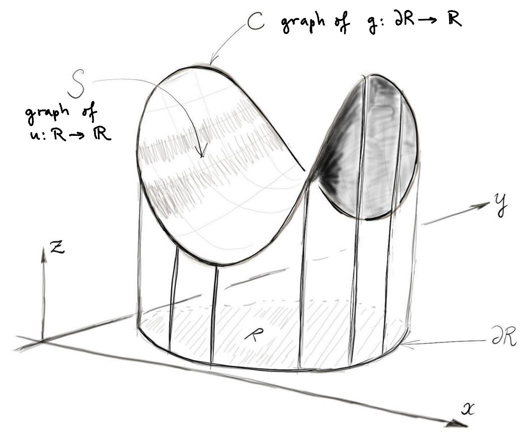
The Minimal Surface equation

Surface area of the graph of a function

Let $R \subset \mathbb{R}^2$ be a "domain" and let $u: R \to \mathbb{R}$ be a C^1 function. Then the area of the graph of u is

$$A[u] = \iint_R \sqrt{1 + u_x^2 + u_y^2}\ dx\, dy = \iint_R \sqrt{1 + u_r^2 + \frac{1}{r^2}u_\theta^2}\ \ r\, d\theta\, dr$$

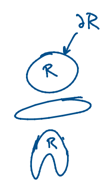


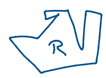
Domain: R is an open subset of the plane $\mathbb{R}^2$

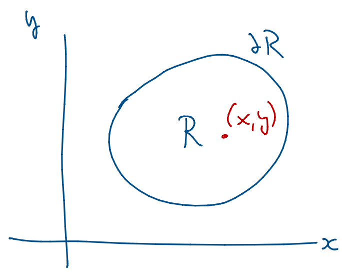
∂R = boundary of R is "somewhat" regular

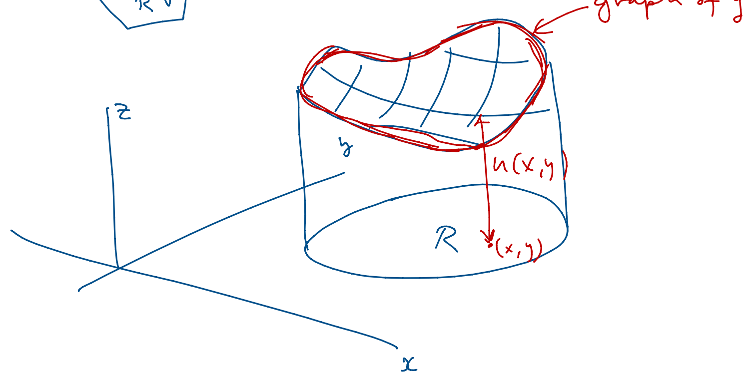
Examples of regular domains: ∂R is a C^∞ differentiable curve

∂R is a polygon



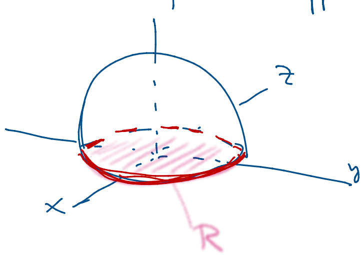






Example

Area of the upper hemisphere $z = \sqrt{1 - x^2 - y^2} = u(x,y)$



$$R = \{(x,y) \mid x^2 + y^2 < 1\}$$

$$\partial R = \{(x,y) \mid x^2 + y^2 = 1\}$$

$$A[u] = \iint_R \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}\ \ dx\, dy$$

$$\frac{\partial u}{\partial x} = \frac{-x}{\sqrt{1 - x^2 - y^2}} \qquad \frac{\partial u}{\partial y} = \frac{-y}{\sqrt{1 - x^2 - y^2}}$$

$$1 + (u_x)^2 + (u_y)^2 = \frac{x^2 + y^2}{1 - x^2 - y^2} + 1 = \frac{1}{1 - x^2 - y^2}$$

$$A[u] = \iint_R \sqrt{\frac{1}{1 - x^2 - y^2}}\ \ dx\, dy = \text{D.I.Y.}\left(\begin{array}{l}\text{use polar}\\ \text{coordinates}\end{array}\right) = 2\pi$$

The Plateau problem

Given a function $g\colon \partial R \to \mathbb{R}$ find a C^1 function $u\colon R \to \mathbb{R}$ such that

- $u(p) = g(p)$ at all points $p \in \partial R$ $\longrightarrow$ $\lim_{q \to p} u(q) = g(p)$ for all $p \in \partial R$.
- $A[u] \le A[v]$ for every C^1 function $v\colon R \to \mathbb{R}$ with $v = g$ on ∂R

The Euler–Lagrange equation.

$$A[u] = \iint_R L(x, y, u(x,y), u_x(x,y), u_y(x,y))\, dx\, dy$$

where $L\colon \underset{(x,y)}{\mathbb{R} \times \mathbb{R}} \times \underset{u}{\mathbb{R}} \times \underset{(v_1,v_2)}{\mathbb{R}^2} \longrightarrow \mathbb{R}$ is given by

$$L(x, y, u, v_1, v_2) = \sqrt{1 + v_1^2 + v_2^2}\ .$$

Or in polar coordinates

$$A[u] = \iint_R L(r, \theta, u, v_r, v_\theta)\, dr\, d\theta$$

where $L(r, \theta, u, v_r, v_\theta) = \sqrt{1 + v_r^2 + \frac{1}{r^2} v_\theta^2} \cdot r$

Suppose $L\colon \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is C^2 and let $u\colon R \to \mathbb{R}$ be a C^1 minimizer of

$$I[u] = \iint_R L\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) dx\, dy\ .$$

i.e. for every C^1 $v\colon R \to \mathbb{R}$ with $u = v$ on ∂R one has $I[v] \ge I[u]$.

Theorem If u is C^2 then u satisfies

$$\frac{\partial}{\partial x}\left(\frac{\partial L}{\partial v_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial v_y}\right) = \frac{\partial L}{\partial u}, \quad \text{and } u(p) = g(p) \text{ for all } p \in \partial R$$

i.e.

$$\frac{\partial}{\partial x}\left(\frac{\partial L}{\partial v_x}(x, y, u(x,y), u_x(x,y), u_y(x,y))\right) + \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial v_y}(x, y, u(x,y), u_x(x,y), u_y(x,y))\right)$$

$$= \frac{\partial L}{\partial u}(x, y, u(x,y), u_x(x,y), u_y(x,y))$$

Example

For $L = L(x,y,u,v_x,v_y) = \sqrt{1+v_x^2+v_y^2}$

we get $\dfrac{\partial L}{\partial u} = 0$

$$\frac{\partial L}{\partial v_x} = \frac{v_x}{\sqrt{1+v_x^2+v_y^2}} \qquad \frac{\partial L}{\partial v_y} = \frac{v_y}{\sqrt{1+v_x^2+v_y^2}}$$

The E-L equation for Minimal Surfaces is

$$\frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{1+u_x^2+u_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{1+u_x^2+u_y^2}}\right) = 0.$$

$$u(x,y) = F(\underbrace{\sqrt{x^2+y^2}}_{=r}) = F(r)$$

where $u_x = \dfrac{\partial u}{\partial x}$ $\quad u_y = \dfrac{\partial u}{\partial y}$

If $|u_x|, |u_y| \ll 1$ then $\sqrt{1+u_x^2+u_y^2} \approx 1$ then the Minimal Surface Equation is approximately

$$\frac{\partial}{\partial x}(u_x) + \frac{\partial}{\partial y}(u_y) = 0, \text{ i.e. } \underline{u_{xx}+u_{yy}=0} \qquad \underline{\text{Laplace Equation}}$$

Derivation of the E-L equation.

Let $u : R \to \mathbb{R}$ be a C^2 minimizer of $I[u]$ with $u=g$ on ∂R

If $\varphi : R \to \mathbb{R}$ is any C^2 function with $\varphi = 0$ on ∂R then

$$I[u+\varepsilon\varphi] \geq I[u] \quad \text{for all } \varepsilon \in \mathbb{R}.$$

Therefore $\quad \dfrac{d}{d\varepsilon} I[u+\varepsilon\varphi]\Big|_{\varepsilon=0} = 0.$

I.e. :

$$\frac{d}{d\varepsilon}\iint_R L\Big(x, y, u(x,y)+\underline{\varepsilon}\varphi(x,y),\ \overbrace{u_x(x,y)+\underline{\varepsilon}\varphi_x(x,y)}^{=\frac{\partial}{\partial x}(u(x,y)+\varepsilon\varphi(x,y))},\ u_y(x,y)+\underline{\varepsilon}\varphi_y(x,y)\Big)\,dx\,dy = 0 \quad \begin{pmatrix}\text{when}\\ \varepsilon=0\end{pmatrix}$$

$L = L(x,y,u,v_x,v_y)$

$$\Rightarrow \iint_R \frac{\partial}{\partial\varepsilon} L(\cdots)\ dx\,dy = 0 \quad \text{when } \varepsilon = 0$$

$$\Rightarrow \iint_R \left[\frac{\partial L}{\partial u}(\cdots)\,\varphi(x,y) + \frac{\partial L}{\partial v_x}(\cdots)\cdot\frac{\partial\varphi}{\partial x}(x,y) + \frac{\partial L}{\partial v_y}(\cdots)\cdot\frac{\partial\varphi}{\partial y}(x,y)\right]dx\,dy = 0$$

when $\varepsilon = 0$.

$$\Rightarrow \iint_R \left[\frac{\partial L}{\partial u}(x,y,u(x,y),u_x(x,y),u_y(x,y))\,\varphi(x,y) + \frac{\partial L}{\partial v_x}(-)\,\frac{\partial \varphi}{\partial x}(x,y) + \frac{\partial L}{\partial v_y}(-)\,\frac{\partial \varphi}{\partial y}(x,y)\right] dx\,dy = 0 \qquad \star$$

Now integrate by parts:

$$\iint_R \frac{\partial L}{\partial v_x}(-)\frac{\partial \varphi}{\partial x} + \frac{\partial L}{\partial v_y}(-)\frac{\partial \varphi}{\partial y}\, dx\,dy =$$

$$= \iint_R \Bigg[\overbrace{\frac{\partial}{\partial x}\Big(\underbrace{\frac{\partial L}{\partial v_x}(-)\cdot\varphi}_{V_1}\Big) + \frac{\partial}{\partial y}\Big(\underbrace{\frac{\partial L}{\partial v_y}(-)\,\varphi}_{V_2}\Big)}^{\operatorname{div}(\vec{V})} - \frac{\partial}{\partial x}\Big(\frac{\partial L}{\partial v_x}(-)\Big)\varphi - \frac{\partial}{\partial y}\Big(\frac{\partial L}{\partial v_y}(-)\Big)\cdot\varphi\Bigg] dx\,dy$$

Green's Thm

$$= \oint_{\partial R} \underbrace{\begin{pmatrix} \frac{\partial L}{\partial v_x}(-)\cdot\varphi \\ \frac{\partial L}{\partial v_y}(-)\cdot\varphi \end{pmatrix}\cdot\vec{n}\, ds}_{=0 \text{ because } \varphi = 0 \text{ on } \partial R} + \iint\Big(-\frac{\partial}{\partial x}\Big(\frac{\partial L}{\partial v_x}\Big) - \frac{\partial}{\partial y}\frac{\partial L}{\partial v_y}(-)\Big)\varphi\; dx\,dy$$

$$= \iint -\left[\frac{\partial}{\partial x}\Big(\frac{\partial \mathcal{L}}{\partial v_x}(-)\Big) + \frac{\partial}{\partial y}\Big(\frac{\partial \mathcal{L}}{\partial v_y}(-)\Big)\right]\varphi(x,y)\,dx\,dy$$

Combine with $\star$ to get

$$0 = \frac{d}{d\varepsilon} I[u+\varepsilon\varphi]\Big|_{\varepsilon=0} = \iint_R \left[\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial x}\Big(\frac{\partial \mathcal{L}}{\partial v_x}\Big) - \frac{\partial}{\partial y}\Big(\frac{\partial \mathcal{L}}{\partial v_y}\Big)\right]\varphi(x,y)\,dx\,dy.$$

for all C^2 $\varphi : R \to \mathbb{R}$ with $\varphi = 0$ on ∂R.

This implies $\quad \frac{\partial L}{\partial u} - \frac{\partial}{\partial x}\Big(\frac{\partial \mathcal{L}}{\partial v_x}\Big) - \frac{\partial}{\partial y}\Big(\frac{\partial \mathcal{L}}{\partial v_y}\Big) \overset{\text{def}}{=} E(x,y) = 0$

Reason

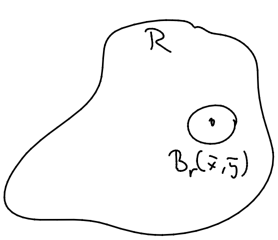
① $\frac{\partial L}{\partial u} - \frac{\partial}{\partial x}\Big(\frac{\partial \mathcal{L}}{\partial v_x}\Big) - \frac{\partial}{\partial y}\Big(\frac{\partial L}{\partial v_y}\Big)$ is continuous because

$$\left.\begin{array}{l} u \in C^2 \longrightarrow u_x, u_y : R \to \mathbb{R} \text{ is } C^1 \\ L \in C^2 \Rightarrow \frac{\partial \mathcal{L}}{\partial v_x} : \mathbb{R}^5 \to \mathbb{R} \text{ is } C^1 \end{array}\right\} \Rightarrow$$

$$\frac{\partial \mathcal{L}}{\partial v_x}(x,y,u(x,y),u_x(x,y),u_y(x,y))$$

$$\Rightarrow \frac{\partial}{\partial x}\Big(\frac{\partial \mathcal{L}}{\partial v_x}(-)\Big) = \frac{\partial^2 \mathcal{L}}{\partial x\,\partial v_x}(-) + \frac{\partial^2 \mathcal{L}}{\partial u\,\partial v_x}(-)\frac{\partial u}{\partial x} + \frac{\partial^2 \mathcal{L}}{\partial v_x\,\partial v_x}\cdot\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 \mathcal{L}}{\partial v_x\,\partial v_y}\,\frac{\partial^2 u}{\partial x\,\partial y}$$

② Suppose that $E(\bar{x},\bar{y}) \neq 0$ for some $(\bar{x},\bar{y}) \in R$.
Then $\exists\, r > 0$ such that $E(x,y) \neq 0$ for all $(x,y) \in B_r(\bar{x},\bar{y})$



Choose

$$\varphi(x,y) = \begin{cases} 0 & \text{if } (x-\bar{x})^2 + (y-\bar{y})^2 \geq r^2 \\ \eta(x,y)^3 & \text{if } (x-\bar{x})^2 + (y-\bar{y})^2 \leq r^2 \end{cases}$$

where

$B_r(x,y)$ $\quad\quad \eta(x,y)$ if $(x-x)^2+(y-y)^2 \le r^2$

where
$$\eta(x,y) = r^2-(x-\bar{x})^2-(y-\bar{y})^2$$

Is φ C^2? If $(x-\bar{x})^2+(y-\bar{y})^2 \neq r^2$ then φ is C^∞.

$$\frac{\partial \varphi}{\partial x} = -3(x-\bar{x})\,\eta(x,y)^2 \longrightarrow 0 \quad \text{as } (x,y)\to \partial B_r(\bar{x},\bar{y})$$

$$\frac{\partial^2 \varphi}{\partial x \partial y} = 2\big(-3(x-\bar{x})\big)\big(-2(y-\bar{y})\big)\,\eta(x,y) \longrightarrow 0 \quad \text{as } (x,y)\to \partial B_r(\bar{x},\bar{y})$$

alternative choice:
$$\varphi(x,y) = \begin{cases} 0 & (x-\bar{x})^2+(y-\bar{y})^2 \geq r^2 \\ e^{\frac{-1}{\eta(x,y)}} & (x-\bar{x})^2+(y-\bar{y})^2 < r^2 \end{cases}$$

Then φ is C^∞

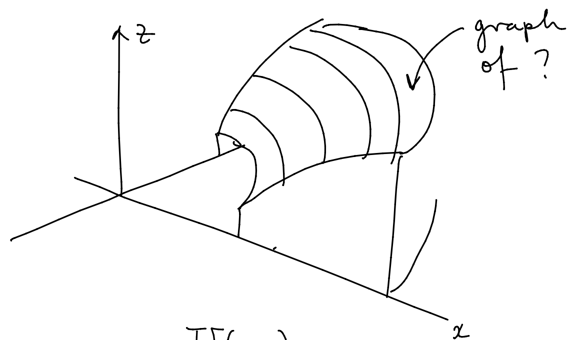
Back to E. Assume $E(x,y) > 0$ for all $(x,y) \in B_r(\bar{x},\bar{y})$.

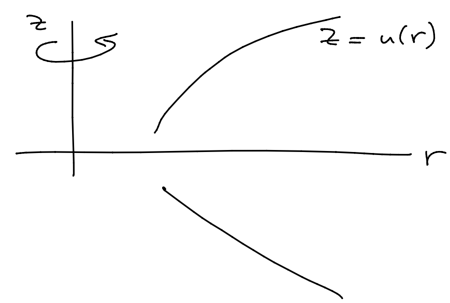
Then
$$\iint_R E(x,y)\,\varphi(x,y)\,dx\,dy > 0$$

which contradicts $\displaystyle\iint_R E(x,y)\,\varphi(x,y)\,dx\,dy = 0$ for all C^2 φ with $\varphi = 0$ on ∂R.

Therefore $E(\bar{x},\bar{y}) = 0$ after all. ////

Example Minimal Surface of Revolution





$$z = \overbrace{u\left(\sqrt{x^2+y^2}\right)}^{=U(x,y)} = u(r) \quad \text{in Polar Coordinates.}$$

Minimal Surface Equation in P.C.

$$A[u] = \iint \sqrt{1+u_r^2+\frac{1}{r^2}u_\theta^2}\; r\,dr\,d\theta$$
$$= \iint L(r,\theta,u,u_r,u_\theta)\,dr\,d\theta$$

where
$$L(r,\theta,u,u_r,u_\theta) = \sqrt{1+u_r^2+\frac{1}{r^2}u_\theta^2}\cdot r$$

where $\quad L(r,\theta,u,u_r,u_\theta) = \sqrt{1+u_r^2+\frac{1}{r^2}u_\theta^2}\cdot r$

Euler-Lagrange Equation:

$$\frac{\partial L}{\partial u} = 0 \qquad \frac{\partial L}{\partial u_r} = \frac{u_r}{\sqrt{\qquad}}\cdot r \qquad \frac{\partial L}{\partial u_\theta} = \frac{1}{r^{\cancel{2}}}\,\frac{u_\theta}{\sqrt{\qquad}}\cdot \cancel{r}$$

$$\frac{\partial}{\partial r}\left(\frac{r u_r}{\sqrt{\qquad}}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\,\frac{u_\theta}{\sqrt{\qquad}}\right) = 0$$

Suppose $u(r,\theta)$ does not depend on θ: $\quad u(r,\theta) = u(r)$
$\quad u_\theta = 0$

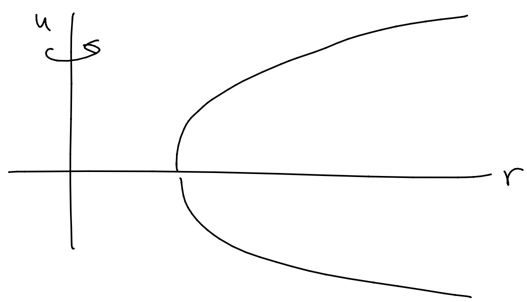
Then

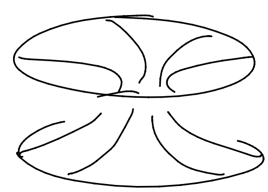
$$\frac{\partial}{\partial r}\left(\frac{r u_r}{\sqrt{1+u_r^2+\underbrace{\cancel{\frac{1}{r^2}u_\theta^2}}_{=0}}}\right) = 0 \iff \frac{r u_r}{\sqrt{1+u_r^2}} = C_1$$

(solve for u_r

$$\iff \frac{\partial u}{\partial r} = \pm\frac{C_1}{\sqrt{r^2-C_1^2}}$$

$$\iff u(r) = \pm C_1 \operatorname{arcosh}\left(\frac{r}{C_1}\right) + C_2 \iff r = C_1\cosh\left(\frac{u-C_2}{C_1}\right)$$





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