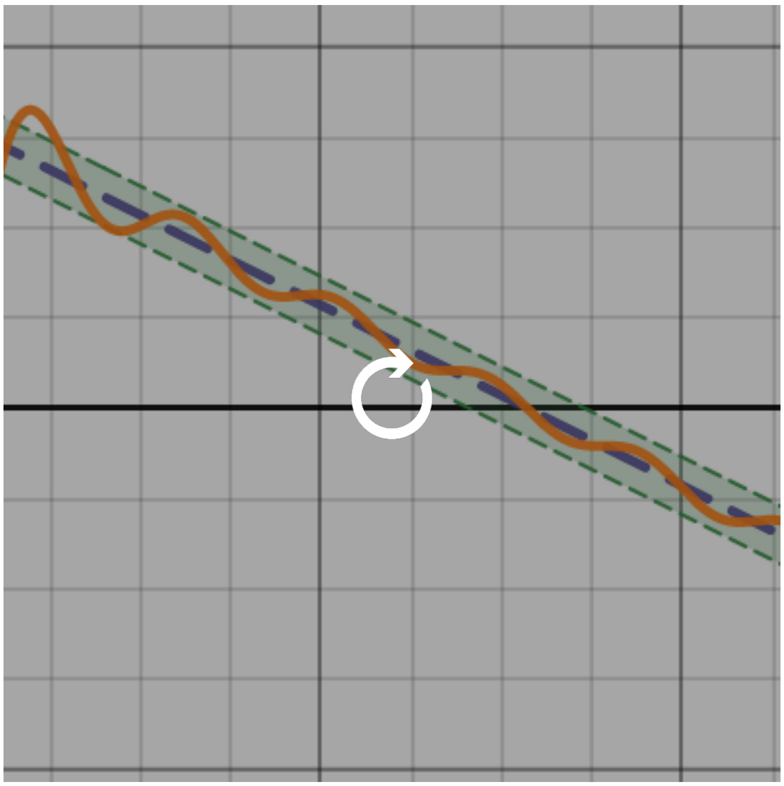
L^2 theory of Fourier series

Uniform vs. pointwise convergence

$f_n \to f$ pointwise means for every x $\lim_{n\to\infty} f_n(x) = f(x)$

$f_n \to f$ uniformly means $\lim_{n\to\infty} \overset{\sup}{\cancel{\max}}_x |f_n(x) - f(x)| = 0$

[Fourier Series, Gibbs Phenomenon]



Student question: what do our convergence theorems say about solutions to the wave equation:

Solving the wave equation

$u_{tt} = u_{xx}$ $\quad$ f, g 2π-periodic.

$u(x,0) = f(x)$

$u_t(x,0) = g(x)$

Formal solution $\quad u(x,t) = \sum_{n=-\infty}^{\infty} \left(\hat{u}_n e^{inx} \overbrace{e^{int}}^{\cos nt + i \sin nt} + \hat{v}_n e^{inx} \overbrace{e^{-int}}^{\cos nt - i \sin nt} \right)$

$$= \sum_{n=-\infty}^{\infty} (u_n + v_n) \cos(nt) e^{inx} + i(u_n - v_n) \sin(nt) e^{inx}$$

$$= \sum_{n=-\infty}^{\infty} a_n \cos(nt) e^{inx} + b_n \sin(nt) e^{inx}$$

$$= \sum_{n=-\infty}^{\infty} a_n \cos(nt)\, e^{inx} + b_n \sin(nt)\, e^{inx}$$

$$\Rightarrow \quad u_t(x,t) = \sum_{-\infty}^{\infty} -n a_n \sin(nt)\, e^{inx} + n b_n \cos(nt)\, e^{inx}$$

Match initial conditions to find a_n, b_n

$$u(x,0) = \sum_{-\infty}^{\infty} a_n e^{inx} = f(x) \Rightarrow a_n = \hat{f}_n$$

$$u_t(x,0) = \sum_{-\infty}^{\infty} n b_n e^{inx} = g(x) \Rightarrow b_n = \frac{1}{n} \hat{g}_n .$$

The solution should be

$$u(x,t) = \sum_{n=-\infty}^{\infty} \left(\hat{f}_n \cos nt \; e^{inx} + \frac{\hat{g}_n}{n} \sin(nt)\, e^{inx} \right)$$

Questions:
1. does the series converge?
2. is the sum a solution? weak? classical?
3. is $u(x,0) = f(x)$?
4. is $u_t(x,0) = g(x)$? more on friday

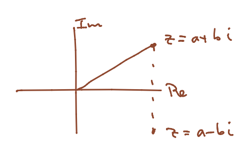
Fourier coefficients as inner products

Def of $\hat{f}_n$ $\qquad \hat{f}_n = \frac{1}{2\pi} \int_0^{2\pi}) \, e^{-inx}\, dx$

Let $\quad R_{per} = \{ f : \mathbb{R} \to \mathbb{C} \mid f$ is) mann integrable & 2π-periodic$\}$

R_{per} is a complex vector ce.

For all $f, g \in R_{per}$ define $\langle f, g \rangle = \int_0^{2\pi} f(x) \overline{g(x)}\, dx$



$a, b \in \mathbb{R}$
$z = a+bi$
$\bar{z} = a-bi$
$z\bar{z} = (a+bi)(a-bi)$
$= a^2 - (bi)^2 = a^2+b^2$
for all $z \in \mathbb{C}$
$z\bar{z} = |z|^2$

Then

symmetric : $\langle f, g \rangle = \overline{\langle g, f \rangle}$

bilinear : $\langle f+g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$

$\langle c f, g \rangle = c \langle f, g \rangle$

bilinear : $\langle f+g, h\rangle = \langle f, h\rangle + \langle g, h\rangle$

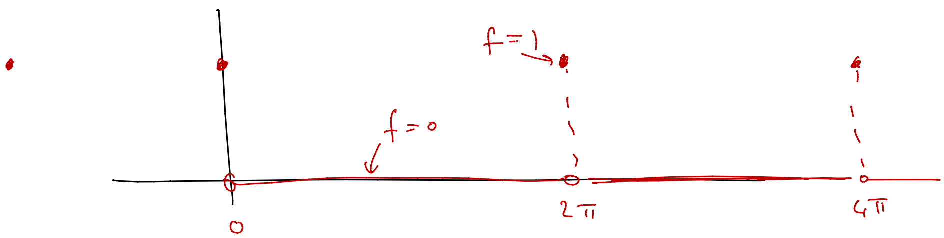
$$\langle cf, g\rangle = c\langle f, g\rangle$$

$$\langle f, g+h\rangle = \langle f, g\rangle + \langle f, h\rangle$$

$$\langle f, cg\rangle = \bar{c}\,\langle f, g\rangle$$

nonnegative $\quad \langle f, f\rangle = \int_0^{2\pi} |f(x)|^2\,dx \geq 0.$

Example $\quad f(x) = \begin{cases} 0 & x \neq 2n\pi \ (\forall n \in \mathbb{Z}) \\ 1 & x = 2n\pi \ (n \in \mathbb{Z}) \end{cases}$

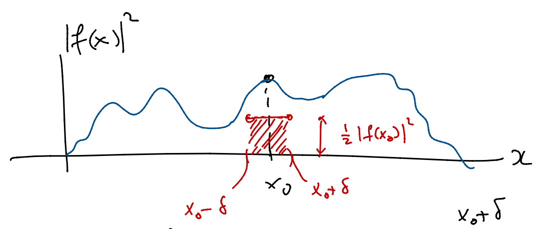


$$\langle f, f\rangle = \int_0^{2\pi} |f(x)|^2\,dx = 0 \quad \text{even though } f \neq 0.$$

Theorem If $f \in \mathcal{R}_{per}$ and f is continuous
then $\langle f, f\rangle = 0 \Rightarrow f = 0$

Proof (outline) Suppose $f(x_0) \neq 0$ and $\langle f, f\rangle = 0$.

f continuous $\Rightarrow$ $\exists\, \delta > 0$ such that $|f(x)| > \frac{1}{\sqrt{2}}|f(x_0)|$
for all $x \in (x_0 - \delta, x_0 + \delta)$



Then $\displaystyle\int_0^{2\pi} |f(x)|^2\,dx \geq \int_{x_0-\delta}^{x_0+\delta} |f(x)|^2\,dx \geq \int_{x_0-\delta}^{x_0+\delta} \tfrac{1}{2}|f(x_0)|^2\,dx = \delta\cdot|f(x_0)|^2 > 0$

i.e. $\langle f, f\rangle > 0$. ////

Define $\quad e_n(x) = e^{inx}$

$$\overline{e^{inx}} = \cos(nx) + i\sin(nx) = e^{-inx} = \cos(nx) - i\sin(nx) = \overline{e^{inx}}$$

Then $\quad \hat{f}_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,e^{-inx}\,dx = \frac{1}{2\pi}\langle f, e_n\rangle$

$$\langle e_k, e_n\rangle = \int_0^{2\pi} e^{ikx}\,\overline{e^{inx}} = \int_0^{2\pi} e^{ikx}\,e^{-inx}\,dx$$

$$\langle e_k, e_n\rangle = \int_0^{2\pi} e^{ikx}\,\overline{e^{inx}} = \int_0^{2\pi} e^{ikx} e^{-inx}\,dx$$

$$= \int_0^{2\pi} e^{i(k-n)x}\,dx \overset{\text{if } k=n}{=} \int_0^{2\pi} 1\,dx = 2\pi$$

$$\overset{\text{if } k\neq n}{=} \left[\frac{e^{i(k-n)x}}{i(k-n)}\right]_{x=0}^{2\pi} = 0$$

So

$$\langle e_k, e_n\rangle = \begin{cases} 2\pi & \text{if } k=n \\ 0 & \text{if } k\neq n \end{cases}$$

If $f(x) = \sum_{-N}^{+N} \hat{f}_k e^{ikx}$ then $f = \hat{f}_{-N} e_{-N} + \dots + \hat{f}_N e_N$

and $\langle f, e_n\rangle = \langle \sum_{k=-N}^{+N} \hat{f}_k e_k, e_n\rangle =$

$$= \sum_{k=-N}^{N} \hat{f}_k \langle e_k, e_n\rangle$$ ← only nonzero term has $k=n$. Assume $-N \leq n \leq +N$

$$= \hat{f}_n \langle e_n, e_n\rangle$$

$$= 2\pi \hat{f}_n$$

Thus

$$\hat{f}_n = \frac{1}{2\pi}\langle f, e_n\rangle$$

Next time consider $\|f\|_2 = \sqrt{\langle f, f\rangle} = \sqrt{\int_0^{2\pi} |f(x)|^2\,dx}$

We will show $\lim_{N\to\infty} \|S_N f - f\|_2 = 0$ for all $f \in R_{per}$.

Recall: $e_n(x) = e^{inx}$ $\quad \langle f, g\rangle = \int_0^{2\pi} f(x)\overline{g(x)}\,dx$ $\quad \|f\|_2 = \sqrt{\langle f, f\rangle}$

$$\langle e_k, e_\ell\rangle = \begin{cases} 2\pi & k=\ell \\ 0 & k\neq \ell \end{cases}$$

$$S_N f(x) = \sum_{k=-N}^{N} \hat{f}_k e_k(x) \qquad \hat{f}_k = \frac{1}{2\pi}\langle f, e_k\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-ikx}\,dx$$

<u>Pythagoras</u>:

(a) If $f \perp g$ then $\|f+g\|_2^2 = \|f\|_2^2 + \|g\|_2^2$

Pythagoras:

(a) If $f \perp g$ then $\|f+g\|_2^2 = \|f\|_2^2 + \|g\|_2^2$

(b) If $f_k \perp f_\ell$ for all $k \neq \ell$ then

$$\|f_1 + \cdots + f_n\|_2^2 = \|f_1\|_2^2 + \cdots + \|f_n\|_2^2$$

Proof of (a)

$$\|f+g\|_2^2 = \langle f+g, f+g\rangle = \langle f,f\rangle + \underbrace{\langle f,g\rangle}_{=0} + \underbrace{\langle g,f\rangle}_{=0} + \langle g,g\rangle$$

because $f \perp g$

$$= \|f\|_2^2 + \|g\|_2^2 .$$

Proof of (b): Use (a) and induction on n. ////

Theorem (Bessel's inequality) $\|S_Nf\|_2^2 + \|f - S_Nf\|_2^2 = \|f\|_2^2$

Proof We show that $S_Nf \perp f - S_Nf$:

$$\langle S_Nf, f - S_Nf\rangle = \left\langle \sum_{-N}^{N} \hat{f}_k e_k, f - S_Nf\right\rangle$$

$$= \sum_{-N}^{N} \hat{f}_k \langle e_k, f - S_Nf\rangle$$

$\langle f, ag\rangle = \bar{a}\langle f,g\rangle$

Next we compute

$$\langle e_k, f - S_Nf\rangle = \langle e_k, f\rangle - \langle e_k, S_Nf\rangle = \langle e_k, f\rangle - \left\langle e_k, \sum_{-N}^{N} \hat{f}_\ell e_\ell\right\rangle$$

$$= \langle e_k, f\rangle - \sum_{-N}^{N} \overline{\hat{f}_\ell}\, \langle e_k, e_\ell\rangle \quad \begin{matrix} = 0 \text{ for } k\neq \ell \\ = 2\pi \text{ for } k=\ell \end{matrix}$$

$$= \langle e_k, f\rangle - \overline{\hat{f}_k}\cdot 2\pi \qquad\qquad \hat{f}_k = \frac{1}{2\pi}\langle f, e_k\rangle$$

$$= \langle e_k, f\rangle - \frac{1}{2\pi}\langle e_k, f\rangle \cdot 2\pi \qquad\qquad \overline{\hat{f}_k} = \frac{1}{2\pi}\overline{\langle f, e_k\rangle} = \frac{1}{2\pi}\langle e_k, f\rangle$$

$$= 0 \quad \checkmark$$

Hence $\langle S_Nf, f - S_Nf\rangle = 0$ and by Pythagoras

$$\|f\|_2^2 = \|f - S_Nf + S_Nf\|_2^2 = \|f - S_Nf\|_2^2 + \|S_Nf\|_2^2$$

////

Bessel's inequality is: $\boxed{\|S_Nf\|_2^2 \leq \|f\|_2^2}$ for all N.

Theorem If $g = \hat{g}_{-N}e_{-N} + \cdots + \hat{g}_N e_N$ then $\|g - f\|_2 \geq \|S_Nf - f\|_2$

" $S_Nf = \hat{f}_{-N}e_{-N} + \cdots + \hat{f}_N e_N$ is the best approximation to f among all trigonometric polynomials of degree $\leq N$, measured with $\|g - f\|_2$ "

Proof

$$\|g - f\|_2^2 = \|g - S_Nf + S_Nf - f\|_2^2 = \|g - S_Nf\|_2^2 + \|S_Nf - f\|_2^2 \geq \|S_Nf - f\|_2^2$$

provided we show $g - S_Nf \perp f - S_Nf$.

$$g - S_Nf = \sum_{-N}^{+N} \hat{g}_k e_k - \sum_{-N}^{+N} \hat{f}_k e_k = \sum_{-N}^{+N} (\hat{g}_k - \hat{f}_k) e_k$$

Each e_k is perpendicular to $f - S_Nf \Rightarrow g - S_Nf \perp f - S_Nf$ ////

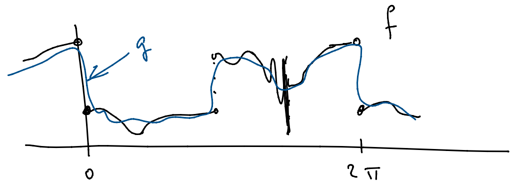
Theorem (L^2 convergence of Fourier series) If $f \in \mathcal{R}_{per}$ then

Theorem (L^2 convergence of Fourier series) If $f \in \mathcal{R}_{per}$ then

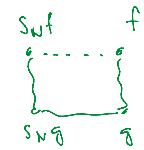
$$\lim_{N\to\infty} \|f - S_N f\|_2 = 0$$

Proof. Assume f is real valued.

Since f is Riemann integrable there exists a C^2, 2π periodic $g : \mathbb{R} \to \mathbb{R}$ with $\|g - f\|_2 < \frac{\varepsilon}{4}$.] will prove this later



$$\int_0^{2\pi} |g(x) - f(x)|^2\, dx \leq \left(\frac{\varepsilon}{4}\right)^2$$



Then

$$\|S_N f - f\|_2 = \|S_N f - S_N g + S_N g - g + g - f\|_2$$

$$\leq \|S_N f - S_N g\|_2 + \|S_N g - g\|_2 + \|g - f\|_2$$

Fact : $f \longmapsto S_N f$ is linear. Therefore $S_N f - S_N g = S_N(f-g)$

Hence

$$\|S_N f - S_N g\|_2 = \|S_N(f-g)\|_2 \overset{\text{Bessel!}}{\leq} \|f - g\|_2 < \frac{\varepsilon}{4}$$

Since g is C^2 we have shown that $S_N g \longrightarrow g$ uniformly.

This means

$$\lim_{N\to\infty} \sup_x |S_N g(x) - g(x)| = 0$$

(i.e. $\lim_{N\to\infty} \|S_N g - g\|_\infty = 0$)

for every x: $|S_N g(x) - g(x)| \leq \|S_N g - g\|_\infty$

Therefore:

$$\|S_N g - g\|_2^2 = \int_0^{2\pi} |S_N g(x) - g(x)|^2\, dx$$

$$\leq \int_0^{2\pi} \|S_N g - g\|_\infty^2\, dx = 2\pi \|S_N g - g\|_\infty^2$$

i.e.

$$\|S_N g - g\|_2 \leq \sqrt{2\pi}\, \|S_N g - g\|_\infty$$

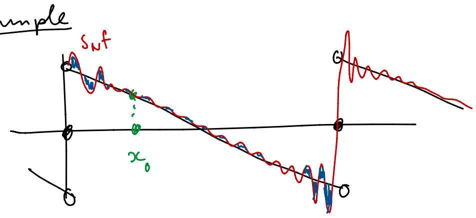
Choose N_ε so large that $\|S_N g - g\|_\infty < \frac{\varepsilon}{2\sqrt{2\pi}}$ for all $N \geq N_\varepsilon$.

Then, if $N \geq N_\varepsilon$:

$$\|S_N f - f\|_2 \leq \|S_N f - S_N g\|_2 + \|S_N g - g\|_2 + \|g - f\|_2$$

$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon \qquad ////$$

Example



$f =$ sawtooth function

$$S_N f(x) = \sin x + \frac{1}{2}\sin 2x + \dots + \frac{1}{N}\sin(Nx)$$

Pointwise convergence at all x_0 where you can verify the Lipschitz condition.

Today's theorem: $\| S_N f - f \|_2 \longrightarrow 0$

i.e. $$\int_0^{2\pi} | S_N f(x) - f(x) |^2 \, dx \longrightarrow 0$$

Theorem If $f \in \mathcal{R}_{per}$ and $\varepsilon > 0$ then there is a $g \in C^\infty$ with

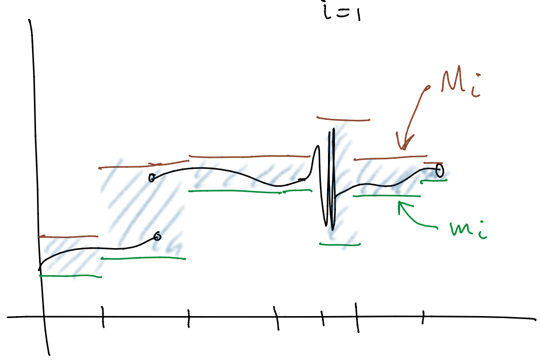
1. g 2π-periodic
2. $\| g - f \|_2 < \varepsilon$
3. $\| g \|_\infty \leq \| f \|_\infty$ and $\| g - f \|_1 < \varepsilon$.

Proof. $f \in \mathcal{R}_{per}$ means: given $\varepsilon > 0$ there exist

$$0 = x_0 < x_1 < \dots < x_N = 2\pi \quad \text{and} \quad m_1 < M_1, m_2 < M_2, \dots, m_N < M_N$$

such that

1. for $x_{i-1} < x \leq x_i$: $m_i \leq f(x) \leq M_i$
2. $$\sum_{i=1}^{N} (M_i - m_i)(x_i - x_{i-1}) < \varepsilon/2$$



$$\mathbb{1}_{(a,b]}(x) = \begin{cases} 1 & \text{if } x \in (a,b] \\ 0 & \text{if } x \notin (a,b] \end{cases}$$

$$\ell(x) = \sum_{i=1}^{N} m_i \, \mathbb{1}_{(x_{i-1}, x_i]}(x) \qquad \text{green graph}$$

$$\ell(x) = \sum_{i=1} m_i \, 1_{(x_{i-1}, x_i]}(x) \qquad \text{green graph.}$$

$$u(x) = \sum_{i=1}^{N} M_i \, 1_{(x_{i-1}, x_i]}(x) \qquad \text{brown graph.}$$

$$\ell(x) \le f(x) \le u(x) \qquad \text{for all } x$$

$$\int_0^{2\pi} \big(u(x) - \ell(x)\big)\, dx = \sum_{i=1}^{N} (M_i - m_i)(x_i - x_{i-1}) < \varepsilon/2$$

To find a C^∞ approximation to f, find $g_1, \dots, g_N$ that are C^∞ and for which

$$\int_0^{2\pi} \left| g_i(x) - 1_{(x_{i-1}, x_i]}(x) \right| dx < \delta \quad \left(\begin{matrix}\text{to be chosen} \\ \text{later}\end{matrix}\right)$$

Then define $\quad g(x) = \sum_{i=1}^{N} m_i \, g_i(x)$

We have

$$\int |g(x) - f(x)|\, dx = \int |g(x) - \ell(x) + \ell(x) - f(x)|\, dx$$

$$\le \int |g(x) - \ell(x)|\, dx + \int \big(\underbrace{f(x)}_{\le u(x)} - \ell(x)\big)\, dx$$

$$\le \int \Big| \sum_{i=1}^{N} m_i \big(g_i(x) - 1_{(x_{i-1}, x_i)}(x)\big) \Big| dx + \int \big(u(x) - \ell(x)\big)\, dx$$

$$\le \sum_{i=1}^{N} |m_i| \underbrace{\int_0^{2\pi} \left| g_i(x) - 1_{(x_{i-1}, x_i)}(x) \right| dx}_{< \delta} + \underbrace{\int_0^{2\pi} \big(u(x) - \ell(x)\big)\, dx}_{< \varepsilon/2}$$

$$\le N \max_{i=1,\dots,N} |m_i| \cdot \delta + \frac{\varepsilon}{2}.$$

We choose $\quad \delta = \dfrac{\varepsilon/2}{N \max\limits_{i=1..N} |m_i|}.$

Then we have found

$$\int_0^{2\pi} |g(x) - f(x)|\, dx < \varepsilon.$$

The choice of m_i: $\quad$ let $\quad m_i = \inf\limits_{x_{i-1} < x < x_i} f(x).$

Then $\quad |m_i| \le \sup\limits_{0 \le x \le 2\pi} |f(x)| = \|f\|_\infty.$

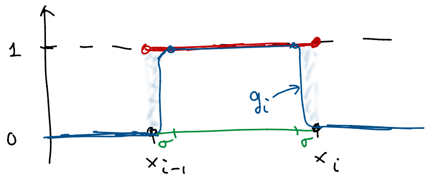
If we construct the g_i so that $\quad \begin{matrix} g_i(x) = 0 \text{ for } x \notin (x_{i-1}, x_i] \\ 0 \le g_i \le 1 \end{matrix} \quad$ then for every x: choose i so that $x \in (x_{i-1}, x_i]$

then $\quad g(x) = \sum_{j=1}^{N} m_j \, g_j(x) = m_i \, g_i(x)$

$$\Rightarrow |g(x)| = |m_i| \, g_i(x) \leq |m_i| \leq \|f\|_\infty$$

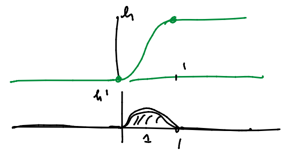
Thus $\quad \|g\|_\infty \leq \|f\|_\infty$.

To complete the construction we have to define g_i

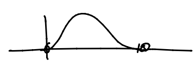


A C^2 function $h : \mathbb{R} \to \mathbb{R}$ with

$$h(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x \geq 1 \end{cases}$$



$$h'(x) = \begin{cases} c\,(x(1-x))^2 & 0 \leq x \leq 1 \\ 0 & x < 0,\ x > 1 \end{cases}$$



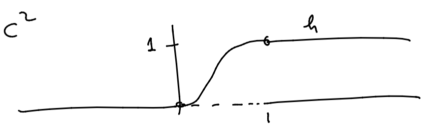
$$h(x) = \begin{cases} \int_0^x c \, t^2(1-t)^2 \, dt = c \int_0^x t^2(1-t)^2 \, dt & 0 \leq x \leq 1 \\ 0 & \text{for } x \leq 0 \\ 1 & \text{for } x \geq 1 \end{cases}$$

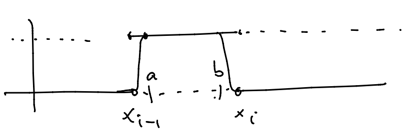
Choose c so that $\quad c \int_0^1 t^2(1-t)^2 \, dt = 1$

Then h is continuous at $x = 1$ and $x = 0$

h' is C^1 everywhere

$\Rightarrow \quad h$ is C^2





$$g_i(x) = h\left(\frac{x - x_{i-1}}{a - x_{i-1}}\right) \cdot h\left(\frac{x_i - x}{x_i - b}\right)$$