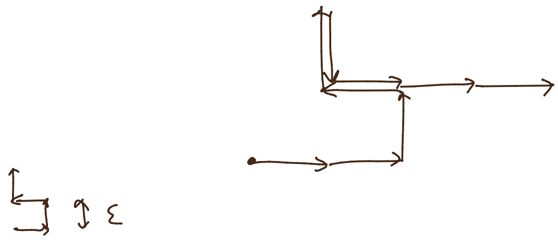
# Random walks

Random walk on a grid.

Brownian motion




$\varepsilon =$ stepsize

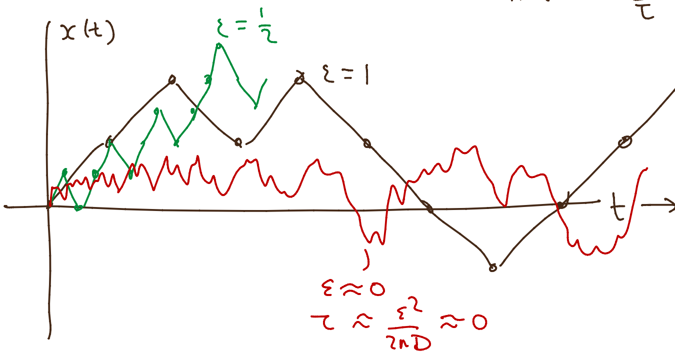
$\tau =$ time between steps

Brownian motion is the limit $\varepsilon \to 0$ & $\tau \to 0$ with $\frac{\varepsilon^2}{\tau} = 2nD$ ($n =$ dimension)

velocity $= \frac{\varepsilon}{\tau}$

$= \frac{\varepsilon}{\varepsilon^2/2nD} = \frac{2nD}{\varepsilon}$

$\to \infty$ as $\varepsilon \to 0$.




<u>Questions:</u> Starting at $x \in R$ what are

1. Average reward on reaching $\partial R$?
2. Average time to reach $\partial R$?
3. Average reward if the interest rate is $r$?

Let $g(p)$ be the reward you get if you first hit $\partial R$ at the point $p \in \partial R$. ($g : \partial R \to \mathbb{R}$)

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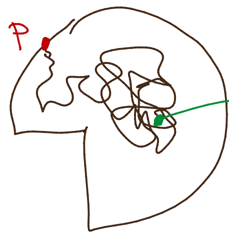
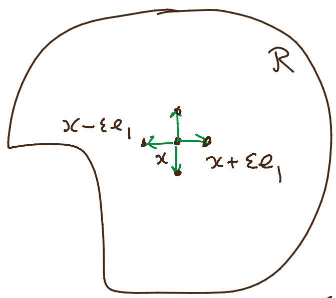



Let $u(x) =$ average reward if you start at $x \in R$

$u : R \to \mathbb{R}$

Then if $x \in \partial R$ : $\quad u(x) = g(x)$ $\quad$ because you have already hit $\partial R$. (at $x$)

If $x \in R$ then




$$u(x) = \underbrace{\tfrac{1}{4}\, u(x+\varepsilon e_1)}_{\substack{\text{average reward}\\ \text{if you start from}\\ x+\varepsilon e_1}} + \tfrac{1}{4}\, u(x-\varepsilon e_1) + \tfrac{1}{4}\, u(x+\varepsilon e_2) + \tfrac{1}{4}\, u(x-\varepsilon e_2)$$

i.e.

$$\frac{1}{4}\Big(u(x+\varepsilon e_1) - 2u(x) + u(x-\varepsilon e_1)\Big) + \frac{1}{4}\Big(u(x+\varepsilon e_2) - 2u(x) + u(x-\varepsilon e_2)\Big) = 0$$

Multiply by 4, divide by $\varepsilon^2$ :

$$\star \quad \frac{u(x+\varepsilon e_1) - 2u(x) + u(x-\varepsilon e_1)}{\varepsilon^2} + \frac{u(x+\varepsilon e_2) - 2u(x) + u(x-\varepsilon e_2)}{\varepsilon^2} = 0$$

Let $\varepsilon \downarrow 0$, use l'Hopital $\quad (x = (x_1, x_2))$

$$\lim_{\varepsilon\to 0} \frac{u(x_1+\varepsilon, x_2) - 2u(x_1,x_2) + u(x_1-\varepsilon, x_2)}{\varepsilon^2} = \lim_{\varepsilon\to 0} \frac{u_{x_1}(x_1+\varepsilon, x_2) - u_{x_1}(x_1-\varepsilon, x_2)}{2\varepsilon}$$

$$= \frac{u_{x_1x_1}(x_1,x_2) + u_{x_1x_1}(x_1,x_2)}{2} = \frac{\partial^2 u}{\partial x_1^2}(x)$$

<u>Conclusion</u> $\star$ implies

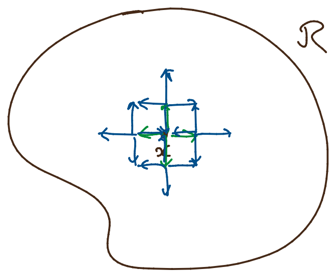
$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = 0 \quad \text{on } R$$

The <u>average reward</u> $u(x)$ is the solution to

$$\begin{cases} \Delta u = 0 & \text{in } R \\ u = g & \text{on } \partial R \end{cases}$$

## ② <u>THE AVERAGE TIME TO $\partial R$</u>

$T(x)$ = average time to $\partial R$, starting at $x \in R$




$$T(x) = \frac{1}{4}\{T(x+\varepsilon e_1) + T(x-\varepsilon e_1) + T(x+\varepsilon e_2) + T(x-\varepsilon e_2)\} + \tau$$

average time for remaining steps. — time to take one step

$$\Rightarrow \quad \frac{T(x+\varepsilon e_1) - 2T(x) + T(x-\varepsilon e_1)}{4} + \frac{T(x+\varepsilon e_2) - 2T(x) + T(x-\varepsilon e_2)}{4} + \tau = 0$$

$$\frac{T(x+\varepsilon e_1) - 2T(x) + T(x-\varepsilon e_1)}{\varepsilon^2} + \frac{T(x+\varepsilon e_2) - 2T(x) + T(x-\varepsilon e_2)}{\varepsilon^2} + \frac{4\tau}{\varepsilon^2} = 0$$

Let $\varepsilon \to 0$, $\quad \varepsilon^2 = 4D\tau \quad (= 2nD\tau, \quad n = \text{dimension} = 2)$

diffusion coefficient — units?? $D = \dfrac{\varepsilon^2}{2n\tau}$ has units $\dfrac{(\text{length})^2}{\text{time}}$ "$\text{cm}^2/\text{sec}$"

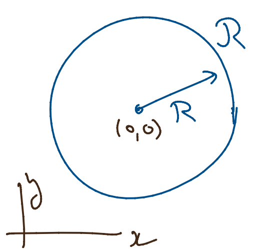
$$\frac{\partial^2 T}{\partial x_1^2} + \frac{\partial^2 T}{\partial x_2^2} + \frac{1}{D} = 0$$

i.e. $\boxed{D\,\Delta T + 1 = 0}$

On $\partial R$: $\boxed{T(x) = 0}$

units of $D\Delta T$: $D\dfrac{\partial^2 T}{\partial x^2} = \dfrac{\text{length}^2}{\text{time}} \cdot \dfrac{\text{time}}{\text{length}^2} = 1$ ✓

<u>Example</u> $R$ = disc of radius $R > 0$




$u(x,y) = R^2 - x^2 - y^2$ satisfies

$$\begin{cases} \Delta u = -2 - 2 = -4 \\ u = 0 \quad \text{on } \partial R \end{cases}$$

if $T = \dfrac{u}{4D}$ then

$$\begin{cases} D\,\Delta T + 1 = \cancel{D}\dfrac{1}{4\cancel{D}}\Delta u + 1 = \dfrac{1}{4}(-4) + 1 = 0 \\ \text{and} \quad T = 0 \quad \text{on } \partial R. \end{cases}$$

Average time to exit is $T(x,y) = \dfrac{R^2 - x^2 - y^2}{4D}$ maximal at $x = y = 0$.

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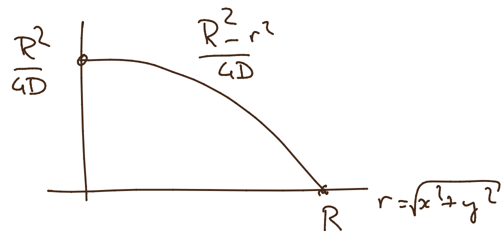



<u>Example continued</u> Add a harmonic function $h(x,y)$ to $T(x,y)$ to make more examples, such as:

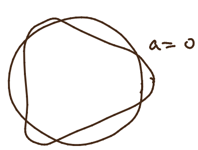
$$h(x,y) = (x+iy)^n + (x-iy)^n \qquad (n = 1, 2, \dots)$$

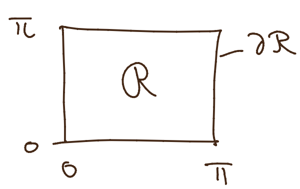
e.g.

$$T(x,y) = \frac{R^2 - x^2 - y^2 + a(x^3 - 3x^2 y)}{4D}$$

satisfies $D\Delta T + 1 = 0$, $T = 0$ on $\partial R_a$

where $R_a = x^2 + y^2 - a(x^3 - 3xy^2) < R^2$





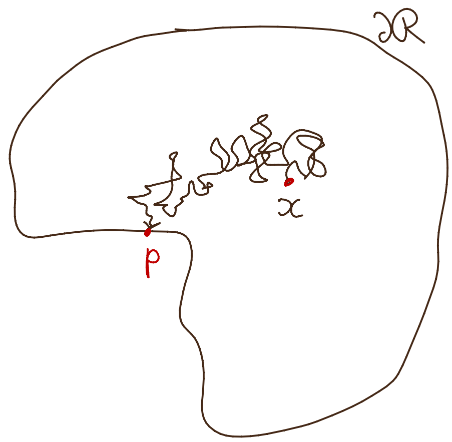
<u>Example</u> $R = [0,\pi] \times [0,\pi]$

To find $T$ expand in a Fourier series:

<u>Try</u>: $$T(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin(mx)\sin(ny)$$

$$\Delta T = -\frac{1}{D} \iff \Delta T = \sum_m \sum_n C_{mn}(-m^2 - n^2)\sin(mx)\sin(ny) \overset{?}{=} -\frac{1}{D}$$

## One more: Expected gain corrected for inflation.




$g : \partial R \longrightarrow \mathbb{R}$

$g(p)$ = reward upon exiting at $p \in \partial R$

$u(x)$ = expected reward upon exiting if you start at $x \in \partial R$

The reward is received in the future. $u(x)$ is measured in current dollars

$r$ = inflation rate.

\$1 now = \$$e^{rt}$ at time $t$ in the future.

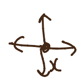
## Equation for $u : R \to \mathbb{R}$.

\$ now

\$ time $\tau$ later

$$u(x) = e^{-r\tau}\,\frac{u(x+\varepsilon e_1) + u(x-\varepsilon e_1) + u(x+\varepsilon e_2) + u(x-\varepsilon e_2)}{4}$$

correction for inflation.




$$4e^{r\tau}u(x) = u(x+\varepsilon e_1) + u(x-\varepsilon e_1) + u(x+\varepsilon e_2) + u(x-\varepsilon e_2)$$

$$(*) \qquad \frac{4(e^{r\tau}-1)}{\varepsilon^2}\,u(x) = \frac{u(x+\varepsilon e_1) - 2u(x) + u(x-\varepsilon e_1) + u(x+\varepsilon e_2) - 2u(x) + u(x-\varepsilon e_2)}{\varepsilon^2}$$

Let $\varepsilon \to 0$ and assume $4D\tau = \varepsilon^2$ $\quad D$ = diffusion coefficient.

Then

$$\lim_{\varepsilon\to 0} \frac{e^{r\tau}-1}{\varepsilon^2} = \lim_{\tau\to 0} \frac{e^{r\tau}-1}{4D\tau} = \frac{r}{4D}$$

$(*)$ with $\varepsilon, \tau \to 0$ gives

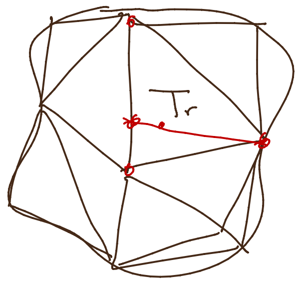
$$\begin{cases} \dfrac{r}{D}u = \Delta u & \text{on } R \\ u = g & \text{on } \partial R \end{cases}$$

Note: $\Delta u = \frac{r}{D}u$ is the Euler-Lagrange equation for

$$I[u] = \iint_R \left(\tfrac{1}{2}|\nabla u|^2 + \tfrac{1}{2}\tfrac{r}{D}u^2\right)dx\,dy$$

## Numerical approximation of the solution

Given a domain $R$ triangulate it




Choose $V =$ all functions that are

1. continuous
2. linear on each triangle ($z = a + bx + cy$)

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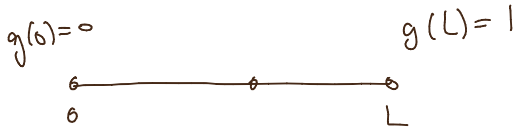
Explicit solution to $\frac{r}{D} u = \Delta u$ if $R = (0, L) \subset \mathbb{R}^1$

The diffeq is $\frac{r}{D} u = \Delta u = u''(x)$.

Have to solve $\begin{cases} u''(x) = \frac{r}{D} u(x) \\ u(0) = 0 \\ u(L) = 1 \end{cases}$




Solution: $u(x) = p e^{ax} + q e^{-ax}$

if $a^2 = \frac{r}{D}$ ie. $a = \sqrt{\frac{r}{D}}$

$u(0) = 0 \Rightarrow p + q = 0$, ie. $q = -p$

$u(L) = 1 \Rightarrow p e^{aL} - p e^{-aL} = 1 \Rightarrow p = \frac{1}{e^{aL} - e^{-aL}}$

$$\Rightarrow \quad u(x) = \frac{e^{ax} - e^{-ax}}{e^{aL} - e^{-aL}} \qquad a = \sqrt{r/D}$$

$$= \frac{\sinh ax}{\sinh aL} \approx e^{a(x-L)} = e^{-a(L-x)}$$