Convergence of sequences of functions

"$f_n \to f$ if the distance between f_n and f goes to zero"

The norm of a function $f: X \longrightarrow \mathbb{R}$ $\quad \|f\|_{\dots}$

Distance $d_{\dots}$ between functions $f, g : X \longrightarrow \mathbb{R}$

$$d(f,g) = \|f-g\|$$

$$\|f\|_\infty = \sup_{x \in X} |f(x)|$$

If $1 \le p < \infty$

$$\|f\|_p = \left(\int_X |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

"root mean square norm"

$p=2$ $\quad \|f\|_2 = \left(\int_X |f(x)|^2 \, dx \right)^{1/2}$

$p=1$ $\quad \|f\|_1 = \int_X |f(x)| \, dx$

<u>Uniform convergence</u> A sequence $f_n : X \to \mathbb{R}$ converges uniformly to $f : X \to \mathbb{R}$

if

1. $\lim_{n \to \infty} \|f_n - f\|_\infty = 0$

2. there exist $\delta_n > 0$ with $\lim_{n \to \infty} \delta_n = 0$

   and $|f_n(x) - f(x)| \le \delta_n$ for all $x \in X$, $n \in \mathbb{N}$

3. For every $\varepsilon > 0$ there is an $N_\varepsilon \in \mathbb{N}$ such that

   for all $n \ge N_\varepsilon$ and $x \in X$: $|f_n(x) - f(x)| \le \varepsilon$

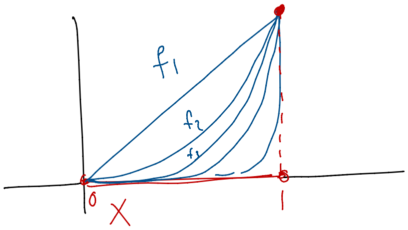
<u>Theorem</u> If $f_n : X \to \mathbb{R}$ are continuous and if $f_n \to f$ uniformly

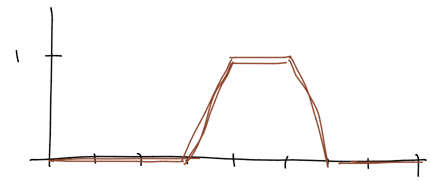
then f is continuous.

<u>Examples</u>

1. $X = [0,1]$ $\quad f_n(x) = x^n$

2. $X = [0,1]$

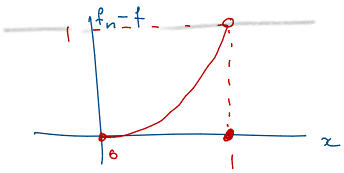




Pointwise limit: $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$

$$\|f_n - f\|_\infty = \sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} \left| x^n - \begin{cases} 0 & \text{if } x<1 \\ 1 & \text{if } x = 1 \end{cases} \right|$$

$$x^n - f(x) = \begin{cases} 1 - 1 = 0 & \text{if } x = 1 \\ x^n - 0 = x^n & \text{if } x \in [0,1) \end{cases}$$



$\sup_{0\le x\le 1} f_n(x) - f(x) = 1$

so $\|f_n - f\|_\infty = 1$ for all n

Therefore $\lim_{n\to\infty} \|f_n - f\|_\infty = 1 \neq 0$, so f_n does not converge uniformly to f

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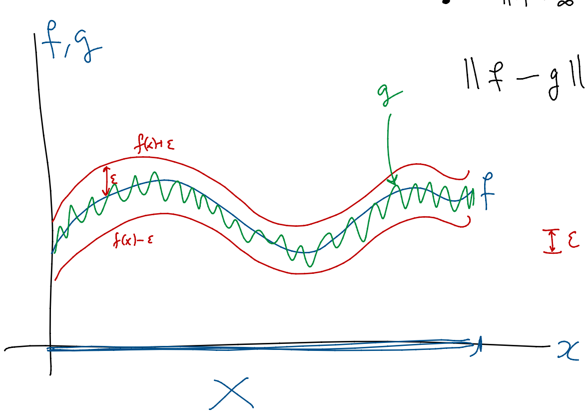
The supremum norm $\|f\|_\infty = \sup_{x\in X} |f(x)|$

- $\|f\|_\infty \le \varepsilon \iff |f(x)| \le \varepsilon$ for all $x \in X$

$\|f - g\|_\infty \le \varepsilon \iff |f(x) - g(x)| \le \varepsilon$ for all $x \in X$

$\Updownarrow$

$f(x) - \varepsilon \le g(x) \le f(x) + \varepsilon$



Theorems

f_n continuous, $f_n \to f$ uniformly $\Rightarrow$ f continuous

$f_n \to f$ uniformly $\Rightarrow$ $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$

$f_n \to f$ uniformly $\not\Rightarrow$ $f_n'(x) \to f'(x)$

Example $f_n(x) = 2^{-n} \sin(2^n x)$ $\quad x \in [0, \pi]$

① $|f_n(x)| \leq 2^{-n} |\sin(2^n x)| \leq 2^{-n}$

$\|f_n\|_\infty = 2^{-n}$ because $f_n(x) = 2^{-n}$ if $x = \frac{\pi}{2^{n+1}} \in [0,\pi]$

Therefore " $\lim_{n\to\infty} f_n = 0$ in the sup norm on $[0,\pi]$ "

" $f_n \to 0$ uniformly on $[0,\pi]$ "

② $f_n'(x) = \frac{d}{dx} 2^{-n} \sin(2^n x) = \cancel{2^{-n}} \cdot \cancel{2^n} \cos(2^n x) = \cos(2^n x)$

$\|f_n'\|_\infty = \sup_{0\leq x\leq\pi} |\cos(2^n x)| \begin{cases} \leq 1 & \text{because } |\cos(..)| \leq 1 \\ \geq 1 & \text{because } \cos(2^n \cdot 0) = 1 \end{cases}$

$= 1$

Therefore f_n' does not converge to 0

Theorem If $f_n : [a,b] \to \mathbb{R}$ is C^1 and if:

① $f_n \to f$ uniformly on $[a,b]$

② $f_n' \to g$ uniformly on $[a,b]$

then f is C^1 and $g(x) = f'(x)$ for all $x \in [a,b]$.

I.e. if $f_n \to f$ uniformly and if $f_n' \to g$ uniformly

then

$$\lim_{n\to\infty} f_n'(x) = f'(x)$$

$$\lim_{n\to\infty} \frac{df_n}{dx} = \frac{d \lim_{n\to\infty} f_n(x)}{dx}$$

A similar theorem holds for partial derivatives of $f_n : \mathbb{R}^k \to \mathbb{R}$

Back to the wave equation.

Given a solutions $u_n : \mathbb{R}^2 \to \mathbb{R}$ of

$$\frac{\partial^2 u_n}{\partial t^2} = \frac{\partial^2 u_n}{\partial x^2}$$

is the limit $u(x,t) = \lim_{n\to\infty} u_n(x,t)$ also a solution?

what kind of limit?

Answer 1 Yes, if $u_n, \frac{\partial u_n}{\partial t}, \frac{\partial^2 u_n}{\partial t^2}, \frac{\partial u_n}{\partial x}, \frac{\partial^2 u_n}{\partial x^2}$ converge uniformly.

Because then:

u_n and $\frac{\partial u_n}{\partial t}$ converge uniformly

$\frac{\partial u_n}{\partial t}$ and $\frac{\partial^2 u_n}{\partial t^2}$ converge uniformly

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\lim u_n\right) \overset{\downarrow}{=} \frac{\partial}{\partial t}\left(\lim \frac{\partial u_n}{\partial t}\right) \overset{\downarrow}{=} \lim \frac{\partial}{\partial t}\frac{\partial u_n}{\partial t} \quad \left(\frac{\partial^2 u_n}{\partial t^2} = \frac{\partial^2 u_n}{\partial x^2}\right)$$

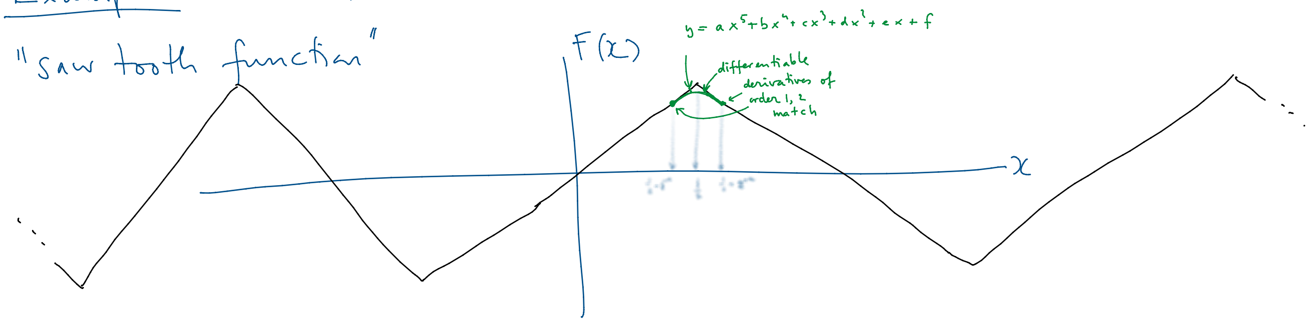
$$= \lim \frac{\partial}{\partial x}\frac{\partial u_n}{\partial x} = \frac{\partial}{\partial x}\left(\lim \frac{\partial u_n}{\partial x}\right) = \frac{\partial}{\partial x}\frac{\partial}{\partial x}\lim u_n$$

$$= \frac{\partial^2 u}{\partial x^2}.$$

Conclusion: $u = \lim u_n$ is a C^2 solution of the W.Eq

Answer 2 No, not necessarily, if we only know $u_n \to u$ uniformly

Example $u(x,t) = F(x+t) + F(x-t)$ where F is the "saw tooth function"



and F_n is "as drawn"

- F_n is C^2 by construction.
- $F_n \to F$ uniformly
(exercise: compute $\sup_x |F_n(x) - F(x)|$

Therefore $u_n(x,t) = F_n(x+t) + F_n(x-t)$

1. is a C^2 solution of the W.Eq
2. $u_n \to u$, where $u(x,t) = F(x+t) + F(x-t)$

So $\lim u_n = u$ exists but it is not a C^2 solution of W.Eq.

Answer 3 Yes: if u_n is a weak solution of the WEq and if $u_n \to u$ uniformly
then u is also a weak solution.

Proof Have to show: $\begin{cases} u \text{ is continuous} \\ u \text{ satisfies } (\bigstar) \end{cases}$

① $\left.\begin{array}{l} u_n \text{ continuous} \\ u_n \to u \text{ unif.} \end{array}\right\} \Rightarrow$ $u = \lim u_n$ is also continuous.

② If $u_n \to u$ uniformly then $\lim\limits_{n\to\infty} u_n(x,t) = u(x,t)$ for all x,t.

Therefore

$$u(x+h,t) + u(x-h,t) = \lim_{n\to\infty} u_n(x+h,t) + u_n(x-h,t)$$

$$= \lim_{n\to\infty} u_n(x,t+h) + u_n(x,t-h) \qquad (u_n \text{ satisfies } \bigstar)$$

$$= u(x,t+h) + u(x,t-h).$$

I.e. u satisfies $(\bigstar)$