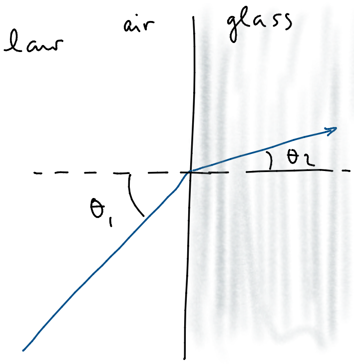
Calculus of variations

Snell's law

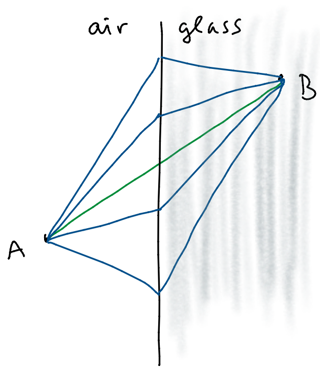


$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$$

v_1, v_2 constants

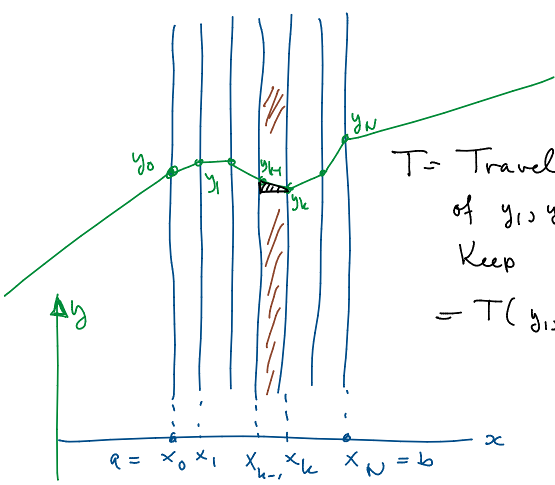
Fermat's principle

v_k = speed of light in the material #k



Light chooses the path that minimizes the travel time from A to B

Fermat's principle for laminated glass

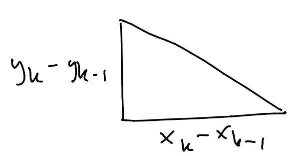


T = Travel time as a function of $y_1, y_2, \ldots, y_{N-1}$

Keep y_0, y_N fixed.

$= T(y_1, \ldots, y_{N-1})$

From x_{k-1} to x_k light speed $= v_k$



Length of path from x_{k-1} to x_k =

$$= \sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2}$$

$$\text{Total time} = T = \sum_{k=1}^{N} \frac{\sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2}}{v_k}$$

Suppose $y_1, \ldots, y_{N-1}$ minimizes $T(y_1, \ldots, y_{N-1})$.

Then ∂T

Suppose $y_1, \ldots, y_{N-1}$ minimizes $T(y_1, \ldots, y_{N-1})$.

Then $\dfrac{\partial T}{\partial y_k} = 0$

Compute $\dfrac{\partial T}{\partial y_k}$:

$$\frac{\partial T}{\partial y_k} = \frac{\partial}{\partial y_k} \sum_{l=1}^{N} \frac{\sqrt{(x_l - x_{l-1})^2 + (y_l - y_{l-1})^2}}{v_l}$$

only the terms with $l = k$ or $l-1 = k$ depend on y_k

$$= \frac{\partial}{\partial y_k} \left\{ \frac{\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}}{v_1} + \frac{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}{v_2} + \cdots + \frac{\sqrt{(x_N - x_{N-1})^2 + (y_N - y_{N-1})^2}}{v_N} \right\}$$

$$= \frac{\partial}{\partial y_k} \left\{ \frac{\sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2}}{v_k} + \frac{\sqrt{(x_{k+1} - x_k)^2 + (y_{k+1} - y_k)^2}}{v_{k+1}} \right\}$$

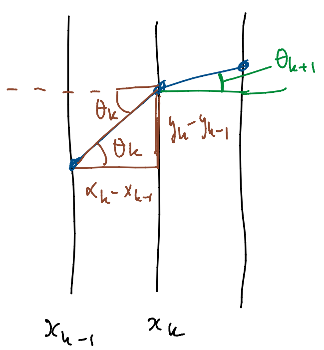
$$= \frac{1}{v_k} \frac{y_k - y_{k-1}}{\sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2}} - \frac{1}{v_{k+1}} \frac{y_{k+1} - y_k}{\sqrt{(x_{k+1} - x_k)^2 + (y_{k+1} - y_k)^2}}$$

Fermat's principle says $(y_0, y_1, \ldots, y_N)$ determines a light ray if

$$\bigstar \quad \frac{1}{v_k} \frac{y_k - y_{k-1}}{\sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2}} - \frac{1}{v_{k+1}} \frac{y_{k+1} - y_k}{\sqrt{(x_{k+1} - x_k)^2 + (y_{k+1} - y_k)^2}} = 0$$

for $k = 1, 2, \ldots, N-1$

Relation with Snell's law

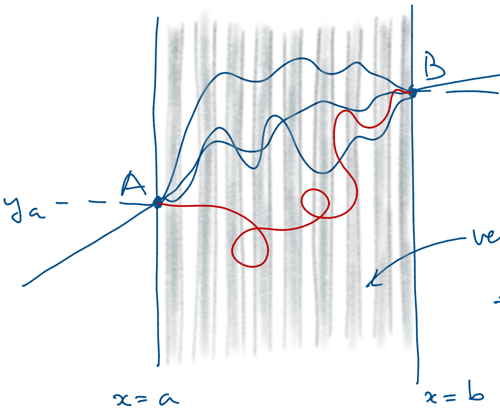


$$\frac{y_k - y_{k-1}}{\sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2}} = \sin \theta_k$$

$$\frac{y_{k+1} - y_k}{\sqrt{\phantom{(x_k - x_{k-1})^2}}} = \sin \theta_{k+1}$$

Equation $\bigstar$ says $\dfrac{\sin \theta_k}{v_k} - \dfrac{\sin \theta_{k+1}}{v_{k+1}} = 0$

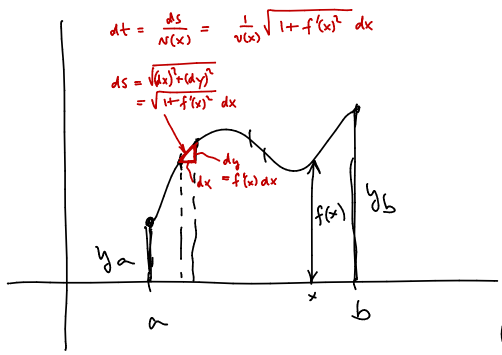
Infinitly thin lamina



Possible light path from A to B: assume it is the graph of $y = f(x)$

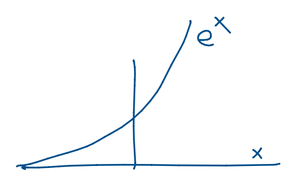
velocity of light $= v(x)$

Compute the travel time associated with the path $y = f(x)$



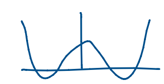
$$T[f] = \int_a^b \frac{1}{v(x)}\sqrt{1+f'(x)^2}\,dx$$

Problem: Find the minimum of $T[f]$ over all f with $f(a) = y_a$, $f(b) = y_b$ (y_a, y_b assumed to be given)



Questions: ① is there a minimizer? → $f(x) = e^x > 0$ $\min\{f(x) \mid x \in \mathbb{R}\}$ is not attained.

② is there only one minimizer?



③ all f? which f?

f must be differentiable. What else do we assume?

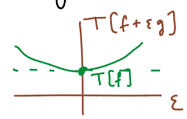
The Euler-Lagrange equation.

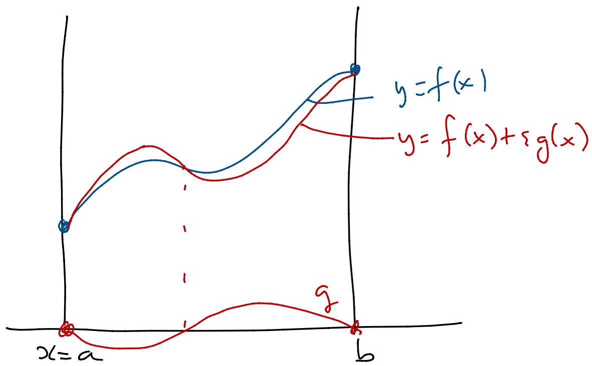
i.e. for all $h:[a,b]\to\mathbb{R}$ with $h(a)=f(a)$ and $h(b)=f(b)$ one has $T[h]\geq T[f]$

Suppose f minimizes $T[f]$ among all differentiable $f:[a,b]\to\mathbb{R}$ with $f(a)=y_a$ and $f(b)=y_b$.

Then for any $g:[a,b]\to\mathbb{R}$ with $g(a)=g(b)=0$ and any $\varepsilon\in\mathbb{R}$ we have

$$T[f+\varepsilon g]\geq T[f]$$





The function $\varepsilon\mapsto T[f+\varepsilon g]$ has a minimum at $\varepsilon=0$

Therefore

$$\left(\frac{d}{d\varepsilon}T[f+\varepsilon g]\right)_{\varepsilon=0}=0$$

i.e. $$\frac{d}{d\varepsilon}\int_{x=a}^{b}\frac{1}{v(x)}\sqrt{1+\left(f'(x)+\varepsilon g'(x)\right)^2}\,dx=0 \quad \text{at } \varepsilon=0$$

Differentiating an integral: $$\frac{d}{d\varepsilon}\int_a^b F(x,\varepsilon)\,dx=\int_a^b\frac{\partial F}{\partial\varepsilon}(x,\varepsilon)\,dx$$

Quick Proof: Let $I(\varepsilon)=\int_a^b F(x,\varepsilon)\,dx$. Then

$$F(x,\varepsilon)=F(x,\varepsilon_0)+\int_{\varepsilon_0}^{\varepsilon}\frac{\partial F}{\partial\varepsilon}(x,\varepsilon')\,d\varepsilon' \qquad \text{(fund. thm. of calc.)}$$

This implies

$$\begin{aligned}
I(\varepsilon)&=\int_a^b F(x,\varepsilon)\,dx=\int_a^b\left(F(x,\varepsilon_0)+\int_{\varepsilon_0}^{\varepsilon}\frac{\partial F}{\partial\varepsilon'}(x,\varepsilon')\,d\varepsilon'\right)dx\\
&=\int_a^b F(x,\varepsilon_0)\,dx+\int_a^b\int_{\varepsilon_0}^{\varepsilon}\frac{\partial F}{\partial\varepsilon'}(x,\varepsilon')\,d\varepsilon'\,dx\\
&\quad\Big\downarrow\ \text{switch order of integration.}\\
&=I(\varepsilon_0)+\int_{\varepsilon_0}^{\varepsilon}\underbrace{\int_a^b\frac{\partial F}{\partial\varepsilon'}(x,\varepsilon')\,dx}_{\text{function of }\varepsilon'}\,d\varepsilon'
\end{aligned}$$

fund thm of calc.

$$\Rightarrow\ I'(\varepsilon)=\left[\int_a^b\frac{\partial F}{\partial\varepsilon'}(x,\varepsilon')\,dx\right]_{\varepsilon'=\varepsilon}=\int_a^b\frac{\partial F}{\partial\varepsilon}(x,\varepsilon)\,d\varepsilon \qquad ////$$

Apply this to compute $\frac{d}{d\varepsilon}T[f+\varepsilon g]$:

Apply this to compute $\frac{d}{d\varepsilon} T[f+\varepsilon g]$:

$$\frac{d}{d\varepsilon} \int_a^b \frac{1}{v(x)} \sqrt{1+\left(f'(x)+\varepsilon g'(x)\right)^2}\, dx$$

$$= \int_a^b \frac{\partial}{\partial \varepsilon}\left(\frac{1}{v(x)} \sqrt{1+\left(f'(x)+\varepsilon g'(x)\right)^2} \right) dx$$

$$= \int_a^b \frac{1}{v(x)} \frac{\frac{\partial}{\partial\varepsilon}\left(1+\left(f'(x)+\varepsilon g'(x)\right)^2\right)}{2\sqrt{\quad \cdots \quad}}\, dx$$

$$= \int_a^b \frac{1}{v(x)} \frac{\cancel{2}\left(f'(x)+\varepsilon g'(x)\right)\cdot g'(x)}{\cancel{2}\sqrt{\qquad}}\, dx$$

Therefore

$$\left(\frac{d}{d\varepsilon} T[f+\varepsilon g]\right)_{\varepsilon=0} = \int_a^b \frac{1}{v(x)} \frac{f'(x)}{\sqrt{1+f'(x)^2}}\, g'(x)\, dx$$

Assuming v is C^1 and f is C^2 we can integrate by parts:

$$\left(\frac{d}{d\varepsilon} T[f+\varepsilon g]\right)_{\varepsilon=0} = \left[\frac{1}{v(x)} \frac{f'(x)}{\sqrt{1+f'(x)^2}}\, g(x)\right]_{x=a}^{b} - \int_a^b \frac{d}{dx}\left(\frac{1}{v(x)} \frac{f'(x)}{\sqrt{1+f'(x)^2}}\right)\cdot g(x)\, dx$$

$g(a) = g(b) = 0$ by assumption.

$$\left(\frac{d}{d\varepsilon} T[f+\varepsilon g]\right)_{\varepsilon=0} = -\int_a^b \frac{d}{dx}\left(\frac{1}{v(x)} \frac{f'(x)}{\sqrt{1+f'(x)^2}}\right) g(x)\, dx$$

FIRST VARIATION OF T

If f minimizes $T[f]$ and if f is C^2 then

$$\int_a^b \frac{d}{dx}\left(\frac{1}{v(x)} \frac{f'(x)}{\sqrt{1+f'(x)^2}}\right) g(x)\, dx = 0$$

for all C^1 $g: [a,b] \to \mathbb{R}$ with $g(a) = g(b) = 0$

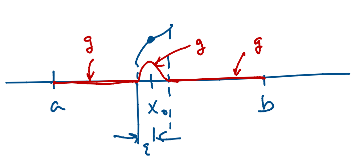
Lemma If $h: [a,b] \to \mathbb{R}$ is continuous and if

$$\int_a^b h(x)\, g(x)\, dx = 0$$

for all C^1 functions $g: [a,b] \to \mathbb{R}$ with $g(a) = g(b) = 0$

then $h(x) = 0$ for all $x \in [a,b]$.

Proof idea:



If $h(x_0) \neq 0$ then choose $\varepsilon > 0$ with $h(x) > 0$ on $(x_0-\varepsilon, x_0+\varepsilon)$

Pick a C^1 function g with $g(x) = 0$ on $[a,b] \setminus (x_0-\varepsilon, x_0+\varepsilon)$ and $g(x) > 0$ on $(x_0-\varepsilon, x_0+\varepsilon)$.

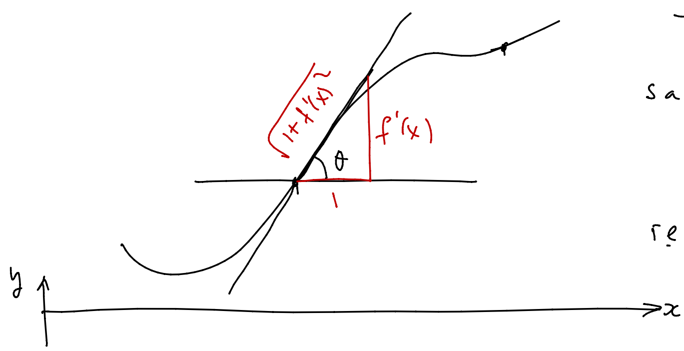
Then $\int_a^b h(x) g(x)\, dx > 0$, a contradiction ////

Conclusion If $f: [a,b] \to \mathbb{R}$ is C^2 and minimizes $T[f]$

<u>Conclusion</u> If $f:[a,b)\to\mathbb{R}$ is C^2 and minimizes $T[f]$ among all functions with $f(a)=y_a$, $f(b)=y_b$ then

$$\frac{d}{dx}\left(\frac{1}{v(x)}\frac{f'(x)}{\sqrt{1+f'(x)^2}}\right)=0 \qquad \text{for all } x\in[a,b].$$

"Euler-Lagrange equation for $T[f]$"



The Euler-Lagrange equation says

$$\frac{1}{v(x)}\frac{f'(x)}{\sqrt{1+f'(x)^2}} = C$$

i.e. $\dfrac{\sin\theta(x)}{v(x)}$ is constant.