Minimizing the Dirichlet Integral

To find a solution of

$$\Delta u = 0 \quad \text{on} \quad \mathcal{R}$$
$$u = g \quad \text{on} \quad \partial\mathcal{R}$$

find a function $u : \mathcal{R} \to \mathbb{R}$ that minimizes

$$\mathcal{D}[u] = \frac{1}{2}\int_{\mathcal{R}} |\nabla u(x)|^2 \, dx \quad \text{with} \quad u = g \text{ on } \partial\mathcal{R}$$

notation
$$\nabla u(x) = \begin{pmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_n} \end{pmatrix} = (u_{x_1}, \cdots, u_{x_n})$$

$$|\nabla u(x)|^2 = \left(\frac{\partial u}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial u}{\partial x_n}\right)^2$$

How do you prove there is a minimizer?

Basic strategy.

① define $D_- = \inf\{D[u] \mid u = g \text{ on } \partial R\}$

② for each $n \in \mathbb{N}$ there is $u_n : R \to \mathbb{R}$ with
$u_n = g$ on ∂R and $D_- \le D[u_n] < D_- + \frac{1}{n}$

u_n is a minimizing sequence.

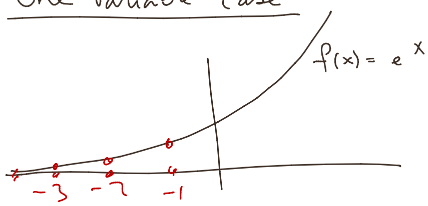
③ Find a subsequence of u_n, say u_{n_k} $(k \in \mathbb{N})$ such that $u_{n_k} \longrightarrow u$

④ Prove $\mathcal{D}$ is continuous and conclude

$$\mathcal{D}[u] = \lim_{k\to\infty} D[u_{n_k}] = D_-$$

because $D_- \le D[u_{n_k}] \le D_- + \frac{1}{n_k} \to D_-$
"sandwich theorem"

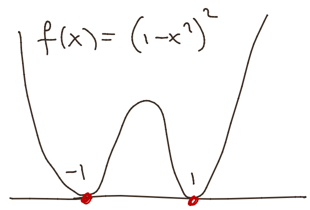
One variable case



$$\inf_{x \in \mathbb{R}} e^x = 0$$

$x_n = -n$ is a minimizing sequence.

but $x_n \to -\infty$ has no convergent subsequence

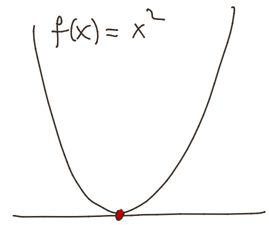


$\{x_n\} = \{1, 1, 1, \dots\}$

$\{y_n\} = \{-1, -1, -1, \dots\}$

$\{z_n\} = \{1, -1, +1, -1, +1, \dots\}$

are minimizing sequences.



Every minimizing sequence x_n satisfies

$x_n \to 0$.

($\inf_{x \in \mathbb{R}} x^2 = 0$,

x_n minimizing $\Leftrightarrow$

$\Leftrightarrow f(x_n) \to 0$

$\Leftrightarrow x_n^2 \to 0$

$\Leftrightarrow x_n \to 0$)

Notation $\int_{\mathcal{R}} f(x)\,dx = \int \cdots \int_{\mathcal{R}} f(x_1,\ldots,x_n)\,dx_1 \ldots dx_n$

$\mathcal{R}$ a bounded open domain in $\mathbb{R}^n$ $(n \geq 1)$

$g : \partial\mathcal{R} \longrightarrow \mathbb{R}$ a function on $\partial\mathcal{R}$.

$C^1_g = \{ u \mid u : \mathcal{R} \to \mathbb{R}$ is a C^1 function with $u = g$ on $\partial\mathcal{R} \}$.

$$D_- = \inf \{ \mathcal{D}[u] \mid u \in C^1_g \}$$

Theorem If $u_k \in C^1_g$ is a sequence with $\mathcal{D}[u_k] \leq D_- + \frac{1}{k}$ then

$$\int_{\mathcal{R}} |\nabla u_k - \nabla u_\ell|^2\,dx \leq \frac{8}{k}$$

for all $\ell \geq k \geq 1$

ie.

$$\int \cdots \int_{\mathcal{R}} \left(\frac{\partial u_k}{\partial x_1} - \frac{\partial u_\ell}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial u_k}{\partial x_n} - \frac{\partial u_\ell}{\partial x_n}\right)^2 dx_1 \ldots dx_n \leq \frac{8}{k}$$

This implies the sequence $\frac{\partial u_1}{\partial x_i}, \frac{\partial u_2}{\partial x_i}, \frac{\partial u_3}{\partial x_i}, \frac{\partial u_4}{\partial x_i}, \ldots$ is a Cauchy sequence for the metric

$$d(f,g) = \left(\int_{\mathcal{R}} |f(x) - g(x)|^2\,dx \right)^{\frac{1}{2}}$$

[Reminder: $f_1, f_2, \ldots \in (X,d)$ is a Cauchy sequence if $d(f_k, f_\ell) \to 0$ if $\ell \geq k \to \infty$.

If (X,d) is complete then every Cauchy sequence converges]

Proof

$$(a-b)^2 + (a+b)^2 = 2(a^2+b^2)$$

$$a^2 - 2ab + b^2 + a^2 + 2ab + b^2 = 2a^2 + 2b^2 \quad ✓$$

$$\Rightarrow |\nabla u - \nabla v|^2 + |\nabla u + \nabla v|^2 = 2|\nabla u|^2 + 2|\nabla v|^2 \text{ for all } x \in \mathcal{R}$$

$$\Rightarrow \left|\frac{\nabla u - \nabla v}{2}\right|^2 + \left|\frac{\nabla u + \nabla v}{2}\right|^2 = \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla v|^2 \text{ on } \mathcal{R}$$

$$\Rightarrow \frac{1}{4}\int_{\mathcal{R}} |\nabla u - \nabla v|^2\,dx = \frac{1}{2}\int_{\mathcal{R}} |\nabla u|^2\,dx + \frac{1}{2}\int_{\mathcal{R}} |\nabla v|^2\,dx - \int_{\mathcal{R}} \left|\nabla\left(\frac{u+v}{2}\right)\right|^2 dx$$

$$= \mathcal{D}[u] + \mathcal{D}[v] - 2\,\mathcal{D}\left[\tfrac{u+v}{2}\right]$$

If $u = u_k$ and $v = u_\ell$ then

$$\frac{1}{4}\int_{\mathcal{R}} |\nabla u_k - \nabla u_\ell|^2\,dx = \underbrace{\mathcal{D}[u_k]}_{\le D_- + \frac{1}{k}} + \underbrace{\mathcal{D}[u_\ell]}_{\le D_- + \frac{1}{\ell}} - 2\,\underbrace{\mathcal{D}\left[\tfrac{u_k+u_\ell}{2}\right]}_{\ge D_-}$$

$$\le D_- + \frac{1}{k} + D_- + \frac{1}{\ell} - 2D_-$$

$$= \frac{1}{k} + \frac{1}{\ell}$$

$$\Rightarrow \int_{\mathcal{R}} |\nabla u_k - \nabla u_\ell|^2\,dx \le \frac{4}{k} + \frac{4}{\ell} \le \frac{8}{k} \quad \text{if } \ell \ge k.$$

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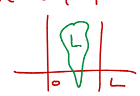
<u>Poincaré inequality</u> If $\mathcal{R}$ is bounded then there is a $C_{\mathcal{R}} \in \mathbb{R}$ such that for all $u \in C^1_0$ (u is a C^1 function with $u = 0$ on $\partial\mathcal{R}$) one has

$$\int_{\mathcal{R}} u^2\,dx \le C_{\mathcal{R}} \int_{\mathcal{R}} |\nabla u|^2\,dx$$

$C_{\mathcal{R}} = 4L^2$ if $\mathcal{R} \subset \{x \mid 0 < x_1 < L\}$

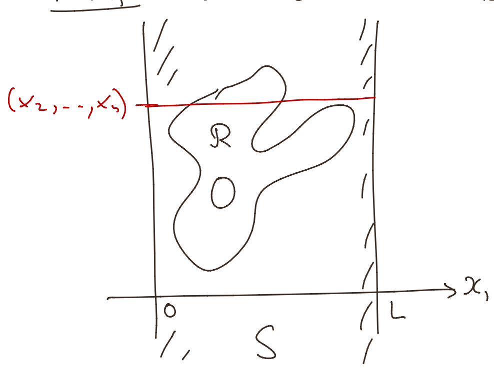


<u>Consequence</u> If $u_k \in C^1_0$ satisfies $\mathcal{D}[u_k] \le D_- + \frac{1}{k}$ then

$$\int_{\mathcal{R}} |u_k - u_\ell|^2\,dx \le C_{\mathcal{R}} \cdot \frac{8}{k} \quad \text{for all } \ell \ge k \ge 1.$$

(because $u_k - u_\ell = 0$ on $\partial\mathcal{R}$, so $\int |u_k - u_\ell|^2 dx \le C_{\mathcal{R}} \int |\nabla(u_k - u_\ell)|^2 dx$
$= C_{\mathcal{R}} \int |\nabla u_k - \nabla u_\ell|^2\,dx$)

<u>Proof</u>. Assume that $\mathcal{R}$ is contained in $\{x \in \mathbb{R}^n \mid 0 < x_1 < L\} = S$



Given $u \in C^1_0$ extend u to a function $u : S \to \mathbb{R}$ by

$$u(x) = \begin{cases} u(x) & \text{if } x \in \mathcal{R} \\ 0 & \text{if } x \notin \mathcal{R}. \end{cases}$$

u is continuous

Let $\bar{x} = (\bar{x}_2, \ldots, \bar{x}_n)$ be any point in $\mathbb{R}^{n-1}$.

Write $u(x_1, \bar{x}_2, \ldots, \bar{x}_n) = u(x_1, \bar{x})$.

Then

$$u(x_1, \bar{x})^2 = \underbrace{u(0, \bar{x})^2}_{=0} + \int_0^{x_1} \frac{d}{ds}\left(u(s, \bar{x})^2\right) ds$$

$$= \int_0^{x_1} 2u(s, \bar{x}) \cdot \frac{\partial u}{\partial x_1}(s, \bar{x})\,ds$$

Integrate w.r.t. x_1, keeping $\bar{x}$ fixed:

$$\int_0^L u(x_1,\bar{x})^2\,dx_1 = \int_0^L \int_0^{x_1} 2u(s,\bar{x})\,u_{x_1}(s,\bar{x})\,ds\,dx_1$$

$0 < s < x_1 < L$

$$= \int_0^L \int_s^L 2u(s,\bar{x})\,u_{x_1}(s,\bar{x})\,dx_1\,ds$$

$$= \int_0^L (L-s)\,2u(s,\bar{x})\,u_{x_1}(s,\bar{x})\,ds$$

$$\le L\int_0^L 2|u(s,\bar{x})|\,|u_{x_1}(s,\bar{x})|\,ds \quad \text{— rename } s = x_1$$

$$\int_0^L u(x_1,\bar{x})^2\,dx_1 \le 2L\int_0^L |u(x_1,\bar{x})|\cdot|u_{x_1}(x_1,\bar{x})|\,dx_1$$

$$\le 2L\left(\int_0^L u(x_1,\bar{x})^2\,dx_1\right)^{\frac{1}{2}}\cdot\left(\int_0^L u_{x_1}(x_1,\bar{x})^2\,dx_1\right)^{\frac{1}{2}}$$

Cauchy – Schwarz:

$$\int f(x)\,g(x)\,dx \le \left(\int f(x)^2\,dx\right)^{\frac{1}{2}}\cdot\left(\int g(x)^2\,dx\right)^{\frac{1}{2}}$$

because $\langle f, g\rangle = \int f(x)g(x)\,dx$ is an inner product

Either $\int_0^L u(x_1,\bar{x})^2\,dx_1 = 0$ or

$$\left(\int_0^L u(x_1,\bar{x})^2\,dx_1\right)^{\frac{1}{2}} \le 2L\left(\int_0^L u_{x_1}(x_1,\bar{x})^2\,dx_1\right)^{\frac{1}{2}}$$

$$\Rightarrow \quad \int_0^L u(x_1,\bar{x})^2\,dx_1 \le 4L^2\cdot\int_0^L u_{x_1}(x_1,\bar{x})^2\,dx_1$$

Integrate with respect to $\bar{x} = (x_2,\dots,x_n)$:

$$\int\!\cdots\!\int_S u(x_1,x_2,\dots,x_n)^2\,dx_1\cdots dx_n \le 4L^2\int\!\cdots\!\int_S \left(\frac{\partial u}{\partial x_1}\right)^2 dx_1\cdots dx_n$$

Since $u, \frac{\partial u}{\partial x_1} = 0$ on $S\setminus\mathcal{R}$:

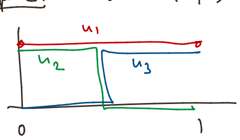
$$\int_{\mathcal{R}} u(x)^2\,dx \le 4L^2\int_{\mathcal{R}}\left(\frac{\partial u}{\partial x_1}\right)^2 dx \le 4L^2\int_{\mathcal{R}}|\nabla u|^2\,dx$$

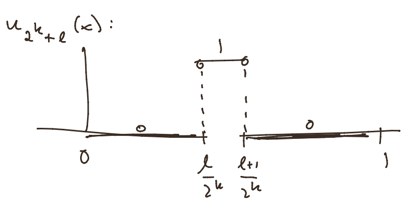
because $|\nabla u|^2 = \left(\frac{\partial u}{\partial x_1}\right)^2 + \dots + \left(\frac{\partial u}{\partial x_n}\right)^2 \ge \left(\frac{\partial u}{\partial x_1}\right)^2$. ////.

Consequence If $u_k \in C^1_g$ is a minimizing sequence for $\mathcal{D}[u]$ then u_k is a Cauchy-Sequence in the L^2 metric

$$d_{L^2}(f,g) = \sqrt{\int_{\mathcal{R}}|f(x)-g(x)|^2\,dx}\,.$$

Example: $\mathcal{R} = (0,1) \subset \mathbb{R}$.





$\lim_{n\to\infty} u_n(x)$ does not exist for any $x \in (0,1)$

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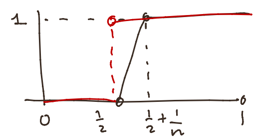
$$d(u_n, 0)^2 = \int_0^1 |u_n(x) - 0|^2\,dx = \int_0^1 u_n(x)^2\,dx = \int_{\ell/2^k}^{(\ell+1)/2^k} 1\,dx = \frac{1}{2^k} \qquad n = 2^k + \ell$$

so $d(u_{2^k+\ell}, 0) = 2^{-k/2} \longrightarrow 0$.

<u>Example</u> $v_n(x)$:



$$f(x) = \begin{cases} 1 & x > \frac{1}{2} \\ 0 & x < \frac{1}{2} \end{cases}$$

$d(v_n, f) \longrightarrow 0$ i.e. $\lim_{n\to\infty} \int_0^1 |v_n(x) - f(x)|^2\,dx = 0$.

<u>Example</u>