Distributions

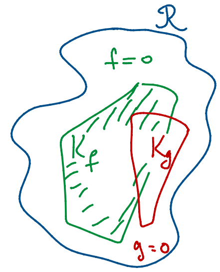
Definition If $\Omega \subset \mathbb{R}^n$ is an open set then $C_c^\infty(\Omega)$ is the set of all functions $f : \Omega \to \mathbb{R}$ that satisfy

$\mathcal{D}(\Omega)$

"test functions"

(1) f is C^∞

(2) there is a compact set $K_f \subset \Omega$ such that $f(x) = 0$ for all $x \in \Omega \setminus K_f$



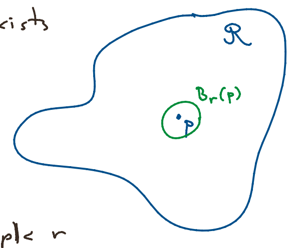
Facts.

* $C_c^\infty(\Omega)$ is a vector space:

  for all $f, g \in C_c^\infty(\Omega)$ and $a, b \in \mathbb{R}$
  $af + bg \in C_c^\infty(\Omega)$. (choose $K_{f+g} = K_f \cup K_g$)

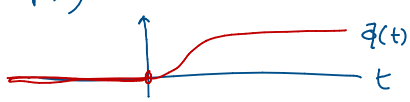
* If $p \in \Omega$ and $\overline{B_r(p)} \subset \Omega$ then there exists a function $\varphi \in C_c^\infty(\Omega)$ with

$$\varphi(x) > 0 \quad \text{for } x \in B_r(p)$$
$$\varphi(x) = 0 \quad \text{for } x \notin B_r(p)$$



Namely:
$$\varphi(x) = \begin{cases} e^{-1/(r^2 - \|x-p\|^2)} & \|x-p\| < r \\ 0 & \|x-p\| \geq r \end{cases}$$

$\varphi(x) = \Phi(r^2 - \|x-p\|^2)$ where $\Phi : \mathbb{R} \to \mathbb{R}$, $\Phi(t) \overset{\text{def}}{=} \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leq 0 \end{cases}$



* If $f, g : \Omega \to \mathbb{R}$ are continuous and if

  $\int_\Omega f(x)\varphi(x)\,dx = \int_\Omega g(x)\varphi(x)\,dx$ holds for all $\varphi \in C_c^\infty(\Omega)$

  then $f(x) = g(x)$ for all $x \in \Omega$.

Definition A distribution of finite order on Ω is a linear map $T : C_c^\infty(\Omega) \to \mathbb{R}$ such that

there exist $C \in \mathbb{R}$, $m \in \mathbb{N}$ such that for all $\varphi \in C_c^\infty(\Omega)$:

$$(*) \quad |T(\varphi)| \leq C \sup_{\substack{i_1 + \dots + i_n \leq m \\ x \in \Omega}} \left| \frac{\partial^{i_1 + \dots + i_n} \varphi}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_n^{i_n}}(x) \right|$$

Examples ① If $f : \Omega \to \mathbb{R}$ is bounded and continuous then

$$T_f(\varphi) = \int_\Omega f(x)\varphi(x)\,dx$$

is a distribution (of order 0)

$T_f(\varphi_1 + \varphi_2) = \int_\Omega f(x)(\varphi_1(x) + \varphi_2(x))\,dx = \int_\Omega f(x)\varphi_1(x)\,dx + \int_\Omega f(x)\varphi_2(x)\,dx$

is a distribution (of order 0)

$$T_f(\varphi_1+\varphi_2) = \int_{\mathcal{R}} f(x)\,(\varphi_1(x)+\varphi_2(x))\,dx = \int_{\mathcal{R}} f(x)\,\varphi_1(x)\,dx + \int_{\mathcal{R}} f(x)\,\varphi_2(x)\,dx$$

$$= T_f(\varphi_1) + T_f(\varphi_2) \qquad \text{linear } \checkmark$$

$$|T(\varphi)| = \left|\int_{\mathcal{R}} f(x)\,\varphi(x)\,dx\right| \le \int_{\mathcal{R}} |f(x)|\,|\varphi(x)|\,dx$$

$$\le \underbrace{\sup_{x\in\mathcal{R}}|f(x)|}_{<\infty \text{ assumption}} \cdot \underbrace{\sup_{x\in\mathcal{R}}|\varphi(x)|}_{\varphi=0 \text{ on } \mathbb{R}^n\setminus K_\varphi} \ \overbrace{\int_{\mathcal{R}} dx}^{n\text{-dimensional volume of } \mathcal{R}} < \infty$$

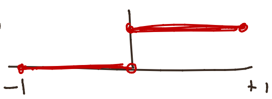
so $\sup\limits_{x\in\mathcal{R}}|\varphi(x)| = \sup\limits_{x\in K_\varphi}|\varphi(x)| = \max\limits_{x\in K_\varphi}|\varphi(x)| < \infty$

$$= C \sup_{x\in\mathcal{R}}|\varphi(x)| \quad \text{where } C = \sup_{\mathcal{R}}|f(x)|\cdot \text{volume}(\mathcal{R})$$

i.e. ✱ with $m=0$ holds. ✓

② The Heaviside function $\mathcal{R} = (-1,+1)$

$$H(\varphi) \stackrel{\text{def}}{=} \int_0^1 \varphi(x)\,dx$$



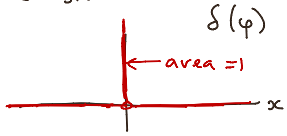
Distributions: Laurent Schwarz ~1940

③ Dirac's δ-function. Let $n=1$, $\mathcal{R}=\mathbb{R}$ and consider

$$\delta(\varphi) \stackrel{\text{def}}{=} \varphi(0) = \int_{-\infty}^{\infty} \underbrace{\delta(x)}_{=0 \text{ when } x\ne 0}\ \underbrace{\varphi(x)}_{=\varphi(0) \text{ when } x=0}\,dx$$

$$= \varphi(0)\int_{-\infty}^{\infty}\delta(x)\,dx = \varphi(0)$$

← Dirac's calculation.



<u>Definition</u> If $T: C_c^\infty(\mathcal{R})\to\mathbb{R}$ is a distribution then its derivative w.r.t. x_k is the distribution

$$(D_kT)(\varphi) = -T\left(\frac{\partial\varphi}{\partial x_k}\right)$$

<u>Examples</u> ① If $f:\mathcal{R}\to\mathbb{R}$ is C^1 then $D_kT_f = T_{\frac{\partial f}{\partial x_k}}$

<u>proof</u>: Let $\varphi\in C_c^\infty(\mathcal{R})$ be given. Then

$$(D_kT_f)(\varphi) \stackrel{\text{def of } D_k}{=} -T_f\left(\frac{\partial\varphi}{\partial x_k}\right) \stackrel{\text{def of } T_f}{=} -\int_{\mathcal{R}} f(x)\,\frac{\partial\varphi}{\partial x_k}\,dx$$

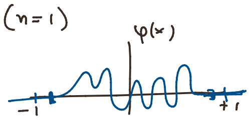
$$= -\int_{\mathcal{R}}\left[\frac{\partial}{\partial x_k}\big(f(x)\,\varphi(x)\big) - \frac{\partial f}{\partial x_k}(x)\,\varphi(x)\right]dx$$

$$= -\int_{\partial\mathcal{R}} f(x)\underbrace{\varphi(x)}_{=0 \text{ on } \partial\mathcal{R}}\,n_k\,da + \underbrace{\int_{\mathcal{R}}\frac{\partial f}{\partial x_k}(x)\,\varphi(x)\,dx}$$

$$= T_{\frac{\partial f}{\partial x_k}}(\varphi) \qquad ////$$

② $DH = \delta \qquad \mathcal{R} = (-1,1)\subset\mathbb{R} \quad (n=1)$

For $\varphi\in C_c^\infty((-1,+1))$ we have



$$(DH)(\varphi) \underset{\text{def of "D"}}{=} -H\left(\frac{d\varphi}{dx}\right) \underset{\text{def of "H"}}{=} -\int_0^1\varphi'(x)\,dx =$$

$$= -\big[\varphi(x)\big]_{x=0}^{1} = -(\underbrace{\varphi(1)}_{=0} - \varphi(0))$$

$$= \varphi(0)$$

$$= \delta(\varphi)$$