Minimizing Dirichlet (again)

Solving $\begin{cases} \Delta u = 0 & \text{in } \mathcal{R} \\ u = g & \text{on } \partial\mathcal{R} \end{cases}$

by minimizing $\mathcal{D}[u] = \frac{1}{2}\int_{\mathcal{R}} |\nabla u|^2 dx$

over all $u \in C^1_g$

Recall: $D_- = \inf\{\mathcal{D}[u] \mid u \in C^1_g\}$

Let $g : \partial\mathcal{R} \to \mathbb{R}$ be given and let $u_k \in C^1_g$ be a sequence with

$$D_- \le \mathcal{D}[u_k] \le D_- + \frac{1}{k}$$

Then we have shown that

$$\|u_k - u_\ell\|_2^2 = \int_{\mathcal{R}} |u_k(x) - u_\ell(x)|^2 dx \le \frac{C}{k}$$

and

$$\left\| \frac{\partial u_k}{\partial x_i} - \frac{\partial u_\ell}{\partial x_i} \right\|_2^2 = \int_{\mathcal{R}} \left| \frac{\partial u_k}{\partial x_i} - \frac{\partial u_\ell}{\partial x_i} \right|^2 dx \le \frac{C}{k}$$

for all $\ell \ge k \ge 1$

Theorem ① There is a distribution $\mathcal{U} : C^\infty_c(\mathcal{R}) \to \mathbb{R}$ such that

$$\mathcal{U}(\varphi) = \lim_{k\to\infty} \int_{\mathcal{R}} u_k(x)\varphi(x)\, dx$$

"$u_k \to \mathcal{U}$ in the sense of distributions."

and

$$D_i\mathcal{U}(\varphi) = \lim_{k\to\infty} \int_{\mathcal{R}} \frac{\partial u_k}{\partial x_i}(x)\, \varphi(x) dx$$

② $\mathcal{U}$ satisfies

$$\Delta\mathcal{U} = \sum_{i=1}^{n} D_i D_i \mathcal{U} = 0$$

$\Delta\mathcal{U} \overset{\text{def}}{=} D_1 D_1 \mathcal{U} + D_2 D_2 \mathcal{U} + \cdots + D_n D_n \mathcal{U}$

Proof of ① . Let $\varphi \in C^\infty_c(\mathcal{R})$ be a test function

Consider the sequence of real numbers

$$a_k \overset{\text{def}}{=} \int_{\mathcal{R}} u_k(x)\varphi(x)\, dx$$

Then
$$\begin{aligned}
|a_k - a_\ell| &= \left| \int u_k(x)\varphi(x)\, dx - \int u_\ell(x)\varphi(x)\, dx \right| \\
&\le \int_{\mathcal{R}} |u_k(x)\varphi(x) - u_\ell(x)\varphi(x)|\, dx \\
&= \int_{\mathcal{R}} |u_k(x) - u_\ell(x)| \cdot |\varphi(x)|\, dx \\
&\le \left(\int_{\mathcal{R}} |u_k(x) - u_\ell(x)|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathcal{R}} \varphi(x)^2 dx \right)^{\frac{1}{2}} \\
&\le \sqrt{\frac{C}{k}}\, \|\varphi\|_{L^2}
\end{aligned}$$

Schwarz

$$\int f g\, dx \le \left(\int f^2 dx \right)^{\frac{1}{2}} \left(\int g^2 dx \right)^{\frac{1}{2}}$$

for all $\ell \ge k \ge 1$.

This implies a_k is a Cauchy sequence in $\mathbb{R}$.

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$\mathbb{R}$ is complete, so $\lim_{k\to\infty} a_k$ exists. We define

$$U(\varphi) = \lim_{k\to\infty} \int_R u_k(x)\,\varphi(x)\,dx$$

$U : C_c^\infty(R) \to \mathbb{R}$ is a distribution:

U is linear: $U(\varphi+\psi) = \lim_{k\to\infty} \int u_k(x)\left[\varphi(x)+\psi(x)\right]dx$

$$= \lim_{k\to\infty} \int u_k(x)\varphi(x)\,dx + \int u_k(x)\psi(x)\,dx$$

$$= U(\varphi) + U(\psi).$$

(similarly: $U(c\varphi) = c\;U(\varphi)$)

$U(\varphi)$ is bounded by

Schwarz $\leq$

$$|U(\varphi)| = \lim_{k\to\infty} \left|\int u_k(x)\,\varphi(x)\,dx\right| \leq \lim_{k\to\infty} \left(\int u_k(x)^2dx\right)^{\frac{1}{2}}\left(\int_R \varphi(x)^2dx\right)^{\frac{1}{2}}$$

$\varphi(x)^2 \leq \|\varphi\|_\infty^2$

$$= \lim_{k\to\infty} \underbrace{\left(\int u_k(x)^2\,dx\right)^{\frac{1}{2}}}_{\|u_k\|_{L^2}} \left(\|\varphi\|_\infty^2 \cdot \int_R dx\right)^{\frac{1}{2}}$$

By the Poincaré inequality:

Can apply Poincaré because $u_1 = u_2 = \dots = g$ on ∂R $\Rightarrow$ $u_k - u_1 = 0$ on ∂R

$$\|u_k\|_{L^2} \overset{\text{triangle} \leq}{\leq} \|u_1\|_{L^2} + \|u_k - u_1\|_{L^2}$$

and

$$\int_R (u_k-u_1)^2\,dx \leq C_P \underbrace{\int_R |\nabla u_k - \nabla u_1|^2\,dx}_{\leq \frac{8}{1} = 8}$$

$$\leq C_P \cdot 8$$

so $\left(\int u_k(x)^2\,dx\right)^{\frac{1}{2}} \leq \|u_1\|_{L^2} + \sqrt{8\,C_P} = C$ (does not depend on k or φ)

Therefore $|U(\varphi)| \leq \left(\|u_1\|_{L^2} + \sqrt{8\,C_P}\right) \cdot \left(\|\varphi\|_\infty^2 \int_R dx\right)^{\frac{1}{2}}$

$$= \left(\|u_1\|_{L^2} + \sqrt{8\,C_P}\right) \cdot \|\varphi\|_\infty \cdot \sqrt{\operatorname{vol} R}$$

$$= C\;\|\varphi\|_\infty$$

Proof of (2) Let $\varphi \in C_c^\infty(R)$ be any testfunction. Then

$$D_- \leq \mathcal{D}[u_k+\varphi] = \tfrac{1}{2}\int_R |\nabla(u_k+\varphi)|^2\,dx$$

$$= \tfrac{1}{2}\int_R |\nabla u_k + \nabla\varphi|^2\,dx$$

$$= \tfrac{1}{2}\int_R \left(|\nabla u_k|^2 + 2\,\nabla u_k\cdot\nabla\varphi + |\nabla\varphi|^2\right)dx$$

$$= \mathcal{D}[u_k] + \int \nabla u_k\cdot\nabla\varphi\,dx + \tfrac{1}{2}\int|\nabla\varphi|^2\,dx$$

$$\leq D_- + \tfrac{1}{k} + \int \nabla u_k\cdot\nabla\varphi\,dx + \tfrac{1}{2}\int|\nabla\varphi|^2\,dx$$

$$\Rightarrow \quad -\int \nabla u_k \cdot \nabla \varphi \, dx \;\leq\; \frac{1}{k} + \frac{1}{2}\int |\nabla \varphi|^2 \, dx$$

Integration by parts implies:

$$-\int_{\mathcal{R}} \nabla u_k \cdot \nabla \varphi \, dx = -\int_{\mathcal{R}} \sum_{i=1}^{n} \frac{\partial u_k}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \, dx$$

$$= -\int_{\mathcal{R}} \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i}\left(u_k \cdot \frac{\partial \varphi}{\partial x_i}\right) - u_k \frac{\partial^2 \varphi}{\partial x_i^2} \right) dx$$

↓ divergence thm.

$$= -\int_{\partial \mathcal{R}} \underbrace{u_k \frac{\partial \varphi}{\partial x_i}}_{=0 \text{ on } \partial\mathcal{R}} n_i \, da + \int_{\mathcal{R}} u_k \underbrace{\sum_{i=1}^{n} \frac{\partial^2 \varphi}{\partial x_i^2}}_{=\Delta\varphi \text{ by def}} dx$$

$$= + \int_{\mathcal{R}} u_k \, \Delta \varphi \, dx$$

so that

if $\varphi \in C_c^\infty(\mathcal{R})$ then $\Delta\varphi \in C_c^\infty(\mathcal{R})$

$$\int_{\mathcal{R}} u_k(x) \, \Delta\varphi \, dx \;\leq\; \frac{1}{k} + \frac{1}{2}\int |\nabla \varphi|^2 \, dx$$

Let $k \to \infty$:

$$U(\Delta\varphi) \;\leq\; \frac{1}{2}\int |\nabla\varphi|^2 \, dx$$

This holds for all $\varphi \in C_c^\infty(\mathbb{R})$, in particular for $t\varphi$ $(t \in \mathbb{R})$:

Therefore, for all $t \in \mathbb{R}$ we have

$$t \; U(\Delta\varphi) = U(\Delta(t\varphi)) \;\leq\; \frac{1}{2}\int |\nabla(t\varphi)|^2 \, dx = \frac{t^2}{2}\int |\nabla\varphi|^2 \, dx$$

Given: $at \leq bt^2$ for all $t \in \mathbb{R}$

If $a > 0$, then $at \leq bt^2$ for all $t > 0$ $\Rightarrow$ $a \leq bt$ for all $t > 0$.
$\Rightarrow$ $a \leq \lim_{t \downarrow 0} bt = 0$. ↯

If $a < 0$, then $at \leq bt^2$ for all $t < 0$ $\Rightarrow$ $a \geq bt$ for all $t < 0$
$\Rightarrow$ $a \geq \lim_{t \uparrow 0} bt = 0$ ↯

$\Rightarrow$ $a = 0$

Conclusion: $\quad U(\Delta\varphi) = 0 \quad$ for all $\varphi \in C_c^\infty(\mathcal{R})$

Finally, in the sense of distributions

$$\Delta U = D_1(D_1 U) + \cdots + D_n(D_n U) = \sum_{i=1}^n D_i D_i U$$

$$\Rightarrow \Delta U(\varphi) = \sum_{i=1}^{n} (D_i D_i U)(\varphi)$$

$\int \frac{\partial^2 u}{\partial x_i^2} \varphi = -\int \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \, dx$

definition of $D_i(D_i U)$

$$= \sum_{i=1}^{n} -D_i U\left(\frac{\partial \varphi}{\partial x_i}\right)$$

def. of D_i again

$$= \sum_{i=1}^{n} + U\left(\frac{\partial}{\partial x_i}\left(\frac{\partial \varphi}{\partial x_i}\right)\right)$$

$$= U\left(\sum_{i=1}^{n} \frac{\partial^2 \varphi}{\partial x_i^2}\right)$$

$$= U(\Delta\varphi)$$

shown above

$$= 0$$

Therefore $\Delta \mathcal{U} = 0$ in the sense of distributions. ////

<u>Questions</u> $\mathcal{U} = g$ on $\partial \mathcal{R}$?

Is $\mathcal{U} = T_u$ for some $u : \mathcal{R} \to \mathbb{R}$?