The heat equation

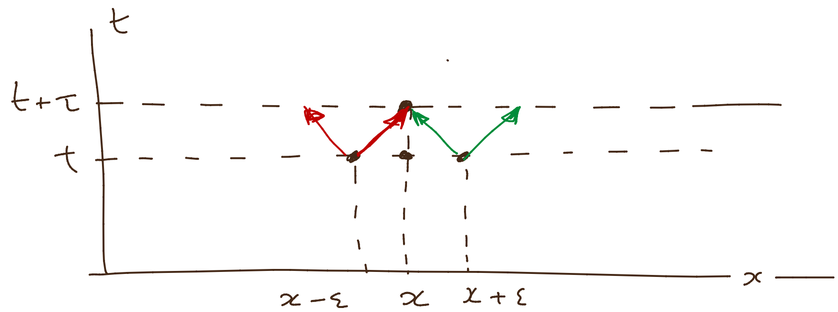
Random walk on the line

at regular time intervals (length τ)

take a random step of size ε

to the left or to the right

$u(t,x) =$ fraction of all random walkers at location x at time t.



$$u(t+\tau,x) = \frac{1}{2}u(t,x-\varepsilon) + \frac{1}{2}u(t,x+\varepsilon)$$

$$\frac{u(t+\tau,x)-u(t,x)}{\tau}\,\frac{\tau}{\varepsilon^2} = \frac{1}{2}\,\frac{u(t,x-\varepsilon)-2u(t,x)+u(t,x+\varepsilon)}{\varepsilon^2}$$

Let $\varepsilon,\tau\to 0$ with $\varepsilon^2 = 2D\tau$ — diffusion coefficient

$$\lim_{\varepsilon\to 0}\frac{u(t,x-\varepsilon)-2u(t,x)+u(t,x+\varepsilon)}{\varepsilon^2} = \lim_{\varepsilon\to 0}\frac{-u_x(t,x-\varepsilon)+u_x(t,x+\varepsilon)}{2\varepsilon}$$

$$= \lim_{\varepsilon\to 0}\frac{u_{xx}(t,x-\varepsilon)+u_{xx}(t,x+\varepsilon)}{2} = u_{xx}(t,x)$$

$$\frac{\partial u}{\partial t}\cdot\frac{1}{2D} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}$$

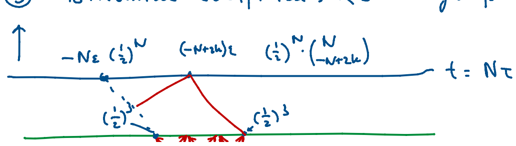
$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} \qquad \underline{\text{the heat equation}}$$

Possible derivations of G:

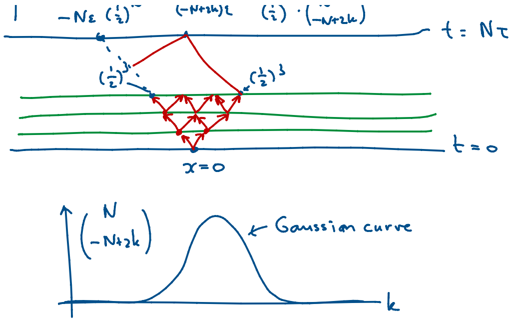
① Use Fourier transform

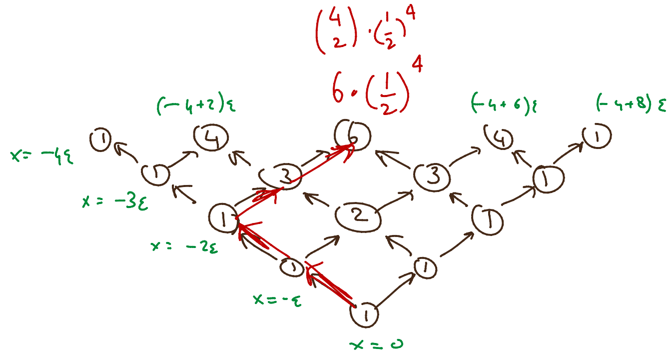
② Assume $G(t,x) = \frac{1}{\sqrt{t}}F\left(\frac{x}{\sqrt{t}}\right)$ find & solve diffeq for F

③ Binomial coefficients & Stirling's formula



$\binom{4}{2}\cdot\left(\frac{1}{2}\right)^4$

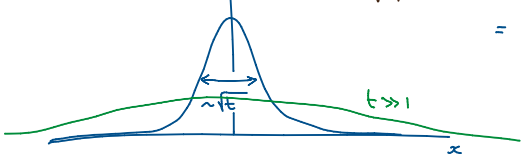




The Fundamental solution

$$G(x,t) = \frac{1}{\sqrt{4\pi t}}\, e^{-x^2/4t}$$

$$= \frac{1}{\sqrt{4\pi t}}\, e^{-\left(\frac{x}{\sqrt{4t}}\right)^2}$$



Satisfies

$$\frac{\partial G}{\partial t} = \frac{\partial^2 G}{\partial x^2} \qquad (1)$$

$$\int_{-\infty}^{\infty} G(x,t)\,dx = 1 \qquad (2)$$

<u>Proof</u> (1) Use $G(x,t) = \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{x^2}{4t}}$ to compute G_t and G_{xx} and note that $G_t = G_{xx}$.

(2) We compute

$$\int_{-\infty}^{\infty} G(t,x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{x^2}{4t}}\,dx \qquad \text{substitute } x = y\sqrt{4t}$$

$$dx = \sqrt{4t}\,dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\cancel{4}\pi \cancel{t}}}\, e^{-\frac{y^2 \cancel{4t}}{\cancel{4t}}}\, \cancel{\sqrt{4t}}\,dy$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2}\,dy$$

$$= \frac{1}{\sqrt{\pi}} \cdot \sqrt{\pi} = 1 \quad \checkmark$$

<u>Theorem 1</u> If $f : \mathbb{R} \to \mathbb{R}$ is bounded and Riemann integrable over every interval $-L \le x \le L$,

Then (*) $u(t,x) = \int_{-\infty}^{\infty} G(t, x-y) f(y)\, dy$

is a solution to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad x \in \mathbb{R},\ t>0$$

$$\lim_{t \downarrow 0} u(t,x) = f(x) \qquad \text{at every } x \in \mathbb{R} \text{ s.t. } f \text{ is continuous at } x$$

Proof. ① $\frac{\partial u}{\partial t} = \frac{\partial}{\partial t} \int_{y=-\infty}^{\infty} G(t, x-y) f(y)\, dy$

$$= \int_{-\infty}^{\infty} \frac{\partial G}{\partial t}(t, x-y) f(y)\, dy$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \frac{\partial}{\partial x} \int_{-\infty}^{\infty} G(t, x-y) f(y)\, dy =$$

$$= \frac{\partial}{\partial x} \int_{-\infty}^{\infty} \frac{\partial G}{\partial x}(t, x-y) f(y)\, dy$$

$$= \int_{-\infty}^{\infty} \frac{\partial^2 G}{\partial x^2}(t, x-y) f(y)$$

equal because $G_t = G_{xx}$

② $\lim_{t \downarrow 0} u(t,x) = f(x)$

We use: $\int_{-\infty}^{\infty} G(t,x)\, dx = 1$

$$\lim_{t \to 0} \int_{-a}^{a} G(t,x)\, dx = 1 \quad \text{and} \quad \lim_{t \to 0} \int_{a}^{\infty} G(t,x)\, dx = 0$$

and $G(t,x) \geq 0$ for all $t>0$, $x \in \mathbb{R}$

$$u(t,x) - f(x) = \int_{-\infty}^{\infty} G(t, x-y) f(y)\, dy - f(x) \underbrace{\int_{-\infty}^{\infty} G(t, x-y)\, dy}_{=1}$$

$$= \int_{-\infty}^{\infty} G(t, x-y) \big(f(y) - f(x)\big)\, dy.$$

f is continuous at $x \Rightarrow$ Given $\varepsilon > 0$ there is a $\delta > 0$ such that

$|f(y) - f(x)| < \varepsilon/2$ for $|x-y| < \delta$

Then $|u(t,x) - f(x)| = \left| \int_{-\infty}^{x-\delta} + \int_{x-\delta}^{x+\delta} + \int_{x+\delta}^{\infty} G(t, x-y)\big(f(y) - f(x)\big)\, dy \right|$

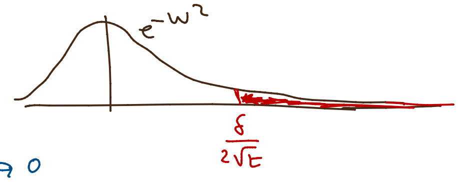
$$\le \int_{-\infty}^{x-\delta} G(t,x-y)\,|f(y)-f(x)|\,dy + \int_{x-\delta}^{x+\delta} G(t,x-y)\,|f(y)-f(x)|\,dy$$
$$+\int_{x+\delta}^{\infty} G(t,x-y)\,|f(y)-f(x)|\,dy$$

$$= I_1 + I_2 + I_3$$

$$I_1 = \int_{-\infty}^{x-\delta} G(t,x-y)\underbrace{|f(y)-f(x)|}_{\le 2\|f\|_\infty}\,\overset{y=x-z}{dy} \le 2\|f\|_\infty \int_\delta^\infty G(t,z)\,dz$$

$$= 2\|f\|_\infty \int_\delta^\infty \frac{1}{\sqrt{4\pi t}}\, e^{-z^2/4t}\,dz \qquad \begin{aligned} dz &= 2\sqrt{t}\,dw \\ z &= 2\sqrt{t}\,w \end{aligned}$$

$$= 2\|f\|_\infty \underbrace{\frac{1}{\sqrt{\pi}} \int_{\frac{\delta}{2\sqrt{t}}}^\infty e^{-w^2}\,dw}_{\to 0 \text{ as } t\to 0}$$



$$< \frac{\varepsilon}{4} \quad \text{if } 0<t\le t_\varepsilon$$

$$I_3 < \frac{\varepsilon}{4} \quad \text{if } 0<t\le t_\varepsilon \qquad \text{(same argument)}$$

$$I_2 = \int_{x-\delta}^{x+\delta} \underbrace{G(t,x-y)}_{>0}\ \underbrace{|f(y)-f(x)|}_{<\frac{\varepsilon}{2}}\,dy$$

$$< \frac{\varepsilon}{2}\int_{x-\delta}^{x+\delta} G(t,x-y)\,dy$$

$$< \frac{\varepsilon}{2}\int_{-\infty}^{\infty} G(t,x-y)\,dy$$

$$= \frac{\varepsilon}{2}.$$

Conclusion: If $0<t<t_\varepsilon$ then

$$|u(t,x)-f(x)| \le I_1+I_2+I_3 < \frac{\varepsilon}{4}+\frac{\varepsilon}{2}+\frac{\varepsilon}{4} = \varepsilon. \qquad ////$$

<u>Theorem 2</u> The function $u(t,x)$ given by (*) is the only solution of $u_t = u_{xx}$, $u(0,x)=f(x)$ for which u, u_t, u_x, u_{xx} are bounded continuous functions.