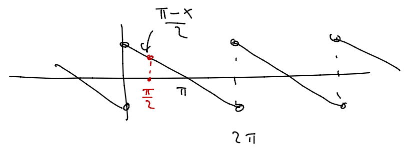
Convergence of Fourier Series

Let $f:\mathbb{R}\to\mathbb{C}$ be 2π periodic

Define
$$\hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikx} f(x)\,dx$$

" Theorem " $\quad f(x) = \sum_{k\in\mathbb{Z}} \hat{f}_k e^{ikx}$

<u>Example</u> $\quad f(x)$:



$$\frac{\pi-x}{2} = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\cdots$$

E.g. if $x=\frac{\pi}{2}$ we get $\quad \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$

(Leibniz' series for $\frac{\pi}{4}$
arctan(x) = Taylor series, set $x=1$)

$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x \Rightarrow \sin^2 x = \frac{1-\cos 2x}{2}$

<u>Example</u> $\quad f(x) = \sin^2 x$

Compute $\quad \hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikx}\sin^2 x\,dx = \cdots\cdots$

$$f(x) = \frac{1}{2} - \frac{1}{2}\cos 2x = \frac{1}{2} - \frac{1}{4}\left(e^{2ix} + e^{-2ix}\right) = -\frac{1}{4}e^{-2ix} + \frac{1}{2} - \frac{1}{4}e^{+2ix}$$

$$\hat{f}_0 = \frac{1}{2} \qquad \hat{f}_{-2} = -\frac{1}{4} \qquad \hat{f}_{+2} = -\frac{1}{4} \qquad \hat{f}_k = 0 \quad \text{if } k\notin\{0,2,-2\}.$$

Define $\quad S_N f(x) = \sum_{k=-N}^{+N} \hat{f}_k e^{+ikx}$ $\quad$ N^{th} partial sum of the Fourier series of f.

The Fourier series converges to f if $\quad \lim_{N\to\infty} S_N f(x) = f(x)$

Formula for $S_N f(x)$:

$$S_N f(x) = \sum_{k=-N}^{N} e^{+ikx}\hat{f}_k = \frac{1}{2\pi}\sum_{k=-N}^{N} e^{+ikx}\int_0^{2\pi} e^{-ik\xi} f(\xi)\,d\xi$$

$$= \frac{1}{2\pi}\sum_{-N}^{N}\int_0^{2\pi} e^{ikx}e^{-ik\xi} f(\xi)\,d\xi$$

$$= \int_0^{2\pi}\frac{1}{2\pi}\sum_{-N}^{N} e^{ikx}e^{-ik\xi} f(\xi)\,d\xi$$

$\downarrow e^{a+b} = e^a e^b$

$$= \int_0^{2\pi}\underbrace{\frac{1}{2\pi}\sum_{-N}^{N} e^{ik(x-\xi)}}_{= D_N(x-\xi)} f(\xi)\,d\xi$$

By def $\quad D_N(s) = \frac{1}{2\pi}\sum_{-N}^{+N} e^{iks}$

$$= D_N(x-\xi)$$

By def. $\quad D_N(s) = \frac{1}{2\pi}\sum_{-N}^{+N} e^{iks}$

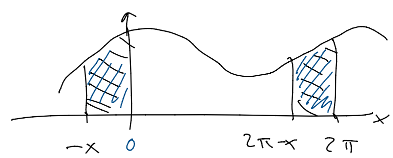
So
$$S_N f(x) = \int_0^{2\pi} D_N(x-\xi)\, f(\xi)\, d\xi$$

Note: $\quad D_N(s+2\pi) = D_N(s) \quad$ for all s $\quad$ (because $e^{ik(s+2\pi)} = e^{iks}\underbrace{e^{i2k\pi}}_{=1} = e^{iks}$)

Therefore we can rewrite the integral by substituting $s = \xi - x$, $ds = d\xi$
$$S_N f(x) = \int_{s=-x}^{2\pi - x} D_N(-s)\, f(x+s)\, ds$$
$\xi = 0 \Rightarrow s = -x$

Other property of D_N: $\quad D_N(-s) = D_N(s) \quad$ (exercise)

Moreover: $\quad D_N(s)\, f(x+s)$ is 2π periodic, so
$$\int_{-x}^{0} D_N(s)\, f(x+s)\, ds = \int_{2\pi - x}^{2\pi} D_N(s)\, f(x+s)\, ds$$



Therefore
$$S_N f(x) = \int_{-x}^{2\pi - x} D_N(s) f(x+s)\, ds = \int_{-x}^{0} + \int_0^{2\pi - x} = \int_{2\pi - x}^{2\pi} + \int_0^{2\pi - x} = \int_0^{2\pi}$$

i.e.
$$\underline{S_N f(x) = \int_0^{2\pi} D_N(s)\, f(x+s)\, ds}$$

D_N is called the Dirichlet kernel.

Formula for $D_N(s)$:
$$D_N(s) = \frac{1}{2\pi}\sum_{-N}^{+N} e^{iks} = \frac{1}{2\pi}\left(e^{-iNs} + e^{-i(N-1)s} + \dots + e^{i(N-1)s} + e^{iNs}\right)$$
$$= \frac{e^{-iNs}}{2\pi}\left\{1 + e^{is} + e^{2is} + \dots + e^{2iNs}\right\}$$
$$= \frac{e^{-iNs}}{2\pi}\left\{1 + e^{is} + (e^{is})^2 + \dots + (e^{is})^{2N}\right\}$$

$$\left(1 + x + x^2 + \dots + x^m = \frac{1 - x^{m+1}}{1-x} = \frac{x^{m+1} - 1}{x - 1}\right)$$

$$= \frac{e^{-iNs}}{2\pi}\, \frac{e^{(2N+1)is} - 1}{e^{is} - 1} = \frac{1}{2\pi}\, \frac{e^{(N+1)is} - e^{-iNs}}{e^{is} - 1} \times \frac{e^{-is/2}}{e^{-is/2}}$$

$$= \frac{1}{\ } \quad e^{(N+\frac{1}{2})is} - e^{-(N+\frac{1}{2})is}$$

$\sin x = \quad e^{ix} - e^{-ix}$

$$= \frac{1}{2\pi} \frac{e^{(N+\frac{1}{2})is} - e^{-(N+\frac{1}{2})is}}{e^{is/2} - e^{-is/2}}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\boxed{D_N(s) = \frac{1}{2\pi} \frac{\sin(N+\frac{1}{2})s}{\sin\frac{s}{2}}}$$

So

$$S_N f(x) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\sin(N+\frac{1}{2})s}{\sin\frac{1}{2}s} \cdot f(x+s)\, ds$$

Property of D_N: $\int_0^{2\pi} D_N(s)\, ds = 1$

(use $D_N(s) = \frac{1}{2\pi}\left(e^{-iNs} + \cdots + e^{-is} + 1 + e^{is} + \cdots + e^{iNs}\right)$)

This implies

$$S_N f(x) - f(x) = \int_0^{2\pi} D_N(s)\, f(x+s)\, ds - f(x)\int_0^{2\pi} D_N(s)\, ds$$

$$= \int_0^{2\pi} D_N(s)\, f(x+s)\, ds - \int_0^{2\pi} D_N(s)\, f(x)\, ds$$

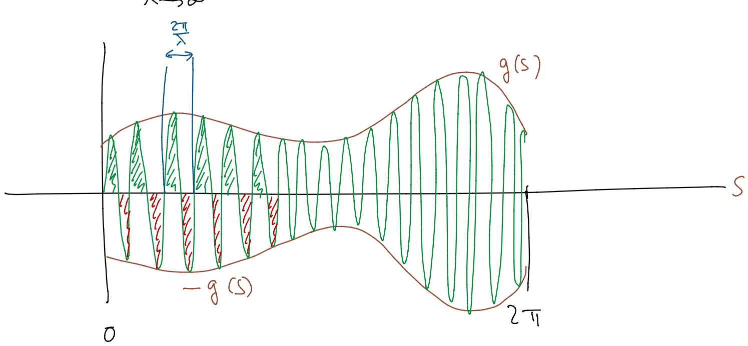
$$= \int_0^{2\pi} D_N(s)\,\{f(x+s) - f(x)\}\, ds$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{\sin(N+\frac{1}{2})s}{\sin(\frac{1}{2}s)}\,\big(f(x+s)-f(x)\big)\, ds$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \sin\underbrace{(N+\tfrac{1}{2})}_{\lambda}s\;\underbrace{\frac{f(x+s)-f(x)}{\sin\frac{s}{2}}}_{\stackrel{\text{def}}{=}\, g(s)}\, ds$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \sin(\lambda s)\, g(s)\, ds$$

Why is $\lim_{\lambda\to\infty} \int_0^{2\pi} \sin(\lambda s)\, g(s)\, ds = 0$?



<u>Riemann (Lebesgue) Lemma</u> If $f : [a,b] \to \mathbb{R}$ is Riemann integrable

then $\lim \int^b \sin(\lambda x)\, f(x)\, dx = 0$

Riemann (Lebesgue) Lemma If $f: [a,b] \to \mathbb{K}$ is Riemann integrable

then
$$\lim_{\lambda \to \infty} \int_a^b \sin(\lambda x)\, f(x)\, dx = 0$$

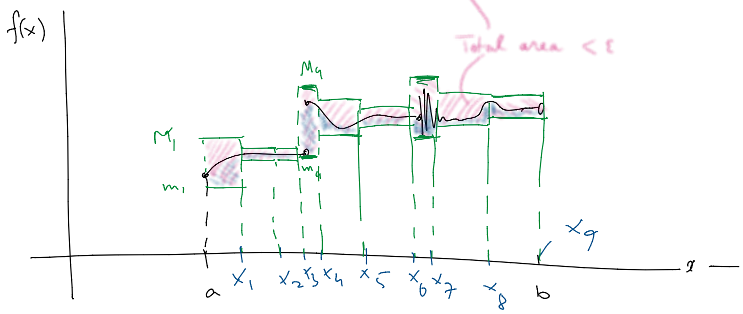
$$\lim_{\lambda \to \infty} \int_a^b \cos(\lambda x)\, f(x)\, dx = 0$$

Proof Recall f is Riemann integrable if for every $\varepsilon > 0$ there exist $a = x_0 < x_1 < \dots < x_n = b$
$m_1 < M_1, \quad m_2 < M_2, \dots, \quad m_n < M_n$

such that ① $m_k < f(x) < M_k$ for $x_{k-1} < x \le x_k$ $(k = 1, \dots, n)$

and ② $\displaystyle\sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) < \varepsilon$

Total area $< \varepsilon$



In particular: if f is bounded on $[a,b]$ and continuous at all but finitely many $x \in [a,b]$ then f is Riemann integrable.

Define $s(x) = m_k$ for $x_{k-1} < x \le x_k$ $(k = 1, 2, \dots, n)$

Then $f(x) \geq s(x)$ for all $x \in (a,b]$

$$\int_a^b (f(x) - s(x))\, dx < \frac{\varepsilon}{2}$$

Consider $\int_a^b \cos(\lambda x)\, f(x)\, dx$:

$$\int_a^b \cos(\lambda x)\, f(x)\, dx = \int_a^b \cos(\lambda x) \left[f(x) - s(x) + s(x) \right] dx$$

$$= \int_a^b \cos(\lambda x) \big(f(x) - s(x) \big)\, dx + \int_a^b \cos(\lambda x) \cdot s(x)\, dx$$

$$= A + B$$

$\left| \int f(x)\,dx \right| \le \int |f(x)|\, dx$

$$|A| = \left| \int_a^b \cos(\lambda x) \big(f(x) - s(x) \big)\, dx \right| \le \int_a^b \underbrace{|\cos \lambda x|}_{\le 1} \cdot \underbrace{\big(f(x) - s(x) \big)}_{\geq 0}\, dx$$

$$\le \int_a^b (f(x) - s(x))\, dx < \varepsilon/2$$

$$\leq \int_a^b \big(f(x) - s(x)\big)\, dx \quad < \varepsilon/2$$

$$B = \int_a^b \cos(\lambda x)\, s(x)\, dx = \qquad s(x) = m_k \text{ if } x \in (x_{k-1}, x_k)$$

$$= \sum_{k=1}^n \int_{x_{k-1}}^{x_k} \cos(\lambda x)\, m_k\, dx$$

do the integral

$$= \sum_{k=1}^n \left[m_k \frac{\sin \lambda x}{\lambda} \right]_{x_{k-1}}^{x_k}$$

$$= \frac{1}{\lambda} \sum_{k=1}^n m_k \big(\sin(\lambda x_{k-1}) - \sin(\lambda x_k)\big)$$

$$|B| \leq \frac{1}{\lambda} \sum_{k=1}^n m_k \left| \sin \lambda x_{k-1} - \sin \lambda x_k \right| \leq \frac{1}{\lambda} \sum_{k=1}^n m_k \cdot 2$$

Conclusion:

$$\left| \int_a^b \cos(\lambda x)\, f(x)\, dx \right| \leq |A| + |B| < \frac{\varepsilon}{2} + \frac{2(m_1 + \dots + m_n)}{\lambda}$$

Choose $\lambda_\varepsilon > 0$ so that for all $\lambda \geq \lambda_\varepsilon$ one has

$$\frac{2(m_1 + \dots + m_n)}{\lambda} < \frac{\varepsilon}{2}.$$

Then for all $\lambda \geq \lambda_\varepsilon$: $\left| \int_a^b \cos(\lambda x)\, f(x)\, dx \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$.

Hence $\lim_{\lambda \to \infty} \int_a^b \cos(\lambda x)\, f(x)\, dx = 0.$ ////

Theorem. If $f : \mathbb{R} \to \mathbb{C}$ is 2π-periodic, Riemann integrable and if for some $x \in \mathbb{R}$ there is a $C > 0$ such that

$$|f(y) - f(x)| \leq C\, |x - y| \qquad \forall y \in \mathbb{R} \qquad \text{(Lipschitz)}$$

then

$$\lim_{N \to \infty} S_N f(x) = f(x)$$

i.e. $$f(x) = \lim_{N \to \infty} \sum_{k=-N}^{+N} \hat{f}_k e^{ikx}$$

Proof

$$S_N f(x) = \int_{-\pi}^{\pi} D_N(s)\, f(x+s)\, ds$$

$$S_N f(x) - f(x) = \int_{-\pi}^{\pi} D_N(s) \big(f(x+s) - f(x)\big)\, ds$$

$$= \frac{1}{2\pi} \int_{-\pi}^{+\pi} \sin\left(N + \tfrac{1}{2}\right) s \cdot \frac{f(x+s) - f(x)}{\sin \frac{s}{2}}\, ds$$

$-\pi$ $\qquad\qquad$ $\sin\frac{\;}{2}$

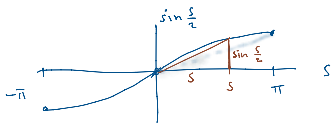
Define $\quad g(s) = \begin{cases} \dfrac{f(x+s)-f(x)}{\sin\frac{s}{2}} & (s \neq 0) \\ 0 & (s = 0) \end{cases}$

<u>Claim</u> ① g is Riemann integrable — 521 exercise

② There is a $C' > 0$ such that $|g(s)| \le C'$ for all $s \in (-\pi, +\pi)$

<u>proof of ②:</u> For all $s \in (-\pi, +\pi)$, $s \neq 0$:

$$|g(s)| = \frac{|f(x+s)-f(x)|}{|\sin\frac{s}{2}|} \underset{\substack{\uparrow \\ \text{(Lipschitz)}}}{\le} \frac{C\,|x+s-x|}{|\sin\frac{s}{2}|} = C\,\frac{|s|}{|\sin\frac{s}{2}|} \underset{\substack{\uparrow \\ \frac{\sin s}{s/2} \ge \frac{2}{\pi} \text{ when } |s|<\pi}}{\le} C\cdot\frac{\pi}{2} = C'$$



$\frac{\sin s/2}{s}$ = slope $\ge \frac{1}{\pi/2} = \frac{2}{\pi}$

Given the claim, the Riemann–Lebesgue lemma implies

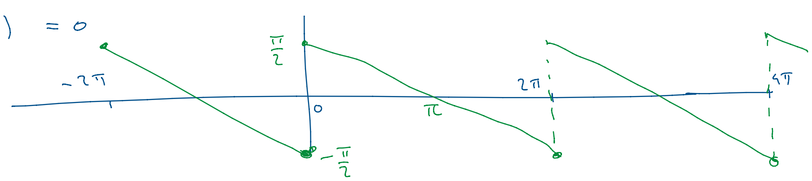
$$\lim_{N\to\infty} \int_{-\pi}^{\pi} \sin\left(N+\tfrac{1}{2}\right)s \;\; g(s)\, ds = 0$$

Hence $\quad \lim_{N\to\infty} S_N f(x) = f(x)$. $\qquad$ ////

<u>Example</u> Consider the saw-tooth function:

$$S(x) = \frac{\pi - x}{2} \qquad 0 < x < 2\pi \qquad S(x+2\pi) = S(x) \quad \forall x$$

$$S(0) = 0$$



Compute Fourier coeffs $\qquad \hat{S}_k = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} e^{-ikx}\,\frac{\pi - x}{2}\,dx = \dots$

The Fourier series for $S(x)$ is

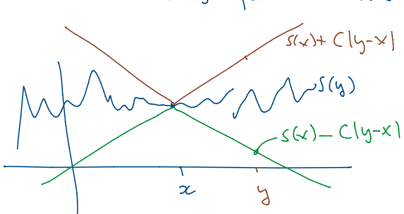
$$(S_N S)(x) = \sum_{-N}^{+N} \hat{S}_k\, e^{ikx} = \sum_{-N}^{+N} \hat{S}_k\,(\cos kx + i \sin kx)$$

$$= \dots = \sin x + \tfrac{1}{2}\sin 2x + \dots + \tfrac{1}{N}\sin(Nx).$$

The function $S(x)$ is Riemann integrable

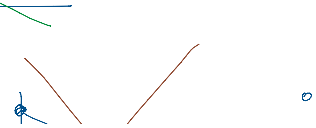
and $\quad |S(y) - S(x)| \le C\,|y-x| \quad$ for all $y \in \mathbb{R}$

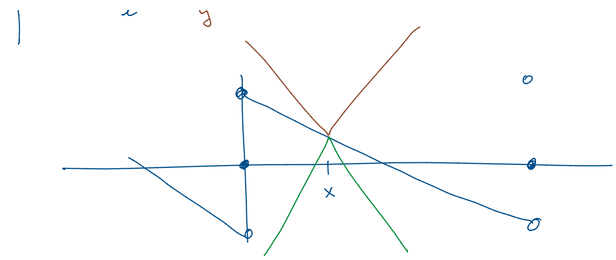
holds for all $x \in (0, 2\pi)$ but not at $x = 0, \pm 2\pi, \pm 4\pi, \dots$



$$|s(y) - s(x)| \le C\,|y-x|$$

$$\Rightarrow \quad s(x) - C|y-x| \le s(y) \le s(x) + C|y-x|$$





The theorem implies $\frac{\pi - x}{2} = \lim_{N\to\infty} \underbrace{\sin x + \frac{1}{2}\sin 2x + \cdots + \frac{1}{N}\sin(Nx)}_{= S_N s(x)}$

The series does not converge uniformly to $s(x)$ because $S_N s$ is continuous and $s(x)$ is not.

Theorem If $f: \mathbb{R} \to \mathbb{C}$ is C^2 and 2π periodic then:

① $|\hat{f}_k| \leq \frac{\|f''\|_\infty}{k^2}$ for all $k \neq 0$

② the series $\sum_{k\in\mathbb{Z}} \hat{f}_k e^{ikx}$ converges to $f(x)$

The convergence is absolute and uniform

Proof ① Let $k \neq 0$. Then

$$2\pi \hat{f}_k = \int_0^{2\pi} e^{-ikx} f(x)\,dx = \underbrace{\left[\frac{e^{-ikx}}{-ik} f(x)\right]_0^{2\pi}}_{=0 \text{ because } e^{ikx} \text{ and } f(x) \text{ are } 2\pi \text{ periodic}} - \int_0^{2\pi} \frac{e^{-ikx}}{-ik} f'(x)\,dx$$

$$= \frac{1}{ik}\int_0^{2\pi} e^{-ikx} f'(x)\,dx$$ integrate by parts again

$$= \left(\frac{1}{ik}\right)^2 \int_0^{2\pi} e^{-ikx} f''(x)\,dx$$

$\left|\int g(x)\,dx\right| \leq \int |g(x)|\,dx$

$$\Rightarrow 2\pi|\hat{f}_k| = \left|\frac{1}{ik}\right|^2 \cdot \left|\int_0^{2\pi} e^{-ikx} f''(x)\,dx\right| \leq \frac{1}{k^2}\cdot\int_0^{2\pi} \underbrace{|e^{ikx}|}_{=1}\cdot\underbrace{|f''(x)|}_{\leq \|f''\|_\infty}\,dx$$

because $\|f''\|_\infty = \sup_{0\leq x\leq 2\pi} |f''(x)|$

$$\Rightarrow 2\pi|\hat{f}_k| \leq \frac{1}{k^2}\|f''\|_\infty \int_0^{2\pi} 1\,dx = \frac{\|f''\|_\infty}{k^2} 2\pi$$

$$\Rightarrow |\hat{f}_k| \leq \frac{\|f''\|_\infty}{k^2}.$$

② The Fourier series of f is $\sum_{-\infty}^{\infty} \hat{f}_k e^{ikx}$

The k^{th} term is bounded by

$$|\hat{f}_k e^{ikx}| \leq \frac{\|f''\|_\infty}{k^2}\cdot|e^{ikx}| = \frac{\|f''\|_\infty}{k^2} \stackrel{\text{def}}{=} M_k$$

The function f is C^2, hence C^1. Therefore for all $x, y \in \mathbb{R}$

$\exists \xi \in \mathbb{R}$: $\quad |f(x) - f(y)| = |f'(\xi)(x-y)| = |f'(\xi)| \cdot |x-y|$ (mean value theorem)

$$\leq \sup_{\xi} |f'(\xi)| \cdot |x-y|$$

$$|f(x) - f(y)| \leq \|f'\|_\infty \cdot |x-y|$$

So f satisfies (Lipschitz) with $C = \|f'\|_\infty$.

Therefore $\quad \lim_{N\to\infty} S_N f(x) = f(x)$

The convergence is uniform: let $\varepsilon > 0$ be given. Then

$$|S_N f(x) - f(x)| = \left| \sum_{-N}^{+N} \hat{f}_k e^{ikx} - \sum_{-\infty}^{\infty} \hat{f}_k e^{ikx} \right|$$

$$= \left| \sum_{-\infty}^{-N-1} \hat{f}_k e^{+ikx} + \sum_{N+1}^{\infty} \hat{f}_k e^{ikx} \right|$$

$$\leq \sum_{-\infty}^{-N-1} |\hat{f}_k| + \sum_{N+1}^{\infty} |\hat{f}_k|$$

$$\leq \sum_{-\infty}^{-N-1} \frac{\|f''\|_\infty}{k^2} + \sum_{N+1}^{\infty} \frac{\|f''\|_\infty}{k^2}$$

$$= 2\|f''\|_\infty \cdot \sum_{N+1}^{\infty} \frac{1}{k^2}$$

The series $\sum_{1}^{\infty} \frac{1}{k^2}$ converges so $\lim_{N\to\infty} \sum_{N+1}^{\infty} \frac{1}{k^2} < \infty$.

Choose N_ε so that $\sum_{N_\varepsilon+1}^{\infty} \frac{1}{k^2} < \frac{\varepsilon}{2\|f''\|_\infty}$. Then

$|S_N f(x) - f(x)| < \varepsilon \quad$ for all $N \geq N_\varepsilon$ and $x \in \mathbb{R}$

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