Laplace's Equation

$\mathcal{R} \subset \mathbb{R}^n$ an open domain.

$u : \mathcal{R} \to \mathbb{R}$ satisfies Laplace's equation if

$$(*) \qquad \underbrace{\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}} = 0$$

This expression is called the Laplacian of u.

$$\Delta u(x) \stackrel{\text{def}}{=} \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}$$

For $n=2$: $\quad u : \mathcal{R} \to \mathbb{R} \quad u = u(x,y)$ is harmonic if

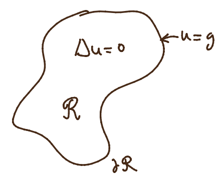
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

A C^2 solution of $(*)$ is a harmonic function

Laplace's equation is the Euler–Lagrange equation for Dirichlet's integral

$$D[u] = \int \cdots \int_{\mathcal{R}} \frac{1}{2}\left(u_{x_1}^2 + \cdots + u_{x_n}^2\right) dx_1 \cdots dx_n$$

Interpretations:



① u = electric potential in a cavity $\mathcal{R}$

② $u(x,y)$ = Temperature at (x,y) if the boundary $\partial\mathcal{R}$ is heated or cooled so the temperature is g on $\partial\mathcal{R}$.

③ diffusion

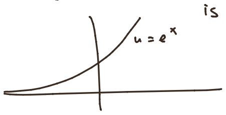
④ Brownian motion

Question. If $g : \partial\mathcal{R} \to \mathbb{R}$ is a given function does there exist a function $u : \mathcal{R} \to \mathbb{R}$ with

$$(B) \qquad \begin{cases} \Delta u = 0 & \text{in } \mathcal{R} \\ u = g & \text{on } \partial\mathcal{R} \end{cases} \qquad ?$$

Dirichlet's answer: yes, just let u be the function on $\mathcal{R}$ with $u = g$ on $\partial\mathcal{R}$ for which $D[u]$ has the smallest possible value.

Objection! The function $u : \mathbb{R} \to \mathbb{R}$, $u(x) = e^x$ is positive but has no minimizer



Some Solutions when $n=2$ $\quad u_{xx} + u_{yy} = 0$

$$x^2 - y^2, \quad xy, \quad x^3 - 3xy^2, \quad xy^2 - yx^2, \ldots$$

$u_{xx} = 2$ $\Rightarrow \Delta u = 2 - 2 = 0$
$u_{yy} = -2$ ✓

$u_{tt} - c^2 u_{xx} = 0$ has general solution

$u(x,t) = F(x+ct) + G(x-ct)$.

Let $c^2 = -1$, ie $c = \sqrt{-1} = i$.

$$u(x,y) = F(x+iy) + G(x-iy) \quad ????$$

$$u_{xx} = \frac{\partial}{\partial x}\frac{\partial}{\partial x} F(x+iy) = F''(x+iy)$$

$$u_{yy} = \frac{\partial}{\partial y}\frac{\partial}{\partial y} F(x+iy) = \frac{\partial}{\partial y}\left(i F'(x+iy)\right) = i^2 F''(x+iy)$$

$$= -F''(x+iy)$$

$$\Rightarrow \quad u_{xx} + u_{yy} = 0.$$

For this to be correct we need a theory of functions of a complex variable $z = x+iy$, and the chain rule $\frac{\partial}{\partial y} F(x+iy) = i F'(x+iy)$ must hold. (Math 623)

You can verify that $F(z) = z^m$, i.e.

$$F(x+iy) = (x+iy)^m$$

satisfies $\frac{\partial}{\partial y} F(x+iy) = i F'(x+iy) = i \frac{\partial}{\partial x} F(x+iy)$.

On the other hand

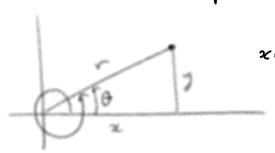
$$F(z) = \bar{z}, \quad F(x+iy) = x - iy$$

does not satisfy $\frac{\partial}{\partial y} F(x+iy) = i \frac{\partial}{\partial x} F(x+iy)$

Explicit formulas for the solution

Let $R = \{(x,y) \in \mathbb{R} : x^2+y^2 < 1\}$

Express Δu in polar coordinates.



$x = r\cos\theta$, $y = r\sin\theta$

Given $u(x,y)$ consider $v(r,\theta) = u(r\cos\theta, r\sin\theta)$

$v_r(r,\theta)$ $\quad u_y(r\cos\theta, r\sin\theta)$

$$v_r = u_x \cdot \cos\theta + u_y \sin\theta$$

$$v_\theta = -r u_x \sin\theta + r u_y \cos\theta$$

$$v_{rr} = (u_{xx}\cos\theta + u_{xy}\sin\theta)\cos\theta + (u_{yx}\cos\theta + u_{yy}\sin\theta)\sin\theta$$

$$= u_{xx}\cos^2\theta + 2u_{xy}\sin\theta\cos\theta + u_{yy}\sin^2\theta$$

$$v_{\theta\theta} = r^2 u_{xx}\sin^2\theta - 2r^2 u_{xy}\sin\theta\cos\theta + r^2 u_{yy}\cos^2\theta$$
$$- r u_x \cos\theta - r u_y \sin\theta$$

$$v_{rr} + \frac{1}{r^2} v_{\theta\theta} = u_{xx} + u_{yy} - \frac{1}{r}(u_x\cos\theta + u_y\sin\theta)$$
$$= \Delta u - \frac{1}{r} v_r$$

Therefore $\quad \Delta u = v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta}$

Special solutions: is $u = r^m e^{in\theta}$ a solution of $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$?

$$u_r = m r^{m-1} e^{in\theta} \qquad u_\theta = in\, r^m e^{in\theta}$$

$$u_{rr} = m(m-1) r^{m-2} e^{in\theta} \qquad u_{\theta\theta} = -n^2 r^m e^{in\theta}$$

$$\Rightarrow \quad \Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} =$$

$$= m(m-1) r^{m-2} e^{in\theta} + \cancel{\tfrac{1}{r}}\, m\, r^{\cancel{m-1}}\, e^{in\theta} - \cancel{\tfrac{1}{r^2}}\, n^2\, r^{\cancel{m}}\, e^{in\theta}$$

r^{m-2} $\quad r^{m-2}$

$$\Rightarrow \quad \Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} =$$

$$= m(m-1) r^{m-2} e^{in\theta} + \cancel{\tfrac{1}{r}}\, m \overset{r^{m-2}}{\cancel{r^{m-1}}} e^{in\theta} - \cancel{\tfrac{1}{r^2}}\, n^2 \overset{r^{m-2}}{\cancel{r^{m}}} e^{in\theta}$$

$$= e^{in\theta} r^{m-2} \{ \overset{m^2 - m + m = m^2}{m(m-1) + m} - n^2 \}$$

$$= e^{in\theta} r^{m-2} (m^2 - n^2)$$

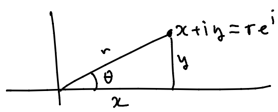
$$= 0 \quad \text{for all } r, \theta \iff m^2 = n^2, \text{ i.e. } m = \pm n.$$

Conclusion: for all $m \in \mathbb{Z}$ the functions

$$r^m e^{im\theta} \quad \text{and} \quad r^m e^{-im\theta}$$

are harmonic. These functions are ounded as $r \to 0$ if $m \geq 0$.

<u>Example</u> $\quad r^m e^{im\theta} = (r e^{i\theta})^m = (r\cos\theta + i r \sin\theta)^m$



$$= (x+iy)^m$$

For $m=2$:
$$(x+iy)^2 = x^2 + 2ixy - y^2$$

$$r^m e^{-im\theta} = (r e^{-i\theta})^m = (x-iy)^m$$

For $m=2$: $\quad (x-iy)^2 = x^2 - 2ixy - y^2$

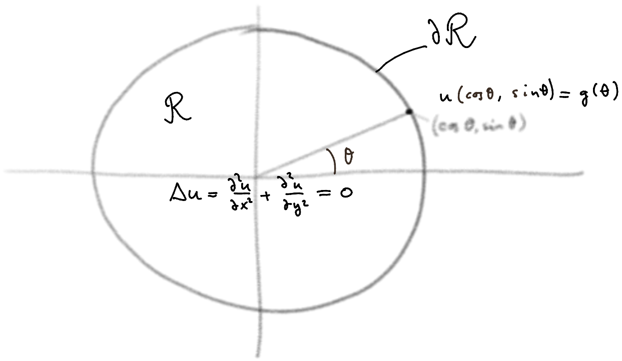
Note:
$$u(x,y) = \frac{1}{2}(x+iy)^2 + \frac{1}{2}(x-iy)^2 = x^2 - y^2$$
$$u(x,y) = \frac{1}{2}(x+iy)^3 + \frac{1}{2}(x-iy)^3 = x^3 - 3xy^2$$

The Poisson kernel

Theorem. If $g: \mathbb{R} \to \mathbb{R}$ is a continuous 2π-periodic function then
$$u(r,\theta) = \int_{-\pi}^{\pi} P(r, \theta - \phi) g(\phi)\, d\phi$$
with

$$P(r,\phi) = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos\phi + r^2} \qquad \text{"The Poisson kernel"}$$

is a solution to the Laplace equation $\Delta u = 0$ on the disc $\mathcal{R} = \{ (x,y) \mid x^2 + y^2 < 1 \}$ with boundary values
$$u(\cos\theta, \sin\theta) = g(\theta)$$



1. Derivation of the Poisson kernel.

Facts: * for each $n \in \mathbb{N}_0$

$$u = r^n e^{in\theta} = (x+iy)^n$$
$$u = r^n e^{-in\theta} = (x-iy)^n$$

satisfy $\Delta u = 0$.

* $\Delta u = 0$ is a linear equation, i.e.

$$\Delta : \underbrace{\{C^2 \text{ functions}\}}_{\text{vector space}} \longrightarrow \underbrace{\{C^0 \text{ functions}\}}_{\text{vector space}}$$

$\Delta(u+v) = \Delta(u) + \Delta(v)$

$\Delta(cu) = c\,\Delta u$ if $c \in \mathbb{C}$

{harmonic functions on $\mathcal{R}$} = Nullspace of Δ
is a linear subspace of {C^2 functions on $\mathcal{R}$}

i.e. If $u_1, \dots, u_n$ are harmonic then $c_1u_1 + \dots + c_nu_n$ also is harmonic for all $c_1, \dots, c_n \in \mathbb{C}$.

* For any $c_{-N}, \dots, c_N \in \mathbb{C}$: $u(r,\theta) = \sum_{-N}^{+N} c_n r^{|n|} e^{in\theta}$ is harmonic.

– Maybe $\lim_{N\to\infty} \sum_{-N}^{+N} c_n r^{|n|} e^{in\theta}$ also is harmonic.

Assume $u(r,\theta) = \sum_{-\infty}^{\infty} c_n r^{|n|} e^{in\theta}$ is harmonic

Try to find c_n so that $u(1,\theta) = g(\theta)$

Solution: Expand $g(\theta) = \sum_{-\infty}^{\infty} \hat{g}_n e^{in\theta}$ and set $c_n = \hat{g}_n$.

So we claim that

$$\boxed{u(r,\theta) = \sum_{-\infty}^{\infty} \hat{g}_n r^{|n|} e^{in\theta}}$$

is the solution to

$$\boxed{\begin{cases} \Delta u = 0 & \text{on } \mathcal{R} \\ u = g & \text{on } \partial\mathcal{R} \end{cases}}$$

Different form of the solution.

Recall $\hat{g}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\phi} g(\phi)\, d\phi$. This leads to

$$u(r,\theta) = \sum_{-\infty}^{\infty} \hat{g}_n r^{|n|} e^{in\theta}$$

$$= \sum_{-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\phi} g(\phi)\, d\phi \right) r^{|n|} e^{in\theta}$$

$$= \frac{1}{2\pi} \sum_{-\infty}^{\infty} \int_{-\pi}^{\pi} r^{|n|} e^{in(\theta-\phi)} g(\phi)\, d\phi$$

$$= \frac{1}{2\pi} \sum_{-\infty}^{\infty} \int_{-\pi}^{\pi} r^{|n|} e^{in(\theta-\phi)} g(\phi)\, d\phi$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta-\phi)} g(\phi)\, d\phi$$

$$= \int_{-\pi}^{\pi} P(r, \theta-\phi)\, g(\phi)\, d\phi$$

where $P(r,\theta) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} r^{|n|} e^{in\theta}$ is the Poisson kernel.

<u>Computation of P:</u>

$$2\pi P(r,\theta) = \text{"} \sum_{-\infty}^{-1} + 1 + \sum_{1}^{\infty} \text{"}$$

$$= \sum_{-\infty}^{-1} r^{-n} e^{in\theta} + 1 + \sum_{1}^{\infty} r^n e^{in\theta}$$

$$= \sum_{1}^{\infty} r^m e^{-im\theta} + 1 + \sum_{1}^{\infty} r^n e^{in\theta}$$

$$\sum_{1}^{\infty} r^n e^{in\theta} = \sum_{1}^{\infty} (re^{i\theta})^n = \frac{re^{i\theta}}{1-re^{i\theta}}$$ if $r<1$ because $|(re^{i\theta})^n| = r^n$

$$\Rightarrow 2\pi P(r,\theta) = \frac{re^{-i\theta}}{1-re^{-i\theta}} + 1 + \frac{re^{i\theta}}{1-re^{i\theta}}$$

$$= 1 + \frac{re^{-i\theta}(1-re^{i\theta}) + re^{i\theta}(1-re^{-i\theta})}{(1-re^{-i\theta})(1-re^{i\theta})}$$ $\quad e^{-i\theta} \cdot e^{i\theta} = 1$

$$= 1 + \frac{re^{-i\theta} - r^2 + re^{i\theta} - r^2}{1 - r(e^{i\theta}+e^{-i\theta}) + r^2}$$ $\quad e^{i\theta} + e^{-i\theta} = 2\cos\theta$

$$= 1 + \frac{2r\cos\theta - 2r^2}{1-2r\cos\theta + r^2}$$

$$= \frac{1-r^2}{1-2r\cos\theta+r^2}$$

<u>Conclusion:</u> $P(r,\theta) = \frac{1}{2\pi} \frac{1-r^2}{1-2r\cos\theta+r^2}$.

2. Proof that $u(r,\theta) = \int_{-\pi}^{\pi} P(r,\theta-\phi)\, g(\phi)\, d\phi$ is a solution.

* Verification that $\Delta u = 0$ in R.

<u>Outline.</u> First show that $P(r,\theta)$ is harmonic, i.e.

$$\frac{\partial^2 P}{\partial r^2} + \frac{1}{r}\frac{\partial P}{\partial r} + \frac{1}{r^2}\frac{\partial^2 P}{\partial \theta^2} = 0. \quad \text{for } r<1.$$

Then justify

$$\Delta u = \Delta \int_{-\pi}^{\pi} P(r,\theta-\phi)\, g(\phi)\, d\phi$$

$$= \int_{-\pi}^{\pi} \underbrace{\Delta P(r,\theta-\phi)}_{=0} g(\phi)\, d\phi = 0 \quad \checkmark$$

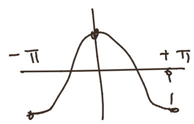
* Verification that $\lim_{r \uparrow 1} u(r,\theta) = g(\theta)$

$$\lim_{r \uparrow 1} u(r,\theta) = \int_{-\pi}^{\pi} P(r, \theta - \phi)\, g(\phi)\, d\phi$$

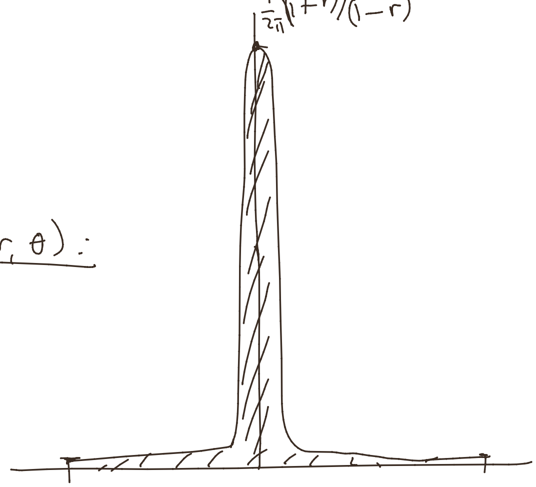
$$P(1, \theta) = \lim_{r \to 1} \frac{1-r^2}{1-2r\cos\theta + r^2} = \frac{1-1^2}{1-2\cos\theta+1} = \frac{0}{2(1-\cos\theta)} = 0$$

unless $\cos\theta = +1$

i.e. unless $\theta = 0$.



Graph of $P(r,\theta)$:



$$P(r,\theta) = \frac{1}{2\pi} \frac{1-r^2}{1-2r\cos\theta+r^2} = \frac{1}{2\pi} \frac{1-r^2}{(1-r\cos\theta)^2 + r^2\sin^2\theta} \geq \frac{1}{2\pi}\frac{1-r^2}{(1+r)^2}$$

$$= \frac{1}{2\pi} \frac{1-r}{1+r}$$

$$P(r,0) = \frac{1}{2\pi}\frac{1-r^2}{1-2r+r^2} = \frac{1}{2\pi}\frac{1-r^2}{(1-r)^2} = \frac{1}{2\pi}\frac{1+r}{1-r}$$

$$P(r, \pm\pi) = \frac{1}{2\pi}\frac{1-r}{1+r}$$

$$\int_{-\pi}^{+\pi} P(r,\theta)\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \frac{1-r^2}{1-2r\cos\theta+r^2}\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{-\infty}^{\infty} r^{|n|} e^{in\theta}\, d\theta = \frac{1}{2\pi}\sum_{-\infty}^{\infty} r^{|n|} \underbrace{\int_{-\pi}^{+\pi} e^{in\theta}\, d\theta}_{\substack{=0 \text{ for } n\neq 0 \\ 2\pi \text{ for } n=0}}$$

$$= \frac{1}{2\pi} r^{|0|}\, 2\pi$$

$$= 1$$

Verification of the boundary values:

$\int^{\pi}$

Verification of the boundary values:

$$u(r,\theta) = \int_{-\pi}^{\pi} P(r,\theta-\phi)\, g(\phi)\, d\phi$$

$$= \int_{-\pi}^{\pi} P(r,\theta-\phi)\big(g(\phi)-g(\theta)\big)\, d\phi + \underbrace{\int_{-\pi}^{\pi} P(r,\theta-\phi)\, g(\theta)\, d\phi}_{= g(\theta) \underbrace{\int_{-\pi}^{\pi} P(r,\theta-\phi)\, d\phi}_{=1}}$$

$$= g(\theta) + \underbrace{\int_{-\pi}^{\pi} P(r,\theta-\phi)\big(g(\phi)-g(\theta)\big)\, d\phi}_{\text{want to show this goes to zero. (on Monday.)}}$$

Theorem. If $g\colon \mathbb{R} \to \mathbb{R}$ is a continuous 2π-periodic function then

$u(r\cos\theta, r\sin\theta) =$

$$u(r,\theta) = \int_{-\pi}^{\pi} P(r,\theta-\phi)g(\phi)\, d\phi$$

with

$$P(r,\phi) = \frac{1}{2\pi}\frac{1-r^2}{1-2r\cos\phi+r^2} = \frac{1}{2\pi}\frac{1-r^2}{(1-r\cos\phi)^2+r^2\sin^2\phi}$$

is a solution to the Laplace equation $\Delta u = 0$ on the disc $\mathcal{R} = \{\,(x,y) \mid x^2+y^2<1\,\}$ with boundary values

$$u(\cos\theta, \sin\theta) = g(\theta)$$

More precisely: $\lim_{r \nearrow 1} u(r\cos\theta, r\sin\theta) = g(\theta)$

Things we know:

$P(r,\theta) > 0$ for all $r<1$, $\theta \in \mathbb{R}$

$\int_{-\pi}^{+\pi} P(r,\theta)\, d\theta = 1$

$\lim_{r \nearrow 1} P(r,\theta) = 0$ for all θ with $0 < |\theta| \le \pi$

$P(r,\theta)$ is a decreasing function of θ for $0<\theta\le\pi$

$P(r,\theta) = P(r,-\theta)$.

$$\big|u(r\cos\theta, r\sin\theta) - g(\theta)\big| = \left|\int_{-\pi}^{\pi} P(r,\theta-\phi)g(\phi)\, d\phi - g(\theta)\underbrace{\int_{-\pi}^{\pi} P(r,\theta-\phi)\, d\phi}_{=1}\right|$$

$$\le \int_{-\pi}^{+\pi} \Big| P(r,\theta-\phi)\,\{g(\phi)-g(\theta)\}\Big|\, d\phi$$

$$= \int_{-\pi}^{\pi} P(r,\theta-\phi)\, |g(\phi)-g(\theta)|\, d\phi \qquad \begin{aligned}\varphi &= \theta-\phi\\ d\varphi &= -d\phi\end{aligned}$$

$$= -\int_{\theta+\pi}^{\theta-\pi} P(r,\varphi)\, |g(\theta-\varphi)-g(\theta)|\, d\varphi$$

$P(r,\varphi)$ and $g(\theta-\varphi)$ are 2π periodic in φ

$$= \int_{-\pi}^{\pi} P(r,\varphi)\, |g(\theta-\varphi) - g(\theta)|\, d\varphi = I$$

$\theta + \pi$

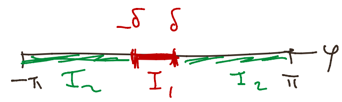
are 2π periodic in φ

$$= \int_{-\pi}^{\pi} P(r,\varphi)\,|g(\theta-\varphi) - g(\theta)|\,d\varphi = I$$

Let $\varepsilon > 0$ be given.

Choose $\delta > 0$ and split the integral: $I = I_1 + I_2$

$$I_1 = \int_{-\delta}^{\delta} P(r,\varphi)\,|g(\theta-\varphi) - g(\theta)|\,d\varphi$$



$$I_2 = \int_{\delta < |\varphi| < \pi} P(r,\varphi)\,|g(\theta-\varphi) - g(\theta)|\,d\varphi$$

$$\int_{\delta < |\varphi| < \pi} = \int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}$$

g is continuous so we can choose $\delta > 0$ so small that

$$|g(\theta-\varphi) - g(\theta)| < \frac{\varepsilon}{2} \text{ for } \varphi \in (-\delta, +\delta).$$

Then

$$I_1 \le \int_{-\delta}^{+\delta} P(r,\varphi)\,\frac{\varepsilon}{2}\,d\varphi < \frac{\varepsilon}{2} \int_{-\pi}^{\pi} \underset{P>0}{P(r,\varphi)}\,d\varphi = \frac{\varepsilon}{2}.$$

For I_2 we use that for $\delta \le |\varphi| \le \pi$:

$0 < P(r,\varphi) = P(r,|\varphi|) \le P(r,\delta)$. and $|g(\theta-\varphi) - g(\theta)| \le |g(\theta-\varphi)| + |g(\theta)| \le 2\|g\|_\infty$

We get

$$I_2 \le \int_{\substack{\delta \le |\varphi| \le \pi \\ (-\pi,-\delta)\cup(\delta,\pi)}} P(r,\delta)\,\left(\|g\|_\infty + \|g\|_\infty\right)\,d\varphi$$

$$= (2\pi - 2\delta) \cdot P(r,\delta)\; 2\|g\|_\infty$$

$$< 4\pi\, P(r,\delta)\,\|g\|_\infty .$$

Since $\lim_{r \uparrow 1} P(r,\delta) = 0$ there is a $\bar r \in (0,1)$ such that

$$P(r,\delta) < \frac{\varepsilon}{8\pi\|g\|_\infty} \quad \text{if } \bar r < r < 1.$$

Therefore $\bar r < r < 1$ implies

$$|u(r\cos\theta, r\sin\theta) - g(\theta)| \le I_1 + I_2 < \frac{\varepsilon}{2} + 4\pi P(r,\delta)\|g\|_\infty$$

$$< \frac{\varepsilon}{2} + 4\pi\|g\|_\infty \cdot \frac{\varepsilon}{8\pi\|g\|_\infty} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$$< \frac{\varepsilon}{2} + 4\pi\|g\|_\infty \cdot \frac{\varepsilon}{8\pi\|g\|_\infty} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

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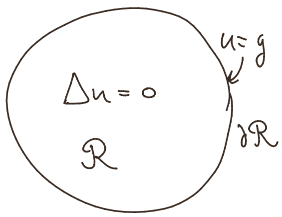

For any continuous $g : \partial\mathcal{R} \to \mathbb{R}$ there is a solution $u : \mathcal{R} \to \mathbb{R}$ of

$$\left.\begin{aligned} \Delta u &= 0 && \text{on } \mathcal{R} \\ u &= g && \text{on } \partial\mathcal{R} \end{aligned}\right\}$$

<u>Missing:</u>

1. Uniqueness?
2. What if $n > 2$?
3. What if $\mathcal{R}$ is not a disc