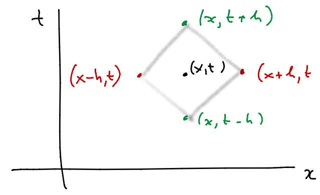
Generalized Solutions

Theorem 1 If $u:\mathbb{R}^2\to\mathbb{R}$ is a C^2 solution to the wave equation $u_{tt}=u_{xx}$ then

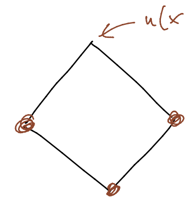
$$u(x+h,t)+u(x-h,t) = u(x,t+h)+u(x,t-h) \qquad (\star)$$

holds for all $x,t\in\mathbb{R}$, $h>0$.



→ Consequence

$u(x,t+h) = u(x-h,t)+u(x+h,t) - u(x,t-h)$



Proof: $u_{tt}=u_{xx}$ and u is $C^2 \Rightarrow$ there exist $F:\mathbb{R}\to\mathbb{R}$ and $G:\mathbb{R}\to\mathbb{R}$ such that $u(x,t)=F(x+t)+G(x-t)$ for all x,t.

$$u(x+h,t)+u(x-h,t) =$$
$$= F(x+h+t)+G(x+h-t)+F(x-h+t)+G(x-h-t)$$

$$u(x,t+h)+u(x,t-h) =$$
$$= F(x+t+h)+G(x-t-h)+F(x+t-h)+G(x-t+h) \qquad ////$$

Definition A continuous function $u:\mathbb{R}^2\to\mathbb{R}$ is a "weak solution" of the wave equation if for all $x,t\in\mathbb{R}$ and $h>0$ one has

$$(\star)\quad u(x+h,t)+u(x-h,t) = u(x,t+h)+u(x,t-h)$$

We have shown that if u is C^2 and $u_{tt}=u_{xx}$ then u is a weak solution to the wave equation.

A classical solution is a C^2 function $u:\mathbb{R}^2\to\mathbb{R}$ that satisfies $u_{tt}=u_{xx}$ at all $(x,t)\in\mathbb{R}^2$

- u classical solution $\Rightarrow$ u weak solution
- u is C^2 and a weak solution $\Rightarrow$ u classical solution

Theorem 2 If $u\in C^2$ satisfies $(\star)$ then $u_{tt}=u_{xx}$ for all x,t

proof. Let $(x,t)\in\mathbb{R}^2$ be given. For all $h>0$ we have

$$u(x+h,t)+u(x-h,t) = u(x,t+h)+u(x,t-h)$$

Differentiate w.r.t. h on both sides

$$\frac{d}{dh}u(x+h,t)+\frac{d}{dh}u(x-h,t) = \frac{d}{dh}u(x,t+h)+\frac{d}{dh}u(x,t-h)$$

$$\Rightarrow\quad u_x(x+h,t)-u_x(x-h,t) = u_t(x,t+h)-u_t(x,t-h)$$

for all $h>0$

$\varphi(x,t,h) \overset{\text{def}}{=} u(x+h,t)+u(x-h,t)$

$\frac{\partial\varphi}{\partial x}\quad \frac{\partial\varphi}{\partial t}\quad \frac{\partial\varphi}{\partial h}$

$\frac{\partial\varphi}{\partial x} = u_x(x+h,t)+u_x(x-h,t)$

$\frac{\partial\varphi}{\partial h} = u_x(x+h,t)-u_x(x-h,t)$

Differentiate again:

$$u_{xx}(x+h,t)+u_{xx}(x-h,t) = u_{tt}(x,t+h)+u_{tt}(x,t-h)$$

for all $h>0$. Take the limit $h\searrow 0$:

$$\cancel{2}u_{xx}(x,t) = \cancel{2}u_{tt}(x,t). \qquad ///$$

Theorem If F, G are continuous, then

$$u(x,t)=F(x+t)+G(x-t)$$

is a weak solution.

Proof We verify

$$u(x+h,t)+u(x-h,t) \overset{?}{=} u(x,t+h)+u(x,t-h) \qquad \star$$

This follows from the proof of Theorem 1 (today) ////

Def A classical solution to the wave equation is a function $u:\mathbb{R}^2\to\mathbb{R}$ for which

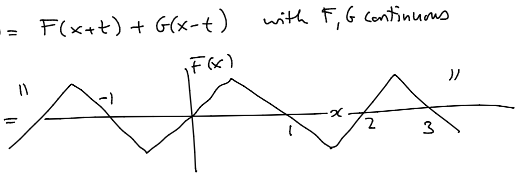
- u is C^2
- $u_{tt}=u_{xx}$ for all x,t.

Every classical solution is a weak solution (Theorem 2)

If u is a weak solution and if u is C^2 then u is a classical solution

Every function $u(x,t)=F(x+t)+G(x-t)$ with F,G continuous is a weak solution

Example Let $F(x) =$



$$G(x)=F(x)$$

Then $u(x,t)=F(x+t)+F(x-t)$ is a weak solution

However, u is not a classical solution because u is not C^2

Theorem If $u:\mathbb{R}^2\to\mathbb{R}$ is a weak solution then then there exist continuous $F, G:\mathbb{R}\to\mathbb{R}$ such that $u(x,t) = F(x+t)+G(x-t)$

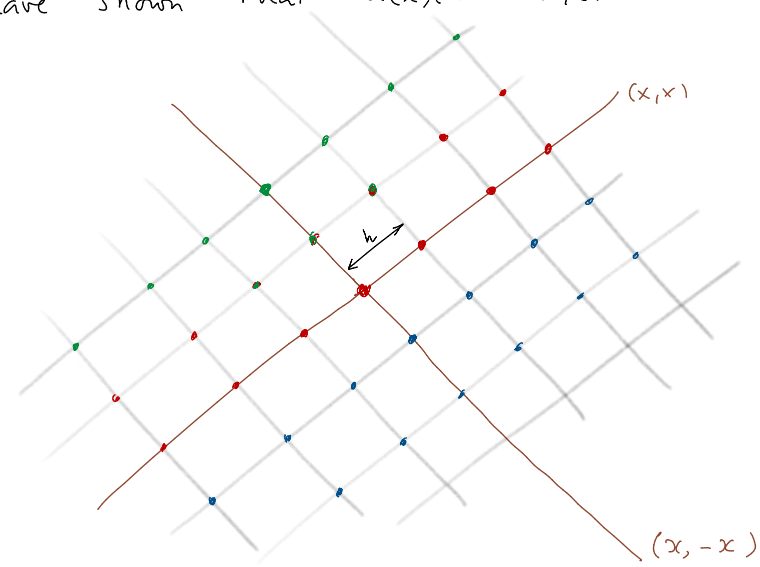
Proof Let u be a given weak solution.

Define
$$F(r) = u\left(\tfrac{r}{2}, \tfrac{r}{2}\right)$$
$$G(r) = u\left(\tfrac{r}{2}, -\tfrac{r}{2}\right) - u(0,0)$$
$$v(x,t) \overset{\text{def}}{=} F(x+t) + G(x-t).$$

Then
$$\begin{aligned} v(x,x) &= F(x+x) + G(x-x) = F(2x) + G(0)\\ &= u(x,x) + u(0,0) - u(0,0)\\ &= u(x,x) \qquad \text{for all } x\in\mathbb{R}\end{aligned}$$

$$\begin{aligned} v(x,-x) &= F(x-x) + G(x-(-x)) = F(0) + G(2x)\\ &= u(0,0) + u(x,-x) - u(0,0)\\ &= u(x,-x)\end{aligned}$$

We have shown that $u(x,t) = v(x,t)$ at these points

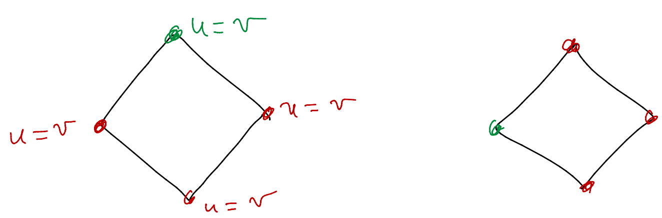


Using (✱) repeatedly we see that $u=v$ at all gridpoints

u is weak solution $\Rightarrow$ u has property (✱)

$\left.\begin{array}{l} v(x,t) = F(x+t)+G(x-t)\\ \text{is a weak solution}\end{array}\right\} \Rightarrow$ v has property (✱)

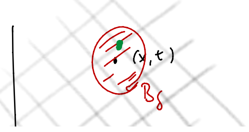
Therefore if $u=v$ at three corners of any 45° square then $u=v$ at the fourth corner of that square



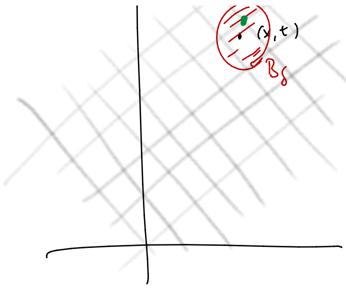
Let $(x,t)\in\mathbb{R}^2$ and $\varepsilon>0$ be given.

We will show $|u(x,t) - v(x,t)| < \varepsilon$

u, v are continuous $\Rightarrow$ there is a $\delta>0$ such that for all $(x',t')\in B_\delta(x,t)$ one has



$$|u(x',t') - u(x,t)| < \frac{\varepsilon}{2}$$



$|u(x',t') - u(x,t)| < \frac{\varepsilon}{2}$

and $|v(x',t') - v(x,t)| < \frac{\varepsilon}{2}$.

Choose $h>0$ so small that $B_\delta(x,t)$ contains a grid point (x',t')

Then:

equal because (x',t') is a gridpoint

$$|u(x,t) - v(x,t)| = |u(x,t) - u(x',t') + v(x',t') - v(x,t)|$$

$$\le |u(x,t) - u(x',t')| + |v(x',t') - v(x,t)|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This holds for every $\varepsilon > 0$. Therefore $u(x,t) = v(x,t)$.

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