The maximum principle

Theorem $R \subset \mathbb{R}^n$ open, bounded

$u : \overline{R} \longrightarrow \mathbb{R}$ continuous

u is C^2 on R

$c : \overline{R} \longrightarrow \mathbb{R}$ continuous

$c(x) \geq 0$ for all $x \in R$

If

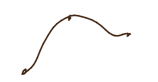
$$\Delta u - c(x)\, u(x) \ \overset{\leq}{\geq}\ 0 \quad \text{on } R$$

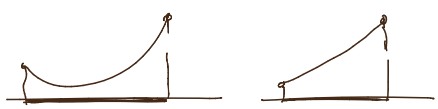
then

$$\underset{x \in \overline{R}}{\overset{\text{min}}{\max}} \{u(x), 0\} = \underset{x \in \partial R}{\overset{\text{min}}{\max}} \{u(x), 0\}$$

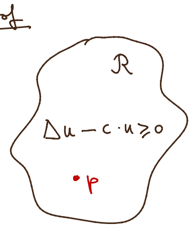
"Non negative $\overset{\text{minima}}{\text{maxima}}$ are attained on the boundary"

Idea $u'' \geq 0$





Proof



Case 1 : assume

$$\Delta u(x) - c(x)\, u(x) > 0 \quad \text{on } R.$$

Let $m = \max_{x \in \overline{R}} u(x)$ and suppose $m \geq 0$.

Choose $p \in \overline{R}$ with $u(p) = m$

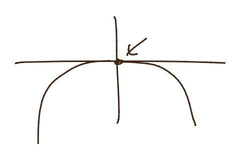
If $p \notin \partial R$ then u has a (local) maximum at p

and hence

$$x_1 \longmapsto u(x_1, p_2, \dots, p_n) \quad \text{has a max at } x_1 = p_1$$

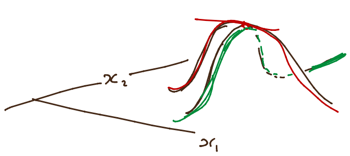
$$\Rightarrow \quad \frac{\partial u}{\partial x_1}(p) = 0 \quad \text{and} \quad \frac{\partial^2 u}{\partial x_1^2}(p) \leq 0$$

$f(x) = -x^4$



similarly:

$$\frac{\partial u}{\partial x_k}(p) = 0 \quad \text{and} \quad \frac{\partial^2 u}{\partial x_k^2}(p) \leq 0 \quad (k = 1, \dots, n)$$



Thus

$$\Delta u(p) = u_{x_1 x_1}(p) + \dots + u_{x_n x_n}(p) \leq 0$$

Moreover $u(p) = m \geq 0$, so

$$\Delta u(p) - c(p)\, u(p) \leq 0 - c(p)\, m \leq 0$$

This contradicts the assumption that $\Delta u - c \cdot u > 0$ everywhere in R.

everywhere in $\mathcal{K}$.

Case 2 Suppose $\Delta u - c(x)\cdot u \geq 0$ on $\mathcal{R}$ (H. Hopf 1930ies)

Then consider $u_\varepsilon(x) = u(x) + \varepsilon\, e^{\lambda x_1}$

$$\Delta(e^{\lambda x_1}) = \frac{\partial^2 e^{\lambda x_1}}{\partial x_1^2} = \lambda^2 e^{\lambda x_1}$$

We have

$$\begin{aligned} \Delta u_\varepsilon - c u_\varepsilon &= \Delta u + \varepsilon \Delta(e^{\lambda x_1}) - cu - \varepsilon c e^{\lambda x_1} \\ &= \Delta u - cu + \varepsilon \lambda^2 e^{\lambda x_1} - \varepsilon c e^{\lambda x_1} \\ &\geq (\lambda^2 - c(x))\, \varepsilon e^{\lambda x_1} \end{aligned}$$

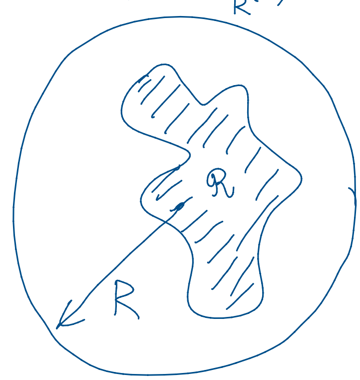
Choose $\lambda = \sqrt{\max\limits_{x \in \overline{\mathcal{R}}} c(x) + 1}$. Then $\lambda^2 - c(x) \geq 1$ for all x.

$\Rightarrow \Delta u_\varepsilon - c u_\varepsilon > 0$ in $\mathcal{R}$.

Case 1 applies to u_ε so for all $x \in \mathcal{R}$

$$u_\varepsilon(x) \leq \max_{y \in \partial \mathcal{R}} u_\varepsilon(y) = \max_{y \in \partial\mathcal{R}} \left(u(y) + \varepsilon e^{\lambda y_1}\right)$$

$\mathcal{R} \subset B_R(0)$



$\mathcal{R}$ is bounded so there is an $R > 0$ such that $\|y\| \leq R$ for all $y \in \partial\mathcal{R}$. Therefore $\varepsilon e^{\lambda y_1} \leq \varepsilon e^{\lambda R}$ for all $y \in \partial\mathcal{R}$.

For all $\varepsilon > 0$ and $x \in \mathcal{R}$:

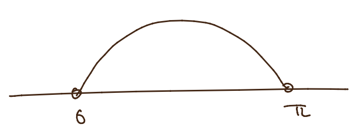
$$u(x) + \varepsilon e^{\lambda x_1} \leq \max_{y \in \partial\mathcal{R}} (u(y)) + \varepsilon e^{\lambda R}$$

Let $\varepsilon \downarrow 0$ and conclude $u(x) \leq \max\limits_{y \in \partial\mathcal{R}} u(y)$. ////.

Examples.

① $n = 1$, $u(x) = \sin(x)$ satisfies $u'' + u = 0$ $x \in (0, \pi)$

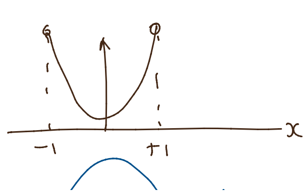
$\mathcal{R} = (0, \pi)$



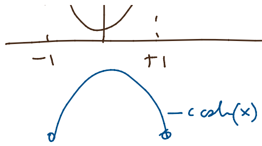
This example shows we need the assumption $c(x) \geq 0$ in the equation $\Delta u - c(x) u \geq 0$

② $n = 1$, $\mathcal{R} = (-1, +1)$ $u = \cosh(x)$ satisfies $u'' - u = 0$

$v = -\cosh(x)$ $v'' - v = 0$



Theorem only says there are no nonnegative interior maxima.



no nonnegative interior maxima.

Assume $c(x) \geq 0$ on $\overline{R}$.

Application Let $u, v : R \to \mathbb{R}$ both be solutions of

$$\Delta u - c(x)\, u = 0 \quad \text{in } R$$
$$\Delta v - c(x)\, v = 0 \quad \text{in } R$$

with $u = v = g$ on ∂R

If u, v are C^2 and continuous on $\overline{R}$ then $u = v$

Proof.

Consider $w(x) = u(x) - v(x)$.

Then $\Delta w - c(x) w = 0$ in R and $w = 0$ on ∂R

Max. principle $\Rightarrow$

$$\max_{x \in \overline{R}} \{w(x), 0\} = \max_{x \in \partial R} \{w(x), 0\} = 0.$$

$$\Rightarrow \quad w(x) \leq 0 \quad \text{for all } x \in R.$$

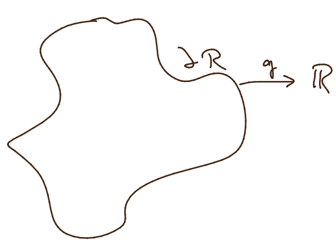
(if $w(x_0) > 0$ for some $x_0 \in R$ then the maximum of w over $\overline{R}$ is positive. Since $w = 0$ on ∂R, w would have to have a positive interior maximum, contradicting the Max Pr.)

$$\Rightarrow \quad u(x) \leq v(x) \quad \text{for all } x \in R$$

Switch u, v to get $v(x) \leq u(x)$ everywhere

Hence $u(x) = v(x)$ for all x. ////

Example



$u(x) =$ expected reward of Brownian motion starting at x when it hits ∂R

$$\Delta u = 0 \quad \text{in } R$$
$$u = g \quad \text{on } R.$$

If $g(x) \leq M$ for all $x \in \partial R$, then $u(x) \leq M$ for all $x \in R$.

Proof consider $v(x) = u(x) - M$.

Then

$$\Delta v = 0 \quad \text{on } R \qquad (\Delta v = \Delta(u - M) = \Delta u - \Delta M = 0)$$

$$v = g - M \leq 0 \quad \text{on } R.$$

Maximum principle $\Rightarrow$ interior maxima of v are ≤ 0

Since $v \leq 0$ on ∂R it follows that $v \leq 0$ on R, i.e. $u \leq M$ on R

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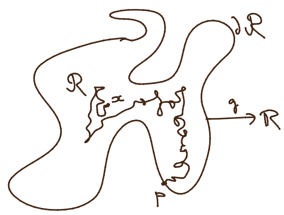
EXAMPLE

$$D\,\Delta u - r\,u = 0 \quad \text{in } R$$
$$u = g \quad \text{on } \partial R$$

interpretation:

$u(x) =$ average reward upon reaching ∂R starting at $x \in R$, corrected for inflation r, assuming diffusion rate D

(in time τ move about $\varepsilon = \sqrt{2nD\tau}$)



① Assume $g(x) \geq 0$ for all $x \in \partial R$.
Then $u(x) \geq 0$ for all $x \in R$

proof Consider $-u(x)$.

- If there is a point $\bar{x} \in R$ with $u(\bar{x}) < 0$ then:

$$\max_{\bar{R}} -u(x) \geq -u(\bar{x}) > 0$$

so $-u$ has a positive maximum.

- for $x \in \partial R$: $-u(x) = -g(x) \leq 0$.

$\Rightarrow$ The maximum of $-u : \bar{R} \to \mathbb{R}$ is attained in R.

But $v = -u$ satisfies

$$\Delta v - c(x)\cdot v = \Delta(-u) - c(x)(-u) = -\Delta u + c(x)u = 0$$

So we have a contradiction with the maximum principle.

② An upperbound for $u(x)$ in terms of $\frac{r}{D}$ and $\sup_{x \in \partial R} g(x)$

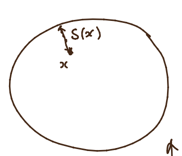
Suppose $0 \leq g(x) \leq G$ for all $x \in \partial R$. Then u satisfies

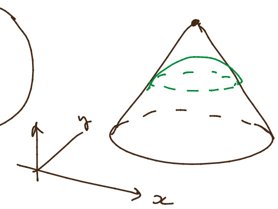
$$\Delta u - \frac{r}{D} u = 0 \quad \text{on } R \qquad u = g \leq G \text{ on } \partial R.$$

Let $S : \bar{R} \to \mathbb{R}$ be a function with

$$(*) \begin{cases} S \in C^2 \\ S = 0 & \text{on } \partial R \\ S > 0 & \text{on } R \end{cases}$$

Example $S(x) =$ distance from x to ∂R





graph of dist to ∂R

Consider

$$w(x) = e^{-\lambda S(x)}$$

Compute $\Delta w - c(x)\,w$: where $c(x) = \frac{r}{D}$

$$\frac{\partial}{\partial x_k} e^{-\lambda S(x)} = -\lambda e^{-\lambda S(x)} \frac{\partial S}{\partial x_k}$$

$$\left(\frac{\partial}{\partial x_k}\right)^2 e^{-\lambda S(x)} = (-\lambda)^2 e^{-\lambda S(x)} \left(\frac{\partial S}{\partial x_k}\right)^2 - \lambda e^{-\lambda S(x)} \frac{\partial^2 S}{\partial x_k^2}$$

Sum over $k = 1, 2, \cdots, n$:

$$\Delta\left(e^{-\lambda S(x)}\right) = \lambda^2 e^{-\lambda S}\left[\left(\frac{\partial S}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial S}{\partial x_n}\right)^2\right] - \lambda e^{-\lambda S}\left(\frac{\partial^2 S}{\partial x_1^2} + \cdots + \frac{\partial^2 S}{\partial x_k^2}\right)$$

$$= \lambda^2 e^{-\lambda S} |\nabla S|^2 - \lambda e^{-\lambda S} \Delta S$$

Conclusion:

$$\Delta e^{-\lambda S} - \frac{r}{D} e^{-\lambda S} = \lambda^2 e^{-\lambda S} |\nabla S|^2 - \lambda e^{-\lambda S} \Delta S - \frac{r}{D} e^{-\lambda S}$$

$$= \left(-\frac{r}{D} - \lambda \Delta S + \lambda^2 |\nabla S|^2\right) e^{-\lambda S}$$

$S : \overline{\mathcal{R}} \to \mathbb{R}$ is C^2 so there is an $M > 0$ with $|\Delta S|, |\nabla S|^2 \le M$ on $\mathcal{R}$.

Therefore

$$\Delta e^{-\lambda S} - \frac{r}{D} e^{-\lambda S} \le \left(-\frac{r}{D} - \lambda M - \lambda^2 M\right) e^{-\lambda S}$$

Choose λ so small that $\lambda M \le \frac{1}{2} \frac{r}{D}$ and $\lambda^2 M \le \frac{1}{2} \frac{r}{D}$

(Or: choose $\lambda < 1$ and $\lambda \le \frac{r}{2MD}$)

Then $\Delta e^{-\lambda S} - \frac{r}{D} e^{-\lambda S} \le 0$ for all $x \in \mathcal{R}$.

<u>Claim:</u> $u \le G e^{-\lambda S}$ for all $x \in \mathcal{R}$.

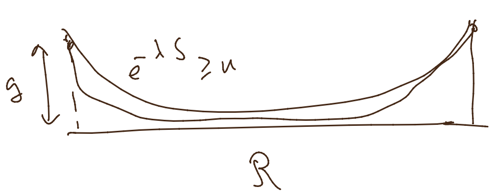
<u>proof</u>: Since $u = g \le G$ on $\partial\mathcal{R}$,

$v = u - G e^{-\lambda S}$ satisfies $v \le 0$ on $\partial \mathcal{R}$

and $\Delta v - \frac{r}{D} v = \underbrace{\Delta u - \frac{r}{D} u}_{=0} - G\Big(\underbrace{\Delta e^{-\lambda S} - \frac{r}{D} e^{-\lambda S}}_{\le 0}\Big) \ge 0.$

Maximum Principle applies to $v \Rightarrow v \le 0$ on $\mathcal{R}$.

$\Rightarrow u \le G e^{-\lambda S}$ on $\mathcal{R}$.



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