial t}(\xi,t) = D\frac{\partial^2 v}{\partial \xi^2}(\xi,t) \qquad \begin{array}{l} -\infty < \xi < \infty \\ 0 < t < T \end{array}$$



$$\Longleftrightarrow \quad -\frac{\partial v}{\partial t}(\xi,t) = D\frac{\partial^2 v}{\partial \xi^2}(\xi,t) \qquad \begin{array}{l} -\infty<\xi<\infty \\ 0<t<T \end{array}$$



$\xi = \ln(x) + Dt$

$0 < x < \infty$ $\quad \Rightarrow \; -\infty < \ln x < \infty$

$\Rightarrow \; -\infty < \ln x + Dt < \infty$

$\Rightarrow \; -\infty < \xi < \infty$

t unchanged $\Rightarrow \quad 0 < t < T$

To find $v(\xi,t)$ we need to know $v(\xi,T)$:

$$v(\xi,T) \underset{\uparrow}{=} u(x,T) = u(e^{\xi - DT}, T) = f(e^{\xi - DT})$$

$\xi = \ln x + DT$

$\Leftrightarrow \xi - DT = \ln x \quad \Leftrightarrow \quad x = e^{\xi - DT}$