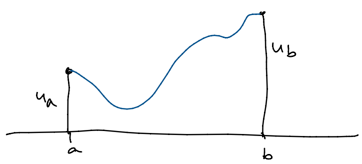
The Euler Lagrange Equation

Problem: Which function $u : [a,b] \to \mathbb{R}$ minimizes

$$J[u] = \int_a^b L(x, u(x), u'(x))\, dx$$

"L" for "Lagrange"

among all C^1 functions $u : [a,b] \to \mathbb{R}$ with $u(a) = u_a$ & $u(b) = u_b$.

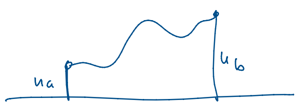


Theorem **If** $u : [a,b] \to \mathbb{R}$ is C^2 and if u minimizes $J[u]$ among all u with $u(a) = u_a$, $u(b) = u_b$, and if $L : [a,b] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ also is C^2, **then**

$$\frac{d}{dx}\left(\frac{\partial L}{\partial v}(x, u(x), u'(x))\right) - \frac{\partial L}{\partial u}(x, u(x), u'(x)) = 0 \leftarrow \boxed{\text{Euler-Lagrange Equation.}}$$

for all $x \in [a,b]$.

Example Suppose the graph of $y = u(x)$ is the shortest path between (a, u_a) and (b, u_b). Suppose u is C^2.



The length of the graph is

$$J[u] = \int_a^b \sqrt{1 + u'(x)^2}\, dx$$

$$= \int_a^b L(x, u(x), u'(x))\, dx$$

where $L(x, u, v) = \sqrt{1 + v^2}$.

Since $u : [a,b] \to \mathbb{R}$ minimizes $J[u]$ the Euler-Lagrange theorem says

$$\frac{d}{dx}\left(\frac{\partial L}{\partial v}(x, u(x), u'(x))\right) - \frac{\partial L}{\partial u}(x, u(x), u'(x)) = 0$$

We have

$$\frac{\partial L}{\partial u} = \frac{\partial \sqrt{1+v^2}}{\partial u} = 0$$

$$\frac{\partial L}{\partial v} = \frac{\partial \sqrt{1+v^2}}{\partial v} = \frac{1}{2\sqrt{1+v^2}} \frac{\partial\, 1+v^2}{\partial v} = \frac{2v}{2\sqrt{1+v^2}} = \frac{v}{\sqrt{1+v^2}}$$

Hence

$$\frac{\partial L}{\partial v}(x, u(x), u'(x)) = \frac{u'(x)}{\sqrt{1+u'(x)^2}} \quad \& \quad \frac{\partial L}{\partial u}(x, u(x), u'(x)) = 0$$

Therefore the Euler-Lagrange equation is

$$\frac{d}{dx}\left(\frac{u'(x)}{\sqrt{1+u'(x)^2}}\right) - 0 = 0$$

$$\Rightarrow \quad \frac{u'(x)}{\sqrt{1+u'(x)^2}} = C \quad \text{is constant}$$

$$u'(x) = \frac{C}{\sqrt{1-C^2}} \quad \text{is also constant.}$$

So the graph is a straight line.

<u>We proved</u> <u>If</u> there is a shortest path and

So the graph is a straight line.

We proved: If there is a shortest path and
If it is the graph of a C^2 function
Then it's a line.

Questions ① Could there be a shorter path that is the graph of a C^1 (but not C^2) function? "Regularity"

② Is there a shortest path? "Existence"

Derivation of the Euler-Lagrange equation.

Suppose $u: [a,b] \to \mathbb{R}$ is C^2 and minimizes $J[u] = \int_a^b L(x, u(x), u'(x))\, dx$ (with $u(a) = u_a$, $u(b) = u_b$).

Let $g: [a,b] \to \mathbb{R}$ be C^2 with $g(a) = g(b) = 0$ and consider

$$J[u+\varepsilon g] = \int_a^b L(x, u(x)+\varepsilon g(x), u'(x)+\varepsilon g'(x))\, dx$$

Then

$$\frac{d}{d\varepsilon} J[u+\varepsilon g] = \frac{d}{d\varepsilon} \int_a^b L(x, u(x)+\varepsilon g(x), u'(x)+\varepsilon g'(x))\, dx$$

$$= \int_a^b \frac{\partial}{\partial \varepsilon} \left\{ L(x, u(x)+\varepsilon g(x), u'(x)+\varepsilon g'(x)) \right\} dx$$

$$= \int_a^b \left\{ \frac{\partial L}{\partial u}(x, u(x)+\varepsilon g(x), u'(x)+\varepsilon g'(x)) \cdot \frac{\partial\, u(x)+\varepsilon g(x)}{\partial \varepsilon} + \frac{\partial L}{\partial v}(\dots) \cdot \frac{\partial\, u'(x)+\varepsilon g'(x)}{\partial \varepsilon} \right\} dx$$

$$= \int_a^b \left\{ \frac{\partial L}{\partial u}(\dots)\, g(x) + \frac{\partial L}{\partial v}(\dots)\, g'(x) \right\} dx$$

integrate by parts

$$= \underbrace{\left[\frac{\partial L}{\partial v}(\dots)\, \cancel{g(x)} \right]_{x=a}^{b}}_{=0 \text{ at } x=a,\, x=b} + \int_a^b \left\{ \frac{\partial L}{\partial u}(\dots)\, g(x) - \frac{d\, \frac{\partial L}{\partial v}(\dots)}{dx}\, g(x) \right\} dx$$

$$= -\int_a^b \left\{ \frac{d\, \frac{\partial L}{\partial v}(\dots)}{dx} - \frac{\partial L}{\partial u}(\dots) \right\} g(x)\, dx.$$

$(\dots) = (x, u(x)+\varepsilon g(x), u'(x)+\varepsilon g'(x))$

For $\varepsilon = 0$ we get

$$\left. \frac{dJ[u+\varepsilon g]}{dx} \right|_{\varepsilon=0} = -\int_a^b \left[\frac{d}{dx}\left(\frac{\partial L}{\partial v}(x, u(x), u'(x)) \right) - \frac{\partial L}{\partial u}(x, u(x), u'(x)) \right] g(x)\, dx$$

(This called the "First Variation of J in the direction of g"
J is a "function of a function", sometimes called "functional".)

End of proof: If u minimizes $J[u]$ then $\left. \frac{dJ[u+\varepsilon g]}{d\varepsilon} \right|_{\varepsilon=0} = 0$ for all C^2 g with $g(a) = g(b) = 0$.

Since L is C^2 and u is C^2 the function $\frac{d}{dx} \frac{\partial L}{\partial v}(\dots) - \frac{\partial L}{\partial u}(\dots)$ is continuous. Therefore

$$\frac{d}{dx} \frac{\partial L}{\partial v}(x, u(x), u'(x)) - \frac{\partial L}{\partial u}(x, u(x), u'(x)) = 0 \quad \text{for } a \le x \le b.$$

is continuous. Therefore

$$\frac{d}{dx}\frac{\partial L}{\partial v}(x,u(x),u'(x)) - \frac{\partial L}{\partial u}(x,u(x),u'(x)) = 0 \quad \text{for } a \le x \le b.$$

////

Back to the example: $L(x,u,v) = \sqrt{1+v^2}$

Euler-Lagrange says:

$$\frac{d}{dx}\left(\frac{\partial L}{\partial v}\right) - \frac{\partial L}{\partial u} = 0. \qquad ??$$

$$\frac{d}{dx}\frac{\partial L}{\partial v} = \frac{\partial}{\partial v}\frac{dL}{\partial x} = \frac{\partial}{\partial v}\frac{\partial\sqrt{1+v^2}}{dx} = \frac{\partial}{\partial v}(0) = 0. \qquad ??$$

$\Rightarrow \quad 0 = 0 \ \checkmark \quad ??$

Interpretation of $\frac{d}{dx}\left(\frac{\partial L}{\partial v}\right)$ is:

① $\frac{\partial L}{\partial v}$: differentiate $L(x,u,v)$ wrt v. function of three variables x,u,v

② $\frac{\partial L}{\partial v}(x,u(x),u'(x))$: substitute $u(x), u'(x)$ in $\frac{\partial L}{\partial v}(x,u,v)$
result: function of one variable (x)

③ $\frac{d}{dx}\left[\frac{\partial L}{\partial v}(x,u(x),u'(x))\right]$ derivative of the function of one variable from ②

$\frac{\partial}{\partial x}\frac{\partial L}{\partial v}$ = second partial derivative of the function $L(x,u,v)$ of three variables.

$= \frac{\partial}{\partial v}\frac{\partial L}{\partial x}$

Theorem. If $\frac{\partial^2 L(x,u,v)}{\partial v^2} > 0$ then the Euler-Lagrange Equation is a second order differential equation of the form

$$u''(x) = F\big(x,u(x),u'(x)\big)$$

Proof The Euler-Lagrange equation is

$$\frac{d}{dx}\left[\frac{\partial L}{\partial v}(x,u(x),u'(x))\right] = \frac{\partial L}{\partial u}(x,u(x),u'(x))$$

$$\Leftrightarrow \frac{\partial^2 L}{\partial x\partial v}(\dots) + \frac{\partial^2 L}{\partial u\partial v}(\dots)\,u'(x) + \underbrace{\frac{\partial^2 L}{\partial v^2}}_{>0}(\dots)\cdot u''(x) = \frac{\partial L}{\partial u}(\dots)$$

$$\Leftrightarrow \quad u''(x) = \frac{\frac{\partial L}{\partial u}(\dots) - \frac{\partial^2 L}{\partial x\partial v}(\dots) - \frac{\partial^2 L}{\partial u\partial v}(\dots)\,u'(x)}{\frac{\partial^2 L}{\partial v^2}(\dots)} \overset{\text{def}}{=} F(x,u(x),u'(x))$$

////

$\int L(x,u,u')\,dx$

and

$\int -L(x,u,u')\,dx$

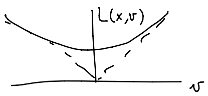
If $L(x,u,v) = a(x,u)v + b(x,u)$ then $\frac{\partial^2 L}{\partial v^2}(x,u,v) = 0$ for all x,u,v and the $\mathcal{E}$-$\mathcal{L}$ equation is

$$\frac{\partial^2 L}{\partial x\partial v}(\dots) + \frac{\partial^2 L}{\partial u\partial v}(\dots)\,u'(x) = \frac{\partial L}{\partial u}(\dots)$$

In the Fermat example $L(x,u,v) = \frac{1}{\cdot}\sqrt{1+v^2}$

$$\frac{\partial^2 L}{\partial x \partial v}(\ldots) + \frac{\partial^2 L}{\partial u \partial v}(\ldots)\, u'(x) = \frac{\partial L}{\partial u}(\ldots)$$

In the Fermat example $L(x,u,v) = \frac{1}{c(x)}\sqrt{1+v^2}$



For fixed x the graph of $L(x,u,v)$ is a hyperbola. $\frac{\partial^2 L}{\partial v^2} > 0$ holds.

Useful fact: if the Lagrangian $L(x,u,v)$ does not depend on u, i.e. if $L(x,u,v) = L(x,v)$, then you can reduce the E-L equation to the first order differential equation

$$\frac{\partial L}{\partial v}(x, u'(x)) = C_1$$

Reason: If $L = L(x,v)$ then $\frac{\partial L}{\partial u}(x,u,v) = 0$ for all (x,u,v). Therefore the E-L equation is

$$\frac{d}{dx}\left(\frac{\partial L}{\partial v}(x,u(x),u'(x))\right) = \underbrace{\cancel{\frac{\partial L}{\partial u}(x,u(x),u'(x))}}_{=0} = 0$$

$\Rightarrow$ Therefore $\frac{\partial L}{\partial v}(x, u'(x))$ is constant: $\exists C_1$ such that $\frac{\partial L}{\partial v}(x, v'(x)) = C_1$

If you can solve $\frac{\partial L}{\partial v}(x,v) = C_1$ for v and the solution is $v = F(x, C_1)$ then the E-L equation is equivalent with

$$u'(x) = F(x, C_1)$$

so

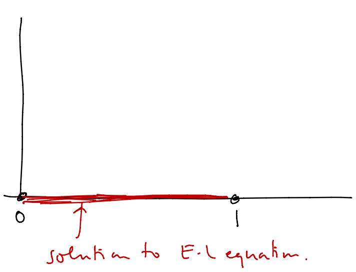
$$u(x) = \int F(x, C_1)\,dx + C_2 = u(x, C_1, C_2)$$

To find the minimizer (if there is one) solve

$$u(a, C_1, C_2) = u_a \qquad u(b, C_1, C_2) = b$$

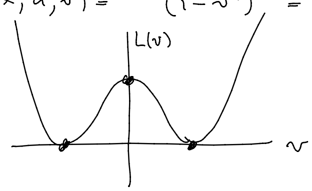
for C_1 and C_2.

Example 1 Minimize $J[u] = \int_0^1 \left(1 - u'(x)^2\right)^2 dx$ with $u(0) = u(1) = 0$.



The Lagrangian is

$$L(x,u,v) = (1-v^2)^2 = L(v)$$



Euler-Lagrange Eqn: $\frac{d}{dx}\left(\frac{\partial L}{\partial v}\right) = \frac{\partial L}{\partial u} \iff \frac{d}{dx}\left(\cancel{2}u'(x)(1-u'(x)^2)\right) = 0$

$$\iff u''(x)(1-u'(x)^2) - 2u'(x)^2 u''(x) = 0$$

$$\iff u''(x)\left(1 - 3u'(x)^2\right) = 0$$

$$\iff u''(x) = 0 \quad \text{or} \quad u'(x) = \pm\tfrac{1}{3}\sqrt{3} \;(\approx \pm 0.57\ldots)$$

u is C^2, and $u''(x) = 0$ or $u'(x) = \pm\frac{1}{3}\sqrt{3}$ for all v implies that $u''(x) = 0$ for all x

Proof by contradiction. Suppose $u''(x_0) \neq 0$.

u'' is continuous $\Rightarrow$ there is an $\varepsilon > 0$ with $u''(x) \neq 0$ for $|x - x_0| < \varepsilon$

$\Rightarrow \quad u'(x) = \pm\frac{1}{3}\sqrt{3}$ for $|x - x_0| < \varepsilon$

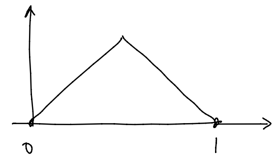
The only C^2 solution with $u(0) = 0$, $u(1) = 0$ is $u(x) = 0$

$\Leftrightarrow \quad u''(x) = 0 \quad$ or $\quad u(x) = \pm \frac{1}{3}\sqrt{3} \quad (\approx \pm 0.57\ldots)$

The only C^2 solution with $u(0) = 0$, $u(1) = 0$ is $u(x) = 0$

$$J[0] = \int_0^1 (1 - 0^2)^2 \, dx = 1$$

Now consider $\quad u(x) = \begin{cases} x & 0 \le x \le \frac{1}{2} \\ 1 - x & \frac{1}{2} \le x \le 1 \end{cases}$

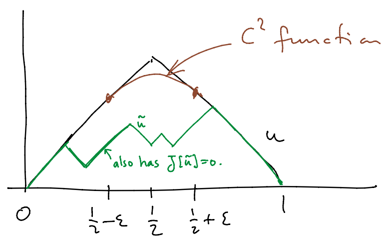


u is continuous

$u'(x)$ exists for all x except $x = \frac{1}{2}$, $\quad u'(x) = \begin{cases} +1 & (0 \le x < \frac{1}{2}) \\ -1 & (\frac{1}{2} < x \le 1) \end{cases}$

$$J[u] = \int_0^1 (1 - (\pm 1)^2)^2 \, dx = 0.$$

For any $\varepsilon > 0$ define $u_\varepsilon(x) = u(x)$ if $|x - \frac{1}{2}| \ge \varepsilon$ and



$\left\{ \begin{array}{l} u_\varepsilon \text{ should be } C^2 \\ \text{on } [0,1] \text{ and} \\ \text{have } -1 \le u_\varepsilon'(x) \le +1 \end{array} \right.$

Then $$J[u_\varepsilon] = \int_0^1 (1 - u_\varepsilon'(x)^2)^2 \, dx = \int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} \underbrace{(1 - u_\varepsilon'(x)^2)^2}_{\le 1 \text{ because } -1 \le u_\varepsilon'(x) \le +1} \, dx \le \int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} 1 \, dx = 2\varepsilon$$

While $J[0] = 1$.

u_{ε_n} where $\varepsilon_n = \frac{1}{2n}$ then $J[u_{\varepsilon_n}] = \frac{1}{n} \to 0$

There is no C^2 minimizer because for every C^2 function u there is an $\varepsilon > 0$ such that $J[u_\varepsilon] < J[u]$.

$u''(x) \ne 0$ for $|x - x_0| < \varepsilon$

$\Rightarrow \quad u'(x) = \pm \frac{1}{3}\sqrt{3}$ for $|x - x_0| < \varepsilon$

$u'(x)$ continuous $\Rightarrow$

either $\quad u'(x) = \frac{1}{3}\sqrt{3}$ for all x with $|x - x_0| < \varepsilon$

or $\quad u'(x) = -\frac{1}{3}\sqrt{3}$ for all x with $|x - x_0| < \varepsilon$

in both cases $\quad u''(x) = \frac{du'(x)}{dx} = \frac{d\,\frac{1}{3}\sqrt{3}}{dx} = 0$

$\Rightarrow$ $u''(x_0) = 0$ Contradiction.

Example 2 (by Weierstrass) Is there a minimizer for $W[u] = \int_0^1 x\, u'(x)^2 \, dx$ with $u(0) = 0$, $u(1) = 1$?