Fourier's solution

$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ is a linear equation i.e.

if $u, v : \mathbb{R}^2 \to \mathbb{R}$ are solutions then so is

$$w(x,t) = a\,u(x,t) + b\,v(x,t)$$

for all $a, b \in \mathbb{R}$

(proof: $(au+bv)_{tt} = (au_t + bv_t)_t = au_{tt} + bv_{tt}$

$= au_{xx} + bv_{xx} = (au_x + bv_x)_x = (au+bv)_{xx}$ ✓)

Similarly: $u_1, \dots, u_N$ solutions $\Rightarrow$ $u = c_1 u_1 + \dots + c_N u_N = \sum_{k=1}^{N} c_k u_k$ also a solution for all $c_1, \dots, c_N \in \mathbb{R}$.

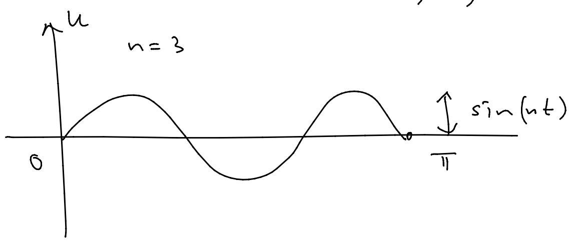
Observation For any $n \in \mathbb{R}$

$u(x,t) = \sin(nx)\,\sin(nt)$

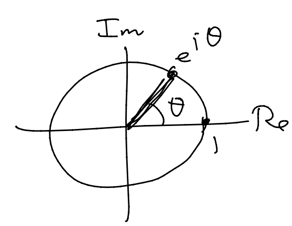
is a solution of the WEqn.

(proof: $u_{xx} = -n^2 \sin(nx)\sin(nt)$
$u_{tt} = -n^2 \sin(nx)\sin(nt)$)

If $n \in \mathbb{N}$ then $u(0,t) = \sin(n \cdot 0)\sin(nt) = 0$
$u(\pi,t) = \sin(n\pi)\sin(nt) = 0$ for all $t \in \mathbb{R}$



Complex exponential $e^{i\theta} = \cos\theta + i\sin\theta$ Euler's def



$e^{i\pi} = -1$ $\qquad$ $e^{in\pi} = (-1)^n \quad (n \in \mathbb{Z})$

$e^{i(\theta+\varphi)} = e^{i\theta} e^{i\varphi}$

$\frac{d}{d\theta}(e^{i\theta}) = i e^{i\theta}$ (verify from Euler's definition)

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad\qquad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$
$$e^{-i\theta} = \cos\theta - i\sin\theta \quad \Longrightarrow \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

<u>Fourier's observation</u> $u(x,t) = e^{inx}e^{int}$ and $v(x,t) = e^{inx}e^{-int}$ are solutions of the W Eqn.

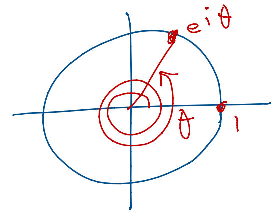
(e.g. $u_{tt} = \left(e^{inx}e^{int}\right)_{tt} = \left(in\, e^{inx}e^{int}\right)_t = (in)^2 e^{inx}e^{int} = -n^2 e^{inx}e^{int}$

$u_{xx} = \left(e^{inx}e^{int}\right)_{xx} = \cdots = -n^2 e^{inx}e^{int}$. ✓

Look for solutions that are periodic with period 2π in x

i.e.

$$u(x+2\pi,t) = u(x,t) \quad \text{for all } x,t.$$



Note: if $u(x,t) = e^{inx}e^{int}$ then

$$u(x+2\pi,t) = e^{inx+2in\pi}e^{int} = e^{inx}\underbrace{e^{2\pi n i}}_{=1 \text{ if } n\in\mathbb{N}}e^{int} = u(x,t)$$

The same is true for $e^{inx}e^{-int} = v(x,t)$: $v(x+2\pi,t) = v(x,t)$ for all x,t provided $n\in\mathbb{N}$

The W. Eqn is linear. Therefore

$$u(x,t) = \sum_{n=-N}^{N}\left(u_n e^{inx}e^{int} + v_n e^{inx}e^{-int}\right)$$

is a solution to the W Eqn for any $u_{-N}, u_{-(N-1)}, \dots, u_{N-1}, u_N \in \mathbb{C}$
$v_{-N}, v_{-(N-1)}, \dots, v_{N-1}, v_N \in \mathbb{C}$

Rewritten using Euler formulas:

$$u_n e^{inx}e^{int} + v_n e^{inx}e^{-int} = e^{inx}\left(u_n e^{int} + v_n e^{-int}\right)$$
$$= e^{inx}\left(u_n(\cos nt + i\sin nt) + v_n(\cos nt - i\sin nt)\right)$$
$$= e^{inx}\left(\underbrace{(u_n+v_n)}_{f}\cos nt + \underbrace{i(u_n-v_n)}_{g_n}\sin nt\right)$$

$u_n + v_n = f_n$
$i u_n - i v_n = g_n$
$\Updownarrow$

$$= e^{\cdots}\Big(\underbrace{(u_n + v_n)}_{f_n} \cos nt + \underbrace{i(u_n - v_n)}_{g_n} \sin nt \Big)$$

$u_n + v_n = f_n$

$i\, u_n - i v_n = g_n$

$\Updownarrow$

$u_n = \frac{f_n - i g_n}{2}$

$v_n = \frac{f_n + i g_n}{2}$

$$= f_n \cos(nt)\, e^{inx} + g_n \sin(nt)\, e^{inx}$$

So our solution $u(x,t)$ is

$$u(x,t) = \sum_{n=-N}^{+N} \left(f_n \cos(nt)\, e^{inx} + g_n \sin(nt)\, e^{inx} \right)$$

To solve the initial value problem, consider

$$u(x,0) = f(x) = \sum_{-N}^{+N} f_n\, e^{inx}$$

$$u_t(x,0) = g(x) = \sum_{-N}^{+N} n g_n\, e^{inx}$$

$\frac{d}{dt} \cos(nt) = -n \sin nt$

$\frac{d}{dt} \sin(nt) = n \cos nt$

Given $f(x)$, we can find f_n by multiplying with e^{-inx} and integrating from 0 to 2π:

$$\int_0^{2\pi} e^{-inx} f(x)\, dx = \int_0^{2\pi} e^{-inx} \sum_{k=-N}^{+N} f_k e^{ikx}\, dx$$

$$= \int_0^{2\pi} \sum_{-N}^{N} e^{-inx} e^{ikx} f_k\, dx$$

$$= \sum_{-N}^{N} f_k \int_0^{2\pi} e^{-inx} e^{ikx}\, dx$$

$$\int_0^{2\pi} e^{-inx} e^{ikx}\, dx = \int_0^{2\pi} e^{i(k-n)x}\, dx \overset{\text{assume } k \neq n}{=} \left[\frac{e^{i(k-n)x}}{i(k-n)} \right]_{x=0}^{2\pi}$$

$$= \frac{\overbrace{e^{i(k-n)2\pi}}^{=1} - \overbrace{e^{i(k-n)\cdot 0}}^{=1}}{i(k-n)} = \frac{1-1}{i(k-n)} = 0$$

If $k = n$ then $\int_0^{2\pi} e^{-inx} e^{ikx}\, dx = \int_0^{2\pi} e^{\overbrace{i(k-n)}^{=0} x}\, dx = \int_0^{2\pi} 1\, dx = 2\pi$

Therefore $\int_0^{2\pi} \cdots \sum^{+N} \underbrace{\int_0^{2\pi} e^{-inx} e^{ikx} \cdots}_{\substack{= 2\pi \text{ for } k=n \\ = 0 \text{ for } k \neq n}}$

Therefore $\displaystyle\int_0^{2\pi} f(x)\, e^{-inx}\, dx = \sum_{-N}^{+N} f_k \overbrace{\int_0^{2\pi} e^{-inx} e^{ikx}\, dx}^{\substack{=2\pi \text{ for } k=n \\ =0 \text{ for } k\neq n}}$

$$= 2\pi\, f_n$$

<u>Conclusion:</u> If $\displaystyle f(x) = \sum_{-N}^{+N} f_n e^{inx}$ then

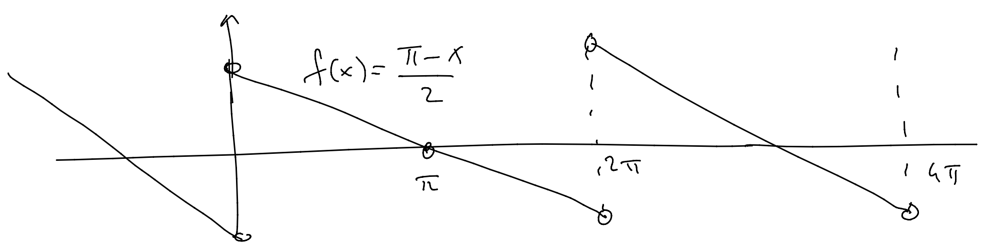
$$f_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-inx} f(x)\, dx \qquad (*)$$

<u>Fourier's claim:</u> "this works for all functions f if $N = \infty$."

Given any 2π-periodic function f define f_n by $(*)$.
Then

$$f(x) = \sum_{-\infty}^{+\infty} f_n e^{inx}$$

<u>Example</u>



Then

$$f_n(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{inx}\, \frac{\pi - x}{2}\, dx = \cdots = \frac{1}{n} \qquad (n \neq 0)$$

Therefore

$$f(x) = \sum_{n\neq 0} f_n e^{inx} = \sum_{n=1}^{\infty} f_n e^{inx} + \sum_{1}^{\infty} f_{-n} e^{-inx}$$

$$= \sin(x) + \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) + \cdots$$